MATH 2030 SOLUTIONS - ASSIGNMENT 1

1. A direction vector for ℓ_1 is $(1-2, 2-1, -1-3) = (-1, 1, -4)$. A direction vector for ℓ_2 is $(-1-0, 1-2, 2-3) = (-1, -1, 1)$. As these vectors are <u>not</u> scalar multiples of each other, the two lines are <u>not</u> parallel.

2. $$\|\underline{u}+\underline{v}\|^2 + \|\underline{u}-\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) + (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v})$$
$$= (\underline{u}+\underline{v})\cdot\underline{u} + (\underline{u}+\underline{v})\cdot\underline{v} + (\underline{u}-\underline{v})\cdot\underline{u} - (\underline{u}-\underline{v})\cdot\underline{v}$$
$$= \underline{u}\cdot\underline{u} + \underline{v}\cdot\underline{u} + \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v} + \underline{u}\cdot\underline{u} - \underline{v}\cdot\underline{u} - \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$
$$= 2\underline{u}\cdot\underline{u} + 2\underline{u}\cdot\underline{v} + 2\underline{v}\cdot\underline{v} - 2\underline{u}\cdot\underline{v}$$
$$= 2\|\underline{u}\|^2 + 2\|\underline{v}\|^2$$

3. A direction vector perpendicular to both $\underline{u}$ and $\underline{v}$ is
$$\underline{u}\times\underline{v} = \left(\begin{vmatrix} 2 & 3 \\ 0 & 1/2 \end{vmatrix}, -\begin{vmatrix} -1 & 3 \\ 2 & 1/2 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} \right)$$
$$= \left(1, \tfrac{13}{2}, -4\right)$$
The vector form of the line is
$$\underline{x}(t) = (0,0,0) + t\left(1, \tfrac{13}{2}, -4\right) = \left(t, \tfrac{13}{2}t, -4t\right)$$
and it has parametric equations
$$x = t, \quad y = \tfrac{13}{2}t, \quad z = -4t, \quad t \in \mathbb{R}$$

4. A general point on the line is $Q(t, 2+t, 1+t)$. Q is closest to P when $\overrightarrow{PQ}$ is perpendicular to the line, which has direction $(1,1,1)$. Thus the point Q on the line is closest to P just in case
$$\overrightarrow{PQ}\cdot(1,1,1) = 0 \iff (t-1, 3+t, t)\cdot(1,1,1) = 0$$
$$\iff t-1+3+t+t = 0$$
$$\iff 3t = -2$$
$$\iff t = -2/3$$
Thus the closest point Q on the line is $\left(\frac{-2}{3}, \frac{4}{3}, \frac{1}{3}\right)$ and the shortest distance is $\|\overrightarrow{PQ}\| = \sqrt{(-2/3-1)^2 + (3-2/3)^2 + (-2/3)^2} = \frac{1}{3}\sqrt{25+49+4} = \frac{\sqrt{78}}{3}$.

5. The planes $x+y+z=2$ and $-x-y+4z=3$ intersect when

$$\begin{cases} x+y+z=2 \\ -x-y+4z=3 \end{cases} \iff \begin{cases} x+y+z=2 \\ 5z=5 \end{cases} \iff \begin{cases} x+y+z=2 \\ z=1 \end{cases} \iff \begin{cases} x+y=1 \\ z=1 \end{cases}$$

So the 2 planes intersect in the line

$$\begin{aligned} x &= t \\ y &= -t+1 \\ z &= 1 \end{aligned} \quad , t \in \mathbb{R}$$

Note that this line satisfies the third equation:

$$x+y+2z = t+(-t+1)+2\cdot 1 = 3$$

Thus all 3 planes contain the line $(x,y,z)=(t,-t+1,1)$ $(t\in\mathbb{R})$.
As the first 2 planes intersect in exactly this line, the three planes intersect in the line $(x,y,z)=(t,-t+1,1)$, $t\in\mathbb{R}$.

<u>OR</u>

$$\begin{cases} x+y+z=2 \\ -x-y+4z=3 \\ x+y+2z=3 \end{cases}$$

$$\iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -1 & -1 & 4 & 3 \\ 1 & 1 & 2 & 3 \end{array}\right] \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \iff \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \iff \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution to intersection of three planes is

$$\begin{aligned} x+y &= 1 \\ z &= 1 \end{aligned}$$

Setting $y=t$, we get

$$\begin{aligned} x &= -t+1 \\ y &= t \\ z &= 1 \end{aligned}$$

so all 3 planes intersect in the line

$$(x,y,z) = (-t+1, t, 1) = (1,0,1)+t(-1,1,0), \quad t\in\mathbb{R}$$

(note that this is the same line as above, with t replaced by $-t$).