MATH 2030 SOLUTIONS-WRITTEN ASSIGNMENT # 2 FALL 2014

1. $A^3 = I \Leftrightarrow A \cdot A \cdot A = I$

$\rightarrow \det(A \cdot A \cdot A) = \det I$

$\rightarrow \det A \cdot \det A \cdot \det A = 1$

$\rightarrow (\det A)^3 = 1$

$\rightarrow \det A = 1^{1/3} = 1$

2. $A\underline{x} = \underline{b}$, where $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}$. Now

$$\det A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -1 - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1-1 = -2 \neq 0$$

so A is invertible and we can apply Cramer's Rule:

$$x = \frac{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -2 & 0 & 0 & -1 \end{vmatrix}}{-2} = \frac{-\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ -2 & 0 & -1 \end{vmatrix}}{-2} = \frac{1}{2}\left(\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} \right)$$

$$= \frac{1}{2}\left(-1 - (-2+2)\right) = -\frac{1}{2}$$

$$y = \frac{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & -2 & 0 & -1 \end{vmatrix}}{-2} = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ -2 & 0 & -1 \end{vmatrix}}{-2} = -\frac{1}{2}\begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = \frac{1}{2}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & -2 & -1 \end{vmatrix}}{-2} = \frac{\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & -2 & -1 \end{vmatrix}}{-2} = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & -2 & -1 \end{vmatrix}}{-2} = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}}{-2} = \frac{\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}}{-2} = \frac{1}{2}$$

$$w = \frac{\begin{vmatrix}1&1&0&0\\0&1&1&1\\0&0&1&2\\1&0&0&-2\end{vmatrix}}{-2} = \frac{\begin{vmatrix}1&1&0&0\\0&1&1&1\\0&0&1&2\\0&-1&0&-2\end{vmatrix}}{-2} = \frac{\begin{vmatrix}1&1&1\\0&1&2\\-1&0&-2\end{vmatrix}}{-2} = \frac{\begin{vmatrix}1&1&1\\0&1&2\\0&1&-1\end{vmatrix}}{-2} = \frac{\begin{vmatrix}1&2\\1&-1\end{vmatrix}}{-2} = \frac{3}{2}$$

$\therefore$ The unique solution is $(x,y,z,w) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

3. $\det(A-\lambda I)=0 \Leftrightarrow \begin{vmatrix}2-\lambda&1&0\\1&2-\lambda&0\\0&0&1-\lambda\end{vmatrix}=0$

$$\Leftrightarrow (2-\lambda)\begin{vmatrix}2-\lambda&0\\0&1-\lambda\end{vmatrix} - \begin{vmatrix}1&0\\0&1-\lambda\end{vmatrix}=0$$
$$\Leftrightarrow (2-\lambda)^2(1-\lambda)-(1-\lambda)=0$$
$$\Leftrightarrow (1-\lambda)\left((2-\lambda)^2-1\right)=0$$
$$\Leftrightarrow (1-\lambda)(\lambda^2-4\lambda+4-1)=0$$
$$\Leftrightarrow (1-\lambda)(\lambda^2-4\lambda+3)=0$$
$$\Leftrightarrow (1-\lambda)(\lambda-1)(\lambda-3)=0$$
$$\Leftrightarrow \lambda=1 \text{ or } 3 \qquad \text{eigenvalues}$$

$\lambda=1$: Solve $(A-1I)\mathbf{x}=\underline{0} \Leftrightarrow \left[\begin{array}{ccc|c}1&1&0&0\\1&1&0&0\\0&0&0&0\end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c}1&1&0&0\\0&0&0&0\\0&0&0&0\end{array}\right]$

$x+y=0$. Free variables are y and z, so set $y=t$, $z=s$.
Then $\begin{cases}x=-t\\y=t\\z=s\end{cases}$, so eigenvectors corresponding to eigenvalue $\lambda=1$ are $(-t,t,s)$, where $s,t\in\mathbb{R}$ and not both are 0.

$\lambda=3$: Solve $(A-3I)\mathbf{x}=\underline{0} \Leftrightarrow \left[\begin{array}{ccc|c}-1&1&0&0\\1&-1&0&0\\0&0&-2&0\end{array}\right]$

$$\Leftrightarrow \left[\begin{array}{ccc|c}1&-1&0&0\\1&-1&0&0\\0&0&1&0\end{array}\right]$$

$$\Leftrightarrow \left[\begin{array}{ccc|c}1&-1&0&0\\0&0&0&0\\0&0&1&0\end{array}\right]$$

$\begin{cases} x-y=0 \\ z=0 \end{cases}$. Free variable is y, so set $y=t$. Then

$\begin{cases} x=t \\ y=t \\ z=0 \end{cases}$, so eigenvectors corresponding to eigenvalue $\lambda=3$ are $(t,t,0)$ where $t\in\mathbb{R}, t\neq 0$.

4. $\det A = \begin{vmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda & 1 \\ 2 & 2 & 1+\lambda \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda & 1 \\ 2 & 1+\lambda \end{vmatrix} = (\lambda-1)[\lambda(1+\lambda)-2]$

$= (\lambda-1)(\lambda^2+\lambda-2)$
$= (\lambda-1)(\lambda-1)(\lambda+2)$
$= (\lambda-1)^2(\lambda+2)$

$\therefore \det A = 0 \iff \lambda=1$ or $\lambda=-2$, so A is not invertible iff $\lambda=1$ or -2.

5. From Theorem 2 of Section 4.3,
$$A^{-1} = \frac{1}{\det A}\operatorname{Adj} A$$
as A is invertible, since $\det A = 1$ or -1 (and therefore $\neq 0$). Now as $\det A = 1$ or -1, $\frac{1}{\det A} = 1$ or -1, so it is enough to show $\operatorname{Adj} A$ has integer entries (since then so will $A^{-1} = \frac{1}{\det A}\operatorname{Adj} A$). $\operatorname{Adj} A$ is filled with cofactors of A, which are determinants of submatrices of A, with a + or − in front. So it is enough to show that if a matrix B has all integer entries, $\det B$ is an integer.

Note that if B has order 2, then $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d are integers, and clearly $\det B = ad-bc$ is also an integer.

If B has order 3, say $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, with all integer entries, then expanding along the top row,

$$\det B = \underset{\text{integer}}{a}\left|\quad\right| - \underset{\text{integer}}{b}\left|\quad\right| + \underset{\text{integer}}{c}\left|\quad\right|, \text{ so by}$$

2×2 matrices with all integer entries, so by first part these are integers

our work for the 2×2 case, $\det B$ is again an integer.

For B of order 4, we similarly use the 3×3 case, and so on. It follows that for any square matrix B, if B has all integer entries, $\det B$ is an integer.

Applying this to the cofactors of A, we see that $\operatorname{Adj} A$ has all integer entries, and hence so does

$$A^{-1} = \frac{1}{\det A} \operatorname{Adj} A$$

as $\det A = 1$ or -1. $\square$