MATH/STAT 2300 ASSIGNMENT 1 – SOLUTIONS WINTER 2016

1. $a_1 = 6$
$a_2 = 15$
$a_6 = 111{,}050{,}740{,}260{,}915$

2. a. $a_{n+1} = \frac{3}{4}a_n$, $a_0 = 500$

n	a_n (amount of Blarbomax in patients blood at end of n^{th} day)
0	500
1	375
2	281.25
3	210.94
4	158.20
5	118.65
6	88.99
7	66.74

b. $a_{n+1} = \frac{1}{4}a_n + 500$, $a_0 = 500$

n	a_n
0	500
1	875
2	1156.25
3	1367.19
4	1525.39
5	1644.04
6	1733.03
7	1799.77

c. In part a, the amount of Blarbomax in the patients blood stream tends to 0, as

$$a_n = 500\left(\tfrac{3}{4}\right)^n$$

so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 500\left(\tfrac{3}{4}\right)^n = 500 \lim_{n\to\infty}\left(\tfrac{3}{4}\right)^n = 500\cdot 0 = 0$.

In part b, the amount of Blarbomax in the patients blood stream tends to 2000.00 mg, an equilibrium value. We can find the equilibrium exactly:

$$a = \left(\tfrac{3}{4}a\right) + 500$$

$$\to \quad \tfrac{1}{4}a = 500$$

$$\to \quad a = \frac{2000}{1} = 2000$$

3. Let a_n be the amount in the account at the end of the n^{th} month (after the payment has come out). Then

$$a_{n+1} = \underbrace{1.0075a_n}_{\text{accumulated amount with interest}} - \underbrace{p}_{\text{payment}}.$$

Solution is of form

$$a_n = C\cdot 1.0075^n - \frac{p}{1-1.0075} = C\cdot 1.0075^n + \frac{p}{.0075}$$

We know $a_0 = 9000$ and want $a_{36} = 0$ so

$$9000 = C\cdot 1.0075^0 + \frac{p}{.0075} = C + \frac{p}{.0075}$$

$$0 = C\cdot 1.0075^{36} + \frac{p}{.0075} = C\cdot 1.0075^{36} + \frac{p}{.0075}$$

Solving, we find $C = -29159.67918$, $p = 286.20$ (to the nearest cent)

$\therefore$ The monthly payment p should be 286.20 (to the nearest penny)

4. To find the equilibrium values, we solve:

$$M = 1.2M - 0.001 \cdot O \cdot M$$
$$O = 0.7O + 0.002 \cdot O \cdot M$$

The solutions are $M = O = 0$, and $M = 150$, $O = 200$.

a. The signs of the coefficients of M and O in the first and second dynamical systems are positive (as in the owl-hawks model), 1.2 and 0.7, indicative of the positive reproductive rates as a factor in the growth of mice and owls (mice have a higher reproductive rate). The -0.001 coefficent indicates that in mice-owl interactions, some proportion of mice are lost (i.e. die), while the 0.002 coefficient indicates that in mice-owl interactions, there is some advantage to the owls (in terms of growth).

b. (See attached worksheet)
Case A, B, C, D - in all cases, don't settle down into any problem.

c. (See attached worksheet)
Model is sensitive to the coefficients and to the starting values.

5. (a) We need to solve

$$R = \tfrac{1}{3}R + \tfrac{1}{3}S + \tfrac{1}{3}T$$
$$S = \tfrac{1}{2}R + \tfrac{1}{6}S$$
$$T = \tfrac{1}{6}R + \tfrac{1}{2}S + \tfrac{2}{3}T$$
$$R + S + T = 1$$

The solution is $R = \frac{1}{3}$, $S = \frac{1}{5}$, $T = \frac{7}{15}$, so the equilibrium is $(\frac{1}{3}, \frac{1}{5}, \frac{7}{15})$.

(b) From the attached worksheet, the equilibrium appears stable

```
# Assignment 1 - MATH and STAT 2300
#
#4
eq1 := M = 1.2 M - 0.001·O·M;
eq2 := O = 0.70·O + 0.002·O·M;
```

$$M = 1.2\,M - 0.001\,\mathrm{O}\,M$$

$$\mathrm{O} = 0.70\,\mathrm{O} + 0.002\,\mathrm{O}\,M \tag{1}$$

```
solve({eq1, eq2}, {O, M});
```

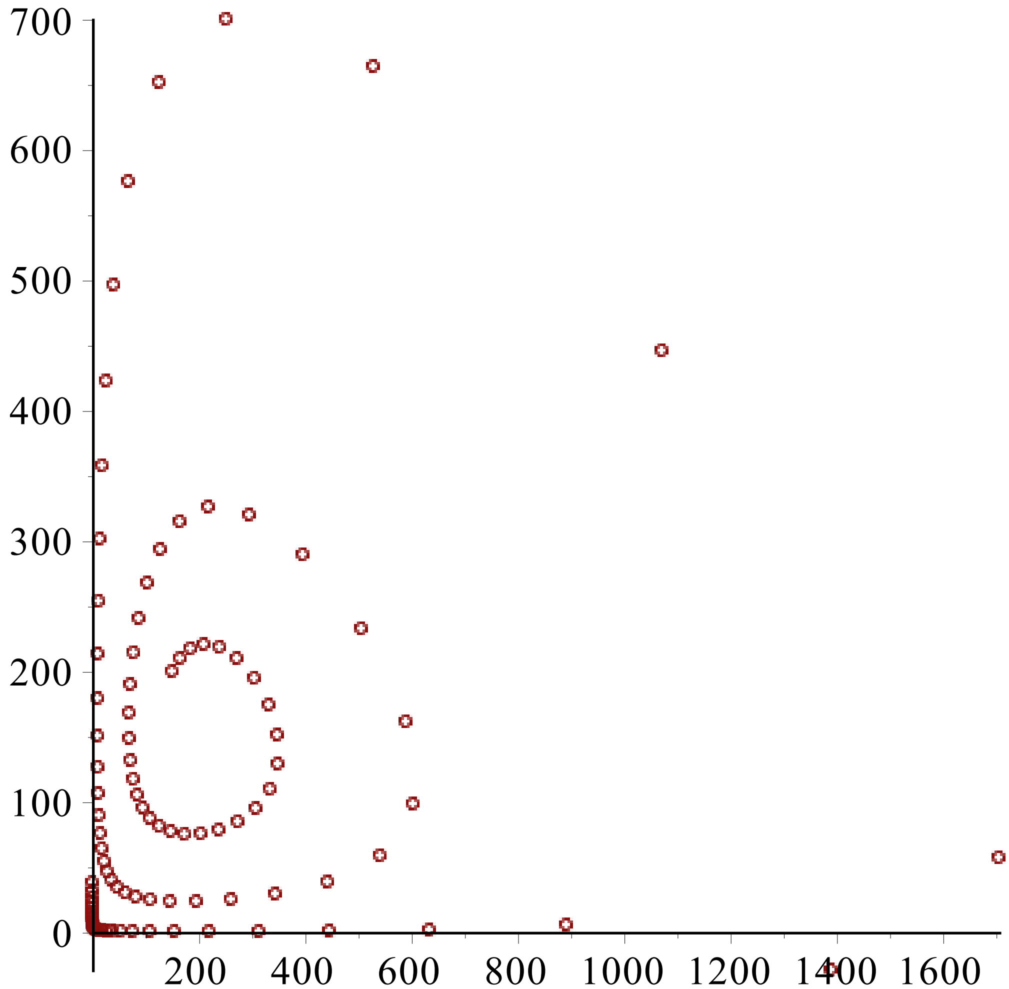
$$\{M = 0., \mathrm{O} = 0.\}, \{M = 150., \mathrm{O} = 200.\} \tag{2}$$

```
funcM := (Ow, Mi) → 1.2·Mi - 0.001·Ow·Mi;
funcO := (Ow, Mi) → 0.7·Ow + 0.002·Ow·Mice;
```

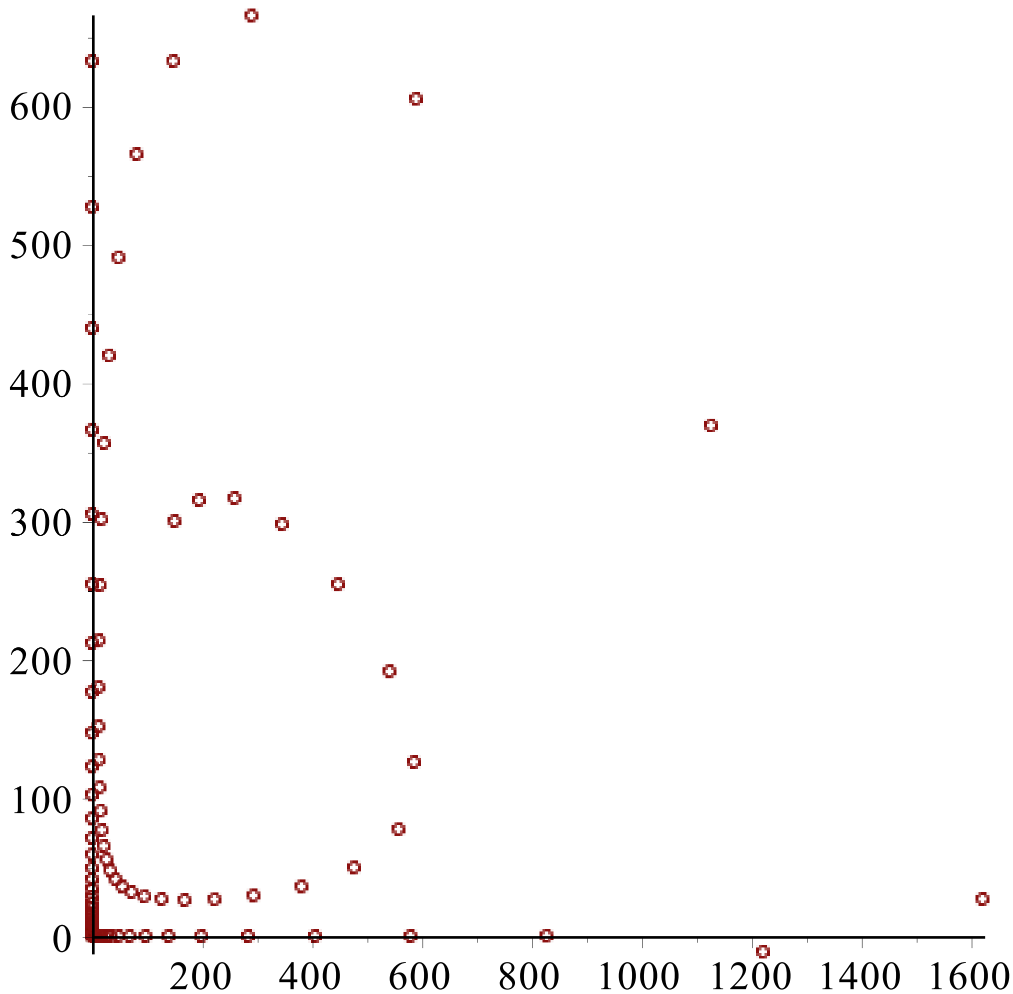
$$(Ow, Mi) \rightarrow 1.2\,Mi + (-1)\cdot 0.001\,Ow\,Mi$$

$$(Ow, Mi) \rightarrow 0.7\,Ow + 0.002\,Ow\,Mice \tag{3}$$

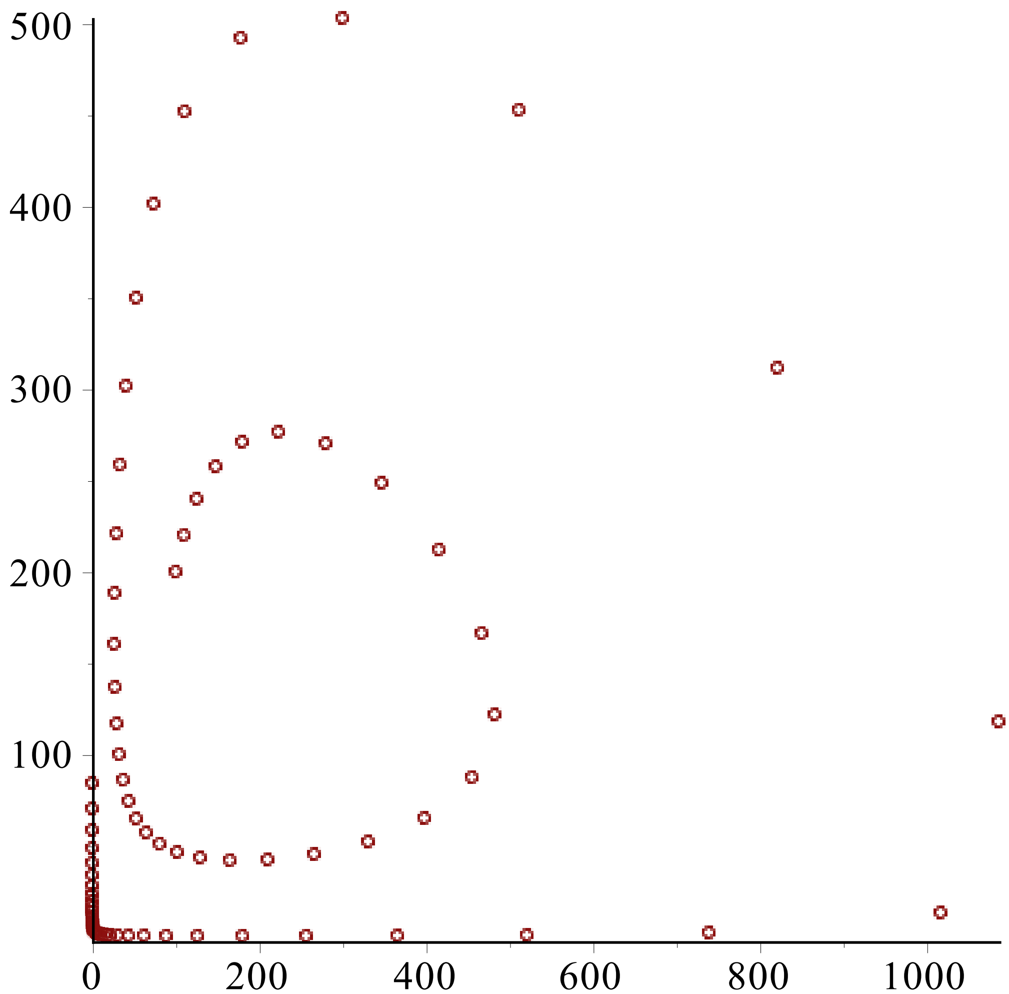
```
Owls := 150 :
Mice := 200 :
# print(0, Owls, Mice);
pts := [Owls, Mice] :
for i from 1 to 100 do
   tempM := funcM(Owls, Mice) :
   Owls := funcO(Owls, Mice) :
   Mice := tempM :
#   print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
od:
pts := [pts] :
plot(pts, symbol = circle, style = point);
```



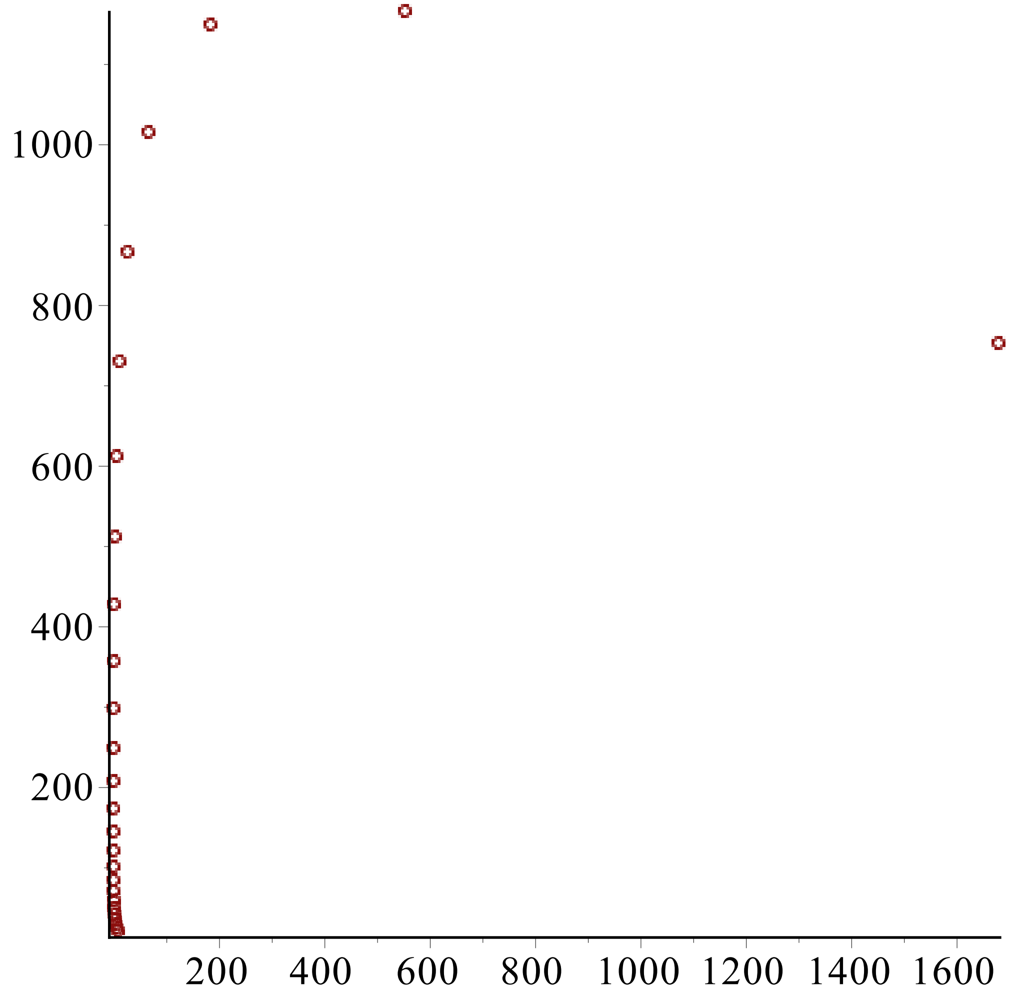

```
Owls := 150 :
 Mice := 300 :
# print(0, Owls, Mice);
 pts := [Owls, Mice] :
 for i from 1 to 100 do
    tempM := funcM (Owls, Mice) :
    Owls := funcO (Owls, Mice) :
    Mice := tempM :
#    print(i, Owls, Mice);
   pts := pts, [Owls, Mice] :
 od:
  pts := [pts] :
 plot(pts, symbol = circle, style = point);
```




```
Owls := 100 :
Mice := 200 :
# print(0, Owls, Mice);
pts := [Owls, Mice] :
for i from 1 to 75 do
   tempM := funcM(Owls, Mice) :
   Owls := funcO(Owls, Mice) :
   Mice := tempM :
#    print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
od:
pts := [pts] :
plot(pts, symbol = circle, style = point);
```




```
Owls := 10 :
 Mice := 20 :
 #print(0, Owls, Mice);
 pts := [Owls, Mice] :
 for i from 1 to 25 do
   tempM := funcM(Owls, Mice) :
   Owls := funcO(Owls, Mice) :
   Mice := tempM :
 #   print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
 od:
  pts := [pts] :
 plot(pts, symbol = circle, style = point);
```



$$\{M = 0., O = 0.\}, \{M = 150., O = 200.\} \tag{4}$$

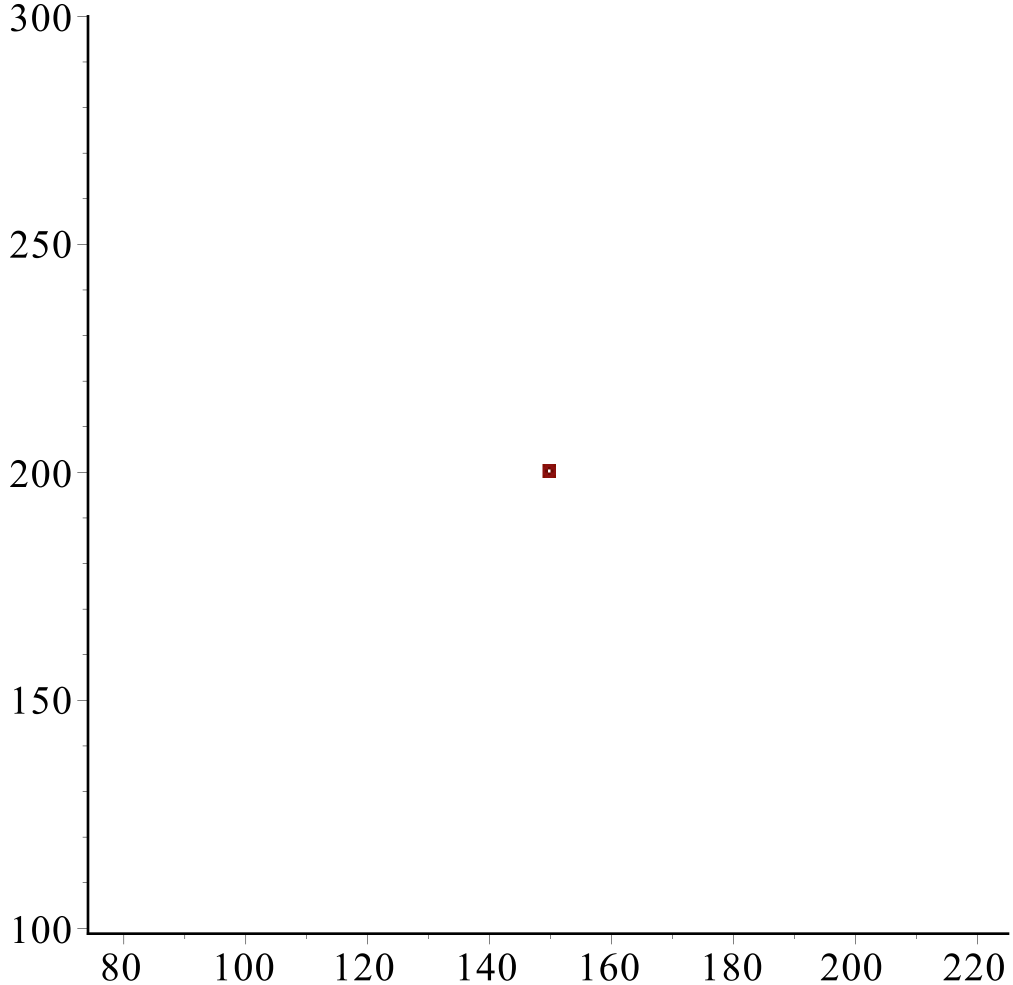
 # Now we change the coefficients slightly in the dynamical system
 # and we try also various star values. In some case (e.g. the first) the behavious is quite different, indicating that
 # the model is sensitive to the coefficients.

$funcM := (Ow, Mi) \rightarrow 1.3 \cdot Mi - 0.002 \cdot Ow \cdot Mi;$
$funcO := (Ow, Mi) \rightarrow 0.8 \cdot Ow + 0.001 \cdot Ow \cdot Mice;$

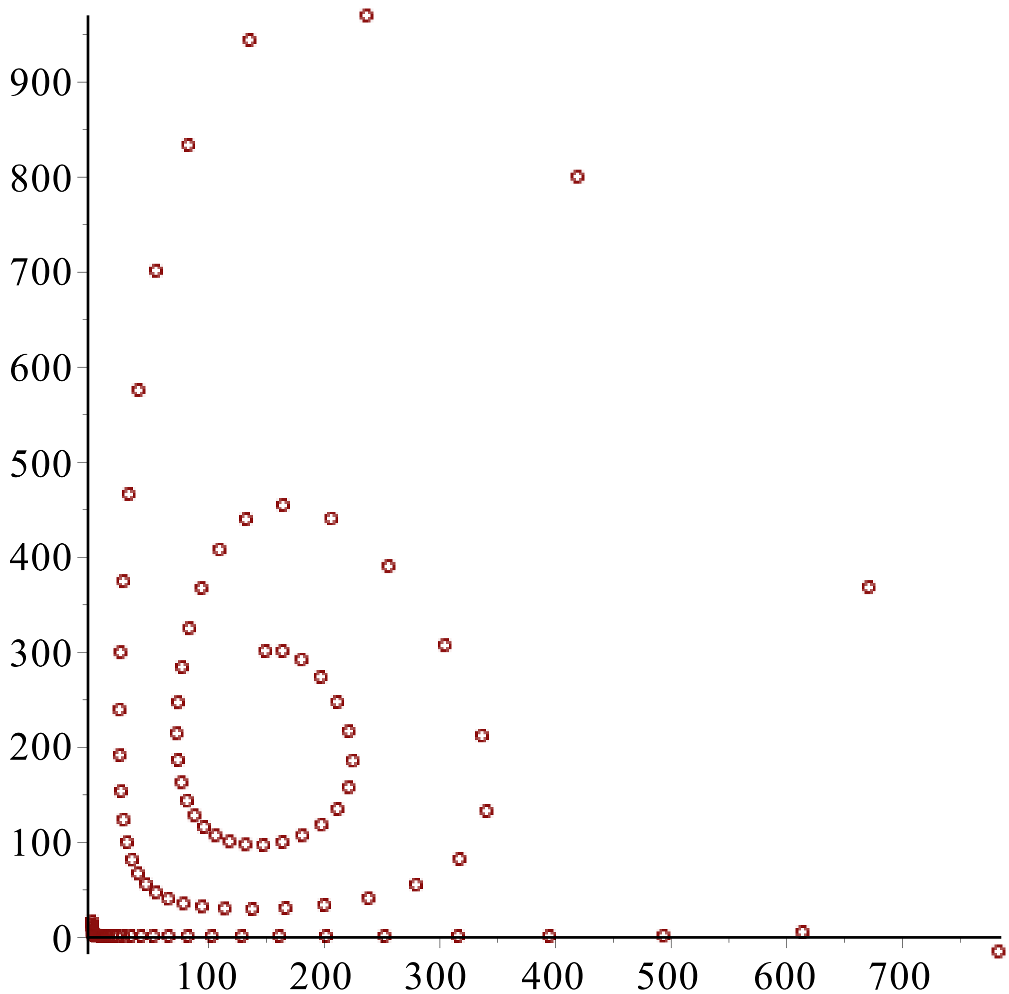
$$(Ow, Mi) \rightarrow 1.3\ Mi + (-1) \cdot 0.002\ Ow\ Mi$$
$$(Ow, Mi) \rightarrow 0.8\ Ow + 0.001\ Ow\ Mice \tag{5}$$

$Owls := 150:$
$Mice := 200:$
$\#print(0, Owls, Mice);$
$pts := [Owls, Mice]:$
for i **from** 1 **to** 100 **do**
$tempM := funcM(Owls, Mice):$
$Owls := funcO(Owls, Mice):$
$Mice := tempM:$
$\# \quad print(i, Owls, Mice);$

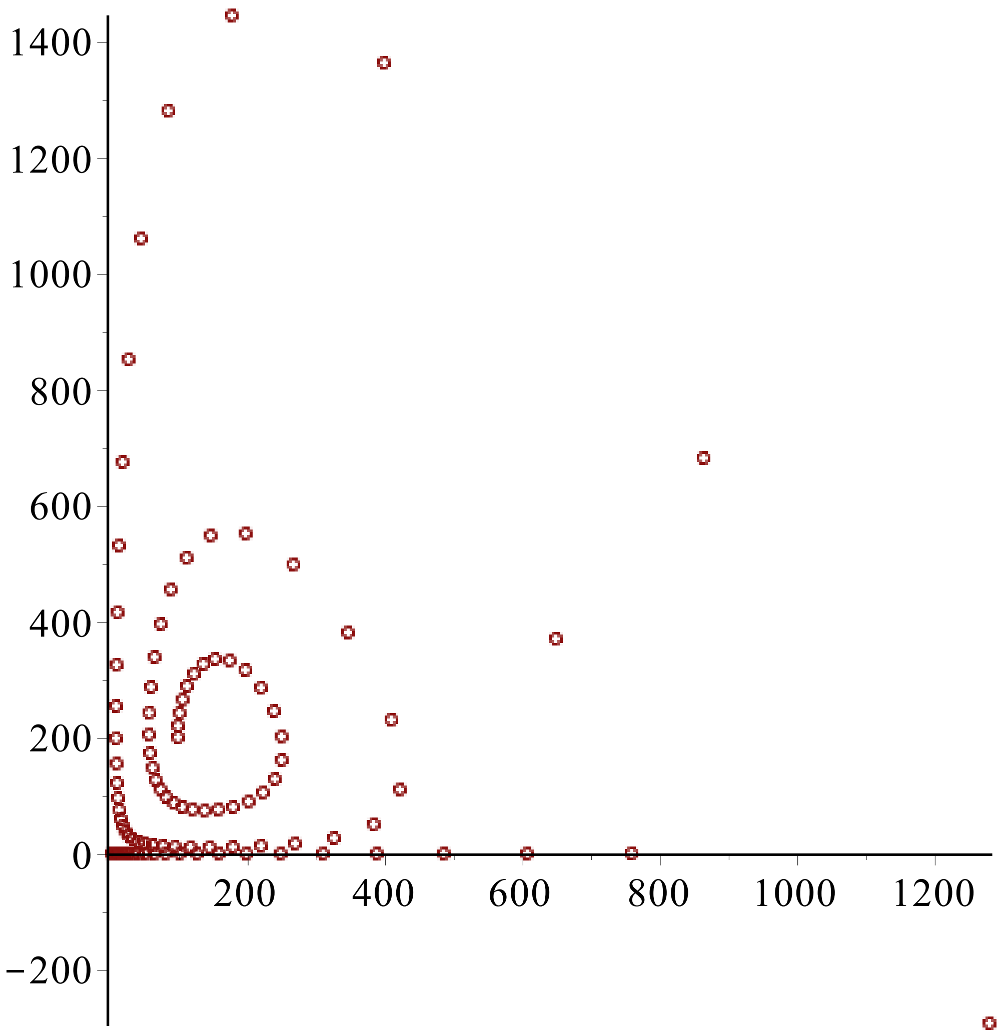
```
  pts := pts, [Owls, Mice] :
 od:
 pts := [pts] :
plot(pts, symbol = circle, style = point);
```



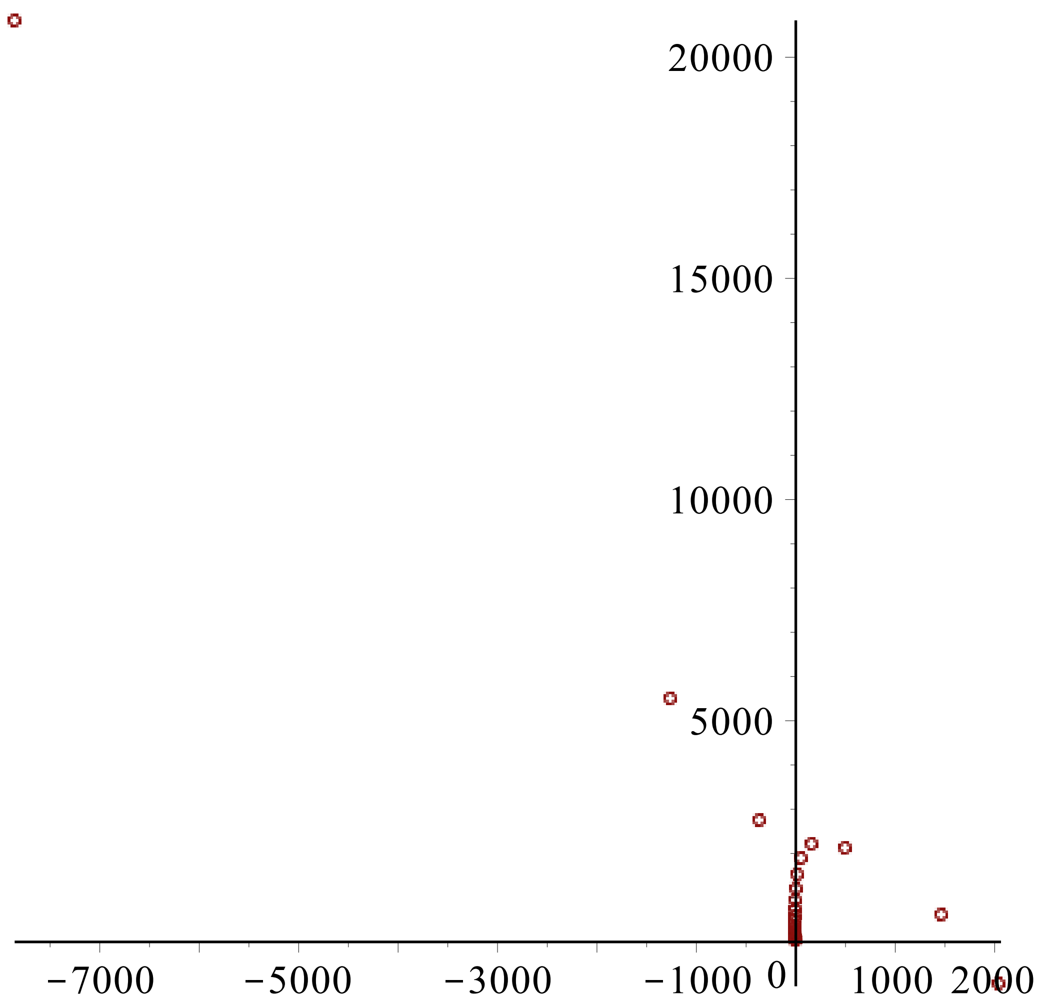
```
Owls := 150 :
 Mice := 300 :
 #print(0, Owls, Mice);
 pts := [Owls, Mice] :
 for i from 1 to 100 do
   tempM := funcM(Owls, Mice) :
   Owls := funcO(Owls, Mice) :
   Mice := tempM :
#   print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
 od:
 pts := [pts] :
plot(pts, symbol = circle, style = point);
```




```
Owls := 100 :
 Mice := 200 :
 #print(0, Owls, Mice);
 pts := [Owls, Mice] :
 for i from 1 to 100 do
   tempM := funcM(Owls, Mice) :
   Owls := funcO(Owls, Mice) :
   Mice := tempM :
#    print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
 od:
 pts := [pts] :
plot(pts, symbol = circle, style = point);
```




```
Owls := 10 :
 Mice := 20 :
 #print(0, Owls, Mice);
 pts := [Owls, Mice] :
 for i from 1 to 25 do
   tempM := funcM(Owls, Mice) :
   Owls := funcO(Owls, Mice) :
   Mice := tempM :
#    print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
 od:
 pts := [pts] :
plot(pts, symbol = circle, style = point);
```



Back to the original model, we'll see how themodel is sensitive to the initial values by taking some values clseo to the
equilibrium vaues and see that we have different behaviour.

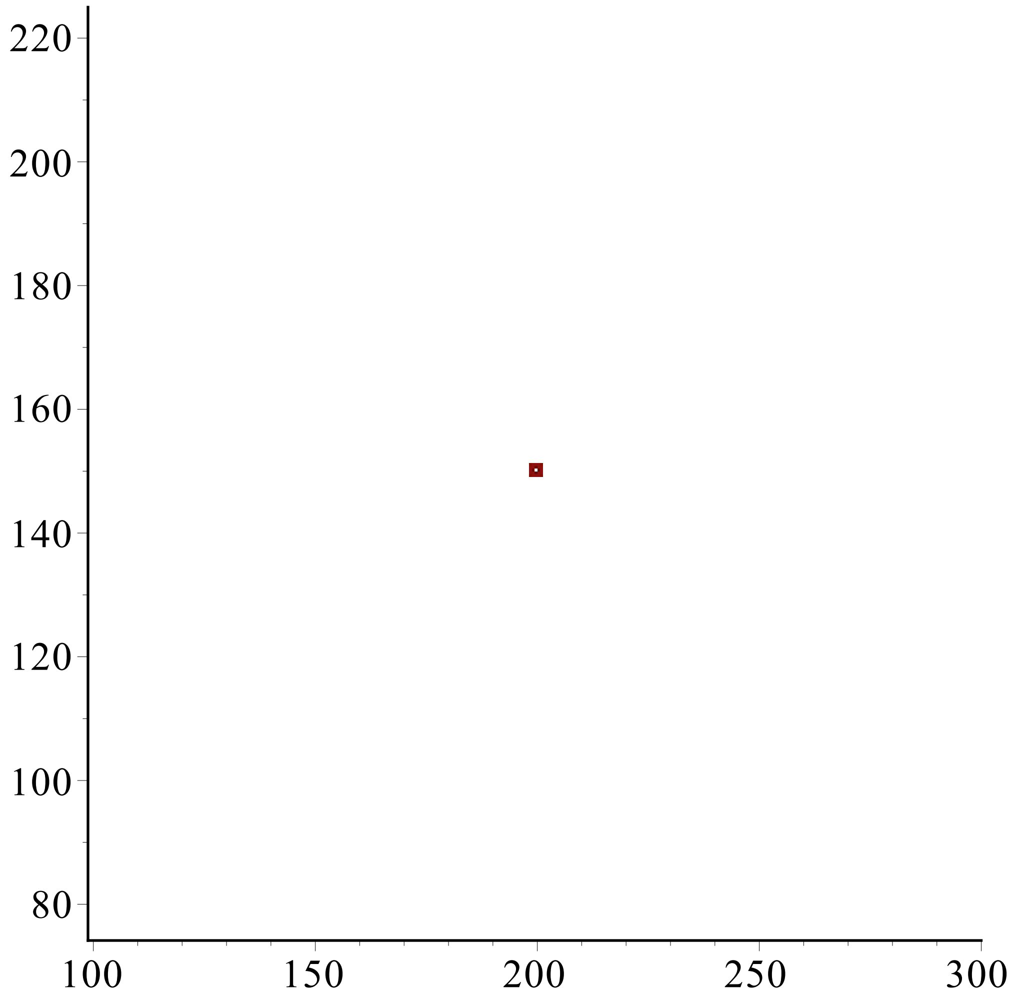
$funcM := (Ow, Mi) \rightarrow 1.2 \cdot Mi - 0.001 \cdot Ow \cdot Mi;$
$funcO := (Ow, Mi) \rightarrow 0.7 \cdot Ow + 0.002 \cdot Ow \cdot Mice;$

$$(Ow, Mi) \rightarrow 1.2\ Mi + (-1) \cdot 0.001\ Ow\ Mi$$

$$(Ow, Mi) \rightarrow 0.7\ Ow + 0.002\ Ow\ Mice \quad \textbf{(6)}$$

```
Owls := 200 :
Mice := 150 :
#print(0, Owls, Mice);
pts := [Owls, Mice] :
for i from 1 to 100 do
  tempM := funcM(Owls, Mice) :
  Owls := funcO(Owls, Mice) :
  Mice := tempM :
#   print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
od:
```

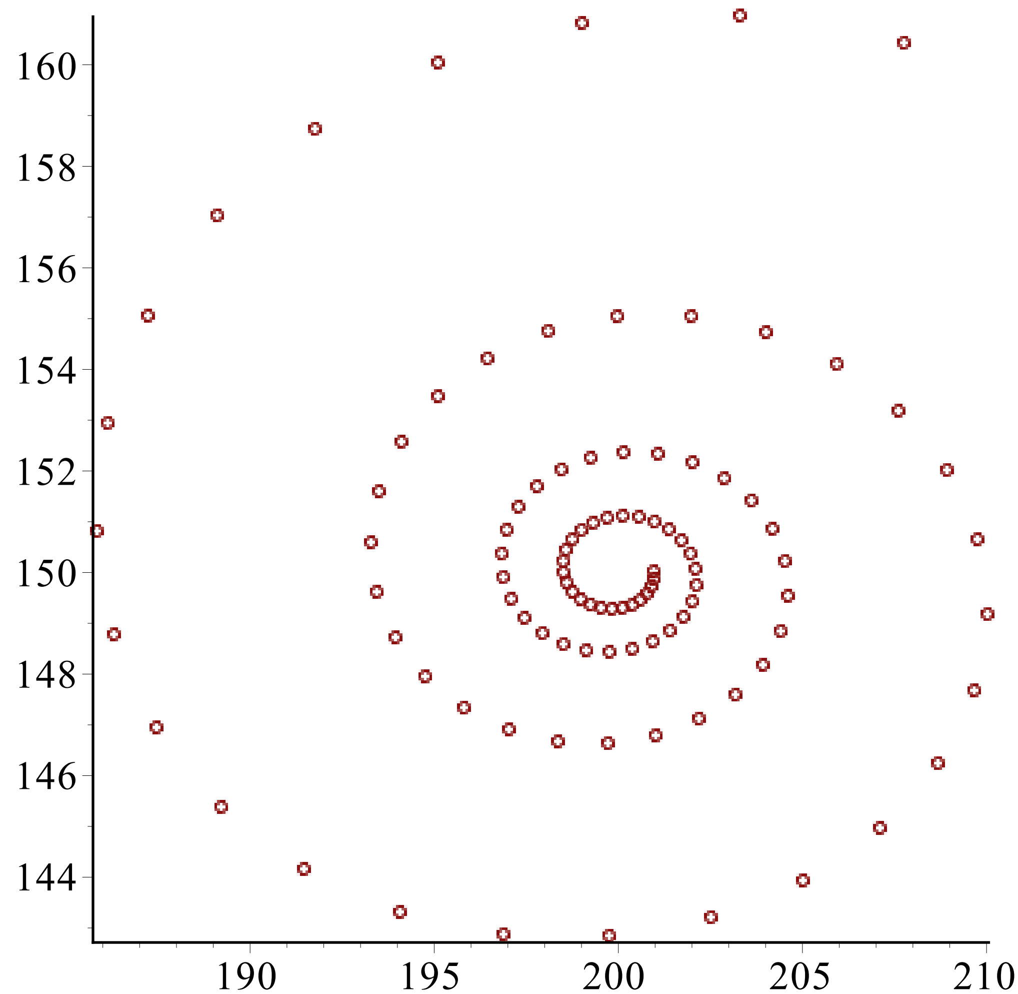
```
pts := [pts]:
plot(pts, symbol = circle, style = point);
```



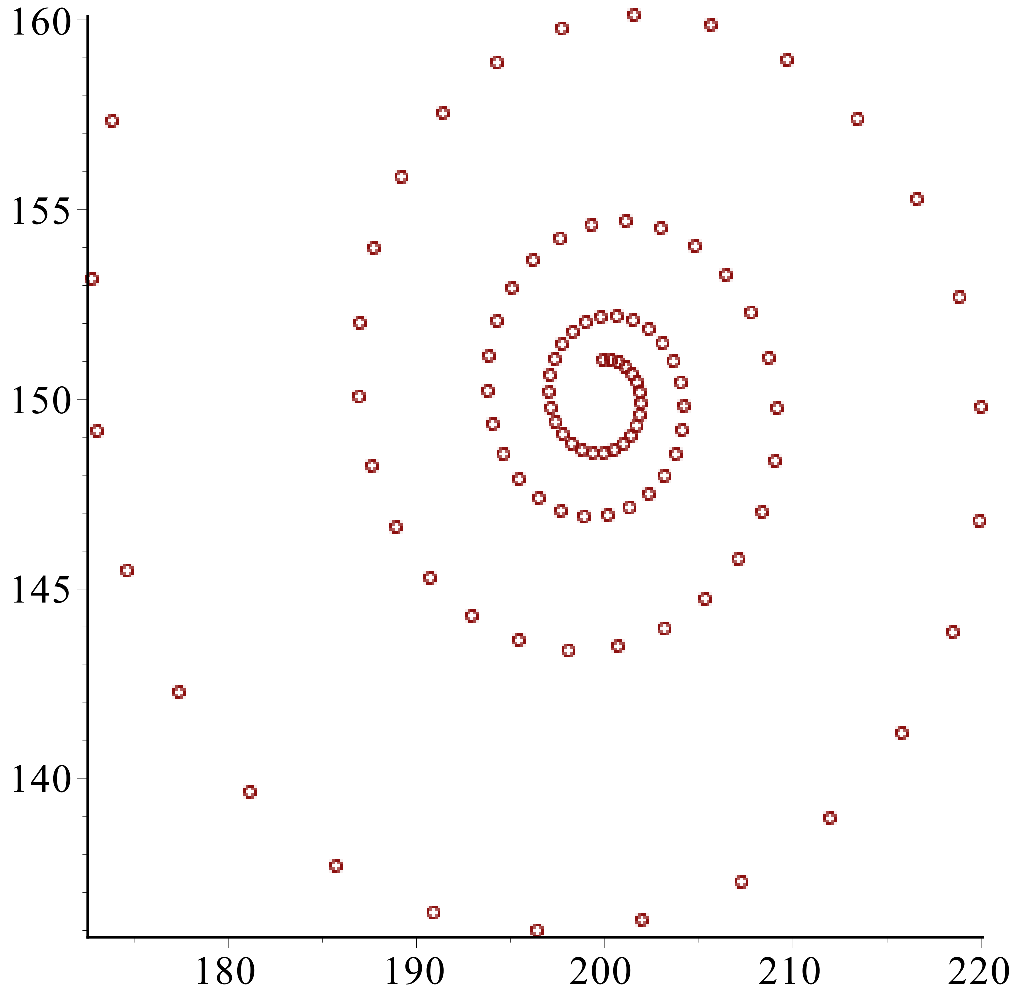

```
# That yields always the same value.

# Now what about points close to this equilibrium value?

Owls := 201 :
Mice := 150 :
#print(0, Owls, Mice);
pts := [Owls, Mice] :
for i from 1 to 100 do
   tempM := funcM(Owls, Mice) :
   Owls := funcO(Owls, Mice) :
   Mice := tempM :
#    print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
od:
pts := [pts] :
plot(pts, symbol = circle, style = point);
```




```
Owls := 200 :
Mice := 151 :
#print(0, Owls, Mice);
pts := [Owls, Mice] :
for i from 1 to 100 do
  tempM := funcM(Owls, Mice) :
  Owls := funcO(Owls, Mice) :
  Mice := tempM :
#   print(i, Owls, Mice);
  pts := pts, [Owls, Mice] :
od:
pts := [pts] :
plot(pts, symbol = circle, style = point);
```



 # In any event, we don't approach the equilibrium, so the model is sensitive to the starting values.

 # Q5
#b

$funcR := (R, S, T) \rightarrow \frac{1}{3} \cdot R + \frac{1}{3} \cdot S + \frac{1}{3} \cdot T :$

$funcS := (R, S, T) \rightarrow \frac{1}{2} \cdot R + \frac{1}{6} \cdot S :$

$funcT := (R, S, T) \rightarrow \frac{1}{6} \cdot R + \frac{1}{2} \cdot S + \frac{2}{3} \cdot T :$

$eq1 := R = funcR(R, S, T) :$
$eq2 := S = funcS(R, S, T) :$
$eq3 := T = funcT(R, S, T) :$
$eq4 := R + S + T = 1 :$
$solve(\{eq1, eq2, eq3, eq4\}, \{R, S, T\});$
$evalf(\%);$

$$\left\{R = \frac{1}{3}, S = \frac{1}{5}, T = \frac{7}{15}\right\}$$

$$\{R = 0.3333333333, S = 0.2000000000, T = 0.4666666667\} \tag{7}$$

 # Let's test out some vauew near the equilibrium point
#

```
R := 0.30 :
S := 0.25 :
T := 1 − R − S :
print(R, S, T);
for i from 1 to 20 do
  tempR := funcR(R, S, T) :
  tempS := funcS(R, S, T) :
  T := funcT(R, S, T) :
  R := tempR :
  S := tempS :
  print(R, S, T);
od:
```

$$0.30, 0.25, 0.45$$
$$0.3333333333, 0.1916666667, 0.4750000000$$
$$0.3333333333, 0.1986111110, 0.4680555556$$
$$0.3333333333, 0.1997685184, 0.4668981481$$
$$0.3333333332, 0.1999614197, 0.4667052469$$
$$0.3333333333, 0.1999935699, 0.4666730967$$
$$0.3333333333, 0.1999989282, 0.4666677383$$
$$0.3333333333, 0.1999998213, 0.4666668451$$
$$0.3333333332, 0.1999999702, 0.4666666963$$
$$0.3333333332, 0.1999999950, 0.4666666715$$
$$0.3333333333, 0.1999999991, 0.4666666673$$
$$0.3333333333, 0.1999999998, 0.4666666666$$
$$0.3333333332, 0.1999999999, 0.4666666665$$
$$0.3333333332, 0.1999999999, 0.4666666665$$
$$0.3333333332, 0.1999999999, 0.4666666665$$
$$0.3333333332, 0.1999999999, 0.4666666665$$
$$0.3333333332, 0.1999999999, 0.4666666665$$
$$0.3333333332, 0.1999999999, 0.4666666665$$
$$0.3333333332, 0.1999999999, 0.4666666665$$
$$0.3333333332, 0.1999999999, 0.4666666665$$
$$0.3333333332, 0.1999999999, 0.4666666665 \tag{8}$$

```
R := 0.34 :
S := 0.19 :
T := 1 - R - S :
print(R, S, T);
for i from 1 to 20 do
  tempR := funcR(R, S, T) :
  tempS := funcS(R, S, T) :
  T := funcT(R, S, T) :
  R := tempR :
  S := tempS :
  print(R, S, T);
od:
```

0.34, 0.19, 0.47

0.3333333333, 0.2016666667, 0.4650000000

0.3333333333, 0.2002777777, 0.4663888890

0.3333333333, 0.2000462962, 0.4666203704

0.3333333333, 0.2000077160, 0.4666589505

0.3333333333, 0.2000012859, 0.4666653806

0.3333333333, 0.2000002142, 0.4666664523

0.3333333333, 0.2000000356, 0.4666666308

0.3333333332, 0.2000000059, 0.4666666606

0.3333333332, 0.2000000009, 0.4666666656

0.3333333333, 0.2000000001, 0.4666666663

0.3333333332, 0.2000000000, 0.4666666665

0.3333333333, 0.1999999999, 0.4666666665

0.3333333332, 0.1999999999, 0.4666666665

0.3333333332, 0.1999999999, 0.4666666665

0.3333333332, 0.1999999999, 0.4666666665

0.3333333332, 0.1999999999, 0.4666666665

0.3333333332, 0.1999999999, 0.4666666665

0.3333333332, 0.1999999999, 0.4666666665

0.3333333332, 0.1999999999, 0.4666666665

0.3333333332, 0.1999999999, 0.4666666665 **(9)**

```
R := 0.50 :
S := 0.25 :
T := 1 - R - S :
print(R, S, T);
for i from 1 to 20 do
  tempR := funcR(R, S, T) :
```

```
  tempS := funcS(R, S, T) :
  T := funcT(R, S, T) :
  R := tempR :
  S := tempS :
  print(R, S, T);
od:
```

$$0.50, 0.25, 0.25$$
$$0.3333333333, 0.2916666667, 0.3750000000$$
$$0.3333333333, 0.2152777777, 0.4513888890$$
$$0.3333333333, 0.2025462962, 0.4641203704$$
$$0.3333333333, 0.2004243826, 0.4662422839$$
$$0.3333333332, 0.2000707304, 0.4665959361$$
$$0.3333333333, 0.2000117883, 0.4666548781$$
$$0.3333333332, 0.2000019646, 0.4666647019$$
$$0.3333333332, 0.2000003274, 0.4666663391$$
$$0.3333333333, 0.2000000545, 0.4666666119$$
$$0.3333333332, 0.2000000090, 0.4666666574$$
$$0.3333333333, 0.2000000014, 0.4666666649$$
$$0.3333333332, 0.2000000002, 0.4666666661$$
$$0.3333333332, 0.2000000000, 0.4666666663$$
$$0.3333333332, 0.1999999999, 0.4666666664$$
$$0.3333333332, 0.1999999999, 0.4666666664$$
$$0.3333333332, 0.1999999999, 0.4666666664$$
$$0.3333333332, 0.1999999999, 0.4666666664$$
$$0.3333333332, 0.1999999999, 0.4666666664$$
$$0.3333333332, 0.1999999999, 0.4666666664$$
$$0.3333333332, 0.1999999999, 0.4666666664 \tag{10}$$