1 Projection onto a vector

- The projection of a vector $\mathbf{v}$ on a vector $\mathbf{w}$ is

$$P_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}'\mathbf{w}}{\mathbf{w}'\mathbf{w}}\mathbf{w}$$

  - eg. suppose $\mathbf{v} = (-1, 3, 4)'$ and $\mathbf{w} = (2, -1, 5)'$. Then $\mathbf{w}'\mathbf{w} = 4 + 1 + 25 = 30$, $\mathbf{v}'\mathbf{w} = -2 - 3 + 20 = 15$ and the projection of $\mathbf{v}$ on $\mathbf{w}$ is (1, -.5, 2.5).
  - eg suppose that $\mathbf{v}$ is as before, but $\mathbf{w} = (1, 0, 0)'$. Then $\mathbf{w}'\mathbf{w} = 1$, $\mathbf{v}'\mathbf{w} = -1$, and the projection of $\mathbf{v}$ on $\mathbf{w}$ is (-1,0,0)'.
  - Similarly the projection of $\mathbf{v}$ on (0,1,0) is (0,3,0) and the projection of $\mathbf{v}$ on (0,0,1) is (0,0,4). Projection of $\mathbf{v}$ on co-ordinate vectors just picks off the associated piece of $\mathbf{v}$.
  - Let $\mathbf{x}' = (x_1, x_2, \ldots, x_n)$ and let $\mathbf{1}$ be the vector of length n, each of whose elements is equal to 1, the so-called "1 vector". Note that $\mathbf{1}'\mathbf{1} = n$ and $\mathbf{x}'\mathbf{1} = \sum_{i=1}^{n} x_i$, so that the projection of $\mathbf{x}$ on $\mathbf{1}$ is

$$P_{\mathbf{1}}\mathbf{x} = \frac{\sum_{i=1}^{n} x_i}{n}\mathbf{1} = \bar{x}\mathbf{1}$$

    The result $\bar{x}\mathbf{1}$ is a vector of length n, each element of which is equal to $\bar{x}$.

- Note that $(\mathbf{v} - P_{\mathbf{w}}\mathbf{v})$ is orthogonal to $P_{\mathbf{w}}\mathbf{v}$, as

$$\begin{aligned}
(\mathbf{v} - P_{\mathbf{w}}\mathbf{v})'P_{\mathbf{w}}\mathbf{v} &= (\mathbf{v} - \frac{\mathbf{v}'\mathbf{w}}{\mathbf{w}'\mathbf{w}}\mathbf{w})'\frac{\mathbf{v}'\mathbf{w}}{\mathbf{w}'\mathbf{w}}\mathbf{w} \\
&= \frac{\mathbf{v}'\mathbf{w}}{\mathbf{w}'\mathbf{w}}\mathbf{v}'\mathbf{w} - \frac{\mathbf{v}'\mathbf{w}}{\mathbf{w}'\mathbf{w}}\frac{\mathbf{v}'\mathbf{w}}{\mathbf{w}'\mathbf{w}}\mathbf{w}'\mathbf{w} \\
&= \frac{\mathbf{v}'\mathbf{w}}{\mathbf{w}'\mathbf{w}}\mathbf{v}'\mathbf{w} - \frac{\mathbf{v}'\mathbf{w}}{\mathbf{w}'\mathbf{w}}\mathbf{v}'\mathbf{w} \\
&= 0
\end{aligned}$$

  so $\mathbf{v}$ is decomposed into the sum of two orthogonal vectors,

$$\mathbf{v} = P_{\mathbf{w}} + (\mathbf{v} - P_{\mathbf{w}}\mathbf{v})$$

2 Projection on a subspace

- Let $\mathbf{v}$ be a vector in R^3, $\mathbf{u_x} = (1,0,0)$ be the unit vector in the x direction and $\mathbf{u_y} = (0,1,0)$ be the unit vector in the y direction.

  Note that $\mathbf{u_x}'\mathbf{u_x} = 1$, and similarly for the unit vector in the y direction, and that $\mathbf{u_x}'\mathbf{u_y} = 0$ - they are orthogonal.

  The projection of $\mathbf{v}$ on $\mathbf{u_x}$ is $P_{\mathbf{u_x}}(\mathbf{v}) = \mathbf{v}'\mathbf{u_x}\mathbf{u_x} = \alpha\mathbf{u_x}$ where $\alpha = \mathbf{v}'\mathbf{u_x}$.

  Similarly, the projection on $\mathbf{u_y}$ is $P_{\mathbf{u_y}}(\mathbf{v}) = \mathbf{v}'\mathbf{u_y}\mathbf{u_y} = \beta\mathbf{u_y}$, where $\beta = \mathbf{v}'\mathbf{u_u}$.

- Let $\mathbf{w} = \alpha\mathbf{u_x} + \beta\mathbf{u_y}$. This is a vector in the (x,y) plane.

  Note that any vector in the (x,y) plane can be written in this fashion. The two orthogonal vectors $\mathbf{u}_x$ and$\mathbf{u}_y$ generate a two dimensional subspace - the (x,y) plane.

$$\begin{aligned}
(\mathbf{v}-\mathbf{w})'\mathbf{w} &= (\mathbf{v} - \alpha\mathbf{u_x} - \beta\mathbf{u_y})'(\alpha\mathbf{u_x} + \beta\mathbf{u_y}) \\
&= \alpha\mathbf{v}'\mathbf{u_x} + \beta\mathbf{v}'\mathbf{u_y} - \alpha^2\mathbf{u_x}'\mathbf{u_x} - \beta^2\mathbf{u_y}'\mathbf{u_y} - 2\alpha\beta\mathbf{u_x}'\mathbf{u_y} \\
&= \alpha\mathbf{v}'\mathbf{u_x} + \beta\mathbf{v}'\mathbf{u_y} - \alpha^2 - \beta^2 \\
&= \alpha^2 + \beta^2 - \alpha^2 - \beta^2 \\
&= 0
\end{aligned}$$

- This means that $(\mathbf{v} - \mathbf{w})$ is perpendicular to $\mathbf{w}$.
- $\mathbf{w}$ is the orthogonal projection of $\mathbf{v}$ on the (x,y) plane.
- Note that $\mathbf{w}$ is the sum of the projections onto the x and y axes. This means that the projection onto the (x,y) plane is the sum of the projection onto the x and y axes.
- We need not use unit vectors in the direction of the co-ordinate axes. Any two orthogonal vectors in the (x,y) plane will work equally well.
- This generalizes - *the perpendicular projection of a vector onto a d dimensional subspace is equal to the sum of the perpendicular projections onto any d orthogonal vectors generating that subspace.*

3 Simple linear regression as a sequence of projections

Let $\mathbf{1}$ and $\mathbf{x}$ be as before, and let $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

We saw that the projection of $\mathbf{x}$ on $\mathbf{1}$ was $\bar{x}\mathbf{1}$. Also:

$$(\mathbf{x} - \bar{x}\mathbf{1})'\mathbf{1} = \sum_{i=1}^{n} x_i - \bar{x}n = 0$$

which means that $\mathbf{1}$ and $(\mathbf{x} - \bar{x}\mathbf{1})$ are a pair of orthogonal vectors which generate the same subspace as do $\mathbf{1}$ and $\mathbf{x}$.

Let's look at the projection of $\mathbf{y}$ onto this two dimensional subspace, which, as we have seen, will be the sum of the projections of $\mathbf{y}$ onto $\mathbf{1}$ and $(\mathbf{x} - \bar{x}\mathbf{1})$.

- The projection of $\mathbf{y}$ onto $\mathbf{1}$ is $\bar{y}\mathbf{1}$.
- The projection of $\mathbf{y}$ onto $(\mathbf{x} - \bar{x}\mathbf{1})$ is

$$\frac{\mathbf{y}'(\mathbf{x} - \bar{x}\mathbf{1})}{(\mathbf{x} - \bar{x}\mathbf{1})'(\mathbf{x} - \bar{x}\mathbf{1})}(\mathbf{x} - \bar{x}\mathbf{1}) \tag{1}$$

  If you look carefully you will see that

$$\mathbf{y}'(\mathbf{x} - \bar{x}\mathbf{1}) = \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}$$

  and

$$(\mathbf{x} - \bar{x}\mathbf{1})'(\mathbf{x} - \bar{x}\mathbf{1}) = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

  .

  so that (1) is $\hat{\beta}_1\mathbf{x} - \hat{\beta}_1\bar{x}\mathbf{1}$, where $\hat{\beta}_1$ is the estimated slope in simple linear regression.

- Adding the two projections, we see that the projection of $\mathbf{y}$ on the two dimensional subspace generated by $\mathbf{1}$ and $(\mathbf{x} - \bar{x}\mathbf{1})$ is

$$\hat{\beta}_1\mathbf{x} + (\bar{y} - \hat{\beta}_1\bar{x})\mathbf{1}$$

But $(\bar{y} - \hat{\beta}_1\bar{x})$ is the formula for $\hat{\beta}_0$ in simple linear regression, so we see that the projection of $\mathbf{y}$ on the two dimensional subspace generated by $\mathbf{1}$ and $(\mathbf{x} - \bar{x}\mathbf{1})$ is

$$\hat{\mathbf{y}} = \hat{\beta}_0\mathbf{1} + \hat{\beta}_1\mathbf{x}$$

so that $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ for $i = 1, 2, \ldots, n$. These are the predicted values in simple linear regression

Taking into account that the subspace generated by $\mathbf{1}$ and $(\mathbf{x} - \bar{x}\mathbf{1})$ is the same subspace as that generated by $\mathbf{1}$ and $\mathbf{x}$, we see that the vector of predicted values in simple linear regression is the projection of $\mathbf{y}$ onto the two dimensional subspace generated by the vectors $\mathbf{1}$ and $\mathbf{x}$.