ACSC/STAT 3703 - Winter 2026 - Assignment 1

Due: Monday, January 19, at 11:59 PM

1. Let X has the gamma(α,θ) distribution with density function

$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\theta^{\alpha}}, \; x > 0$$

where both α and θ are positive.

   (a) Show that the moment generating function of X is $M_X(t) = (1 - \theta t)^{-\alpha}$.

   (b) By suitable manipulation of the moment generating function, find the first three moments of the gamma(α,θ) distribution.

   (c) Suitably manipulate these moments to find the skewness of the gamma(α,θ) distribution.

2. Let X have a two parameter Pareto distribution with survivor function

$$S(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

where x, θ and α are all positive,

   (a) Derive the density function of X.

   (b) Derive the hazard function of X.

   (c) Derive the mean excess loss function $e_X(d)$ of X.

3. Let X have density function

$$f(x) = \frac{2}{x^3}, \; x \geq 1$$

   (a) Derive the distribution function of X.

   (b) Find the mean of X

   (c) Find the median of X

   (d) Find the mode of X

4. Let X have the Weibull density function with survivor function

$$S(x) = e^{-\left(\frac{x}{\theta}\right)^{\tau}}$$

where τ, θ and x are all positive.

   (a) Derive the density function of X.

   (b) Derive the formula for π_p, the p'th percentile of X. That is, solve $p = F(\pi_p)$.

   (c) If the 25'th percentile is 1000 and the 75'th percentile is 10000, solve for τ.

   (d) Calculate the hazard function $h(x)$.

   (e) On the basis of the hazard function, would you say that the Weibull is heavy tailed or light tailed? Why?

5. If $X_1, X_2, \ldots X_{100}$ are i.i.d. gamma distributed with mean 5000 and variance 25×10^6.

   (a) Find the parameters α and θ of the gamma distribution.

   (b) Use the central limit theorem to approximate the probability that $\bar{X} = \frac{\sum_{i=1}^{100} X_i}{100}$ is greater than 5250. (You can leave the answer as $P(Z > c)$ for the appropriate value of c.)

6. Let X have an exponential distribution with density function

$$f(x) = \lambda e^{-\lambda x}$$

where x and λ are both positive.

(a) Find the hazard function $h(x)$ of X.

(b) Verify that

$$S(x) = e^{-\int_{y=0}^{x} h(y)dy}$$

(This is the general relationship between the survivor function and the hazard function of a non-negative random variable.)

(c) Show that for any $x > 0$ and $c > 0$ that $P(X > x + c | X > c) = P(X > x)$. (This is referred to as the memoryless property of the exponential distribution.)

(d) Suppose that $\lambda = 1$.

i. Find $VaR_{.999}(X)$, the Value at Risk of X at the 99.9% level.

ii. Find $TVaR_{.999}(X)$, the Tail Value at Risk of X at the 99.9% security level.