ACSC/STAT 3703 - Midterm 2 Solutions - out of 30 points

NAME: ____________________ Banner Number: ____________________

Formulas:

- If X has an exponential distribution with mean θ, then the distribution function of X is $F_X(x) = 1 - e^{-x/\theta}$, and $E[X \wedge c] = \theta F_X(c)$.
- The p.g.f of a Poisson random variable with mean λ is $P_X(z) = e^{\lambda(z-1)}$.
- For any pair of random variables X and Y, $E[Y] = E[E[Y|X]]$ and $V[Y] = E[V[Y|X]] + V[E[Y|X]]$.

1. The number of claims N made to an insurance company in a given year is a random variable with mean 100 and standard deviation 10. The individual claims $X_1, X_2, \ldots$ are independent of one another and of N. The mean claim size is $\mu_X = 500$ and the standard deviation of the claim size is $\sigma_X = 10$. The total of the claims made against the company in one year is $S_N = X_1 + X_2 + \ldots + X_N$. (Hint: for both parts of this question, use the mean and variance formulas above.)

(3) (a) What is the mean of S_N?

$$E[S] = E[X]E[N] = 500(100) = 50,000$$

(b) What is the standard deviation of S_N?

$$V[S] = E[V[S|N]] + V[E[S|N]] = E[NV(X)] + V(NE[X])$$

$$= E[N]V(X) + (E[X])^2 V[N] = 100(100) + 500^2(100) = 25,010,000$$

(3) standard deviation is $\sqrt{25,010,000} \approx 5001$.

2. Suppose that X is a discrete random variable with probability mass function given by $P(X = k) = p_k, k = 0, 1, 2, \ldots$. Let X^T be the 0 truncated verstion of X, for which

$$P(X^T = k) = p_k^T = \frac{p_k}{1 - p_0}, k = 1, 2, \ldots$$

and X^M be the 0 modified version of X, for which

$$P(X^M = k) = (1 - p_0^M)p_k^T, k = 1, 2, \ldots$$

(a) Prove that $E(X^M) = E(X)\frac{1-p_0^M}{1-p_0}$.

$P_k^M = (1 - p_0^M)p_k^T = \frac{1-p_0^M}{1-p0}p_k$, for k=1,2, ...

(3) The result follows immediately.

(b) The following table gives the probability distribution of a random variable X. Fill in the missing entries for the probability distributions of the zero truncated and zero modified random variables.

$$\frac{1 - p_0^M}{1 - p0} = \frac{p_2^M}{p_2} = (1/6)/(1/4) = 2/3$$

so $p_k^M = p_k(2/3)$, for k=1,2,3. Then $p_0^M = 1 - p_1^M - p_2^M - p_3^M$.
for $k = 1, 2, 3$, $p_k^T = p_k/(1 - p_0) = p_k(4/3)$, giving the following

(4)

k	$P(X = k)$	$P(X^T = k)$	$P(X^M = k)$
0	1/4	0	1/2
1	3/8	1/2	1/4
2	1/4	1/3	1/6
3	1/8	1/6	1/12

3. Losses follow an exponential distribution with mean λ. (Recall that the loss elimination ratio associated with an ordinary deductible d is $\frac{E(X \wedge d)}{E(X)}$. What is the size of the ordinary deductible needed to give a loss elimination ratio of 1/2?

$$\frac{E(X \wedge d)}{E(X)} = 1 - e^{-d/\lambda} = 1/2 \rightarrow d = -\lambda log_e(1/2) \approx .693\lambda$$

(5)

4. From theorem 6.1 we know that if $X_1, X_2, \ldots, X_n$ are i.i.d. Poisson with mean λ then $S_n = \sum_{i=1}^{n} X_i$ is Poisson with mean $n\lambda$. The theorem was proven by finding the probability generating function of S_n, beginning with the common pgf of the X's.

   Now consider $S_N = \sum_{i=1}^{N} X_i$, where N is independent of $X_1, X_2, \ldots$ and the X's are i.i.d. The p.g.f. of S_N is derived in section 9.3.1 and is given by $P_{S_N}(z) = P_N(P_X(z)))$ where $P_N(z) = E[z^N]$ is the p.g.f. of N and $P_X(z) = E[z^X]$ is the common p.g.f. of the X's.

   Suppose that the X_j's are i.i.d. Poisson(λ), independent of N, and N is Poisson with mean γ.

   (a) What is the p.g.f. of S_N?

$$P_S(z) = P_N(P_X(z)) = P_N(e^{\lambda(z-1)}) = e^{\gamma(e^{\lambda(z-1)}-1)}$$

(3)

   (b) Does S_N have a Poisson distribution? Why?

   No, becuase the above pgf of S cannot be written as $e^{\theta(z-1)}$ for any

(2) θ.

5. The amount of a loss X has density function

$$f_X(x) = \frac{1}{100}, \quad 0 < x < 100$$

Insurance pays 80% of the amount of a loss in excess of an ordinary deductible of $d = 20$. The maximum insured loss is $u = 60$.

(a) What is $E[Y_L]$, the expected value of the loss?

Use the result of theorem 8.7, which states that

$$E(Y^L) = \alpha(1+r)\left[E(X \wedge \frac{u}{1+r}) - E(X \wedge \frac{d}{1+r})\right]$$

where r is the inflation rate, 0 in this case, and α is the co-payment
(5) proportion - the proportion paid by the insurance company.

$$E[X \wedge c] = \int_{x=0}^{c} x f(x) dx + c \int_{x=c}^{100} f(x) dx$$
$$= c^2/200 + c(100-c)/100 = c - c^2/200$$

so

$$E(Y^L) = .8(60-60^2/200-20+20^2/200) = .8(60-18-20+2) = 24(.8) = 19$$

(b) What is the expected value of the payment $E[Y_P]$? (Hint: Recall that a payment is made only if $X > 20$.)

$$E[Y^P] = \frac{E[Y^L]}{P(X > 20)} = 19.2/.8 = 24$$

(2)

(answer to part a divided by .8)