1. Q1 Winter 2023

$X_1, \dots, X_n \sim N(\mu, \sigma^2)$

$$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2 = \frac{n}{n-1} \sum \frac{X_i^2}{n} - \frac{n \bar{X}^2}{n-1}$$

$$\frac{n}{n-1} \frac{\sum X_i^2}{n} \xrightarrow{P} E X_1^2$$

$$\frac{n}{n-1} \bar{X}^2 \xrightarrow{P} (E(X_1))^2$$

$$\Rightarrow S^2 \xrightarrow{P} \sigma^2$$

$$\Rightarrow \frac{S^2}{\sigma^2} \xrightarrow{P} 1 \Rightarrow \frac{S}{\sigma} \xrightarrow{P} 1 \Rightarrow \frac{\sigma}{S} \xrightarrow{P} 1 \quad (1)$$

By CLT $\sqrt{n} \frac{(\bar{X} - \mu)}{\sigma} \xrightarrow{\mathcal{L}} Z \sim N(0,1)$ (2)

$$\Rightarrow \sqrt{n} \frac{(\bar{X} - \mu)}{S} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \frac{\sigma}{S} \xrightarrow{\mathcal{L}} Z$$

B.7.1

$P(|X_n^* - X^*| > \epsilon)$

$= P(|F_n^{-1}(u) - F^{-1}(u)| > \epsilon)$

$= P(F_n^{-1}(u) - F^{-1}(u) > \epsilon \;\cup$
$\quad F_n^{-1}(u) - F^{-1}(u) < -\epsilon)$

$\leq P(F_n^{-1}(u) - F^{-1}(u) > \epsilon)$
$\quad + P(F_n^{-1}(u) - F^{-1}(u) < -\epsilon)$

$= P(F_n^{-1}(u) > F^{-1}(u) + \epsilon)$
$\quad + P(F_n^{-1}(u) < F^{-1}(u) - \epsilon)$

$= P(u > F_n(F^{-1}(u) + \epsilon))$
$\quad + P(u < F_n(F^{-1}(u) - \epsilon))$

by [illegible]

$\longrightarrow P(u > F(F^{-1}(u) + \epsilon))$
$\quad + P(u < F(F^{-1}(u) - \epsilon))$

$= 0$ because $F(F^{-1}(u) + \epsilon) > u$
and $F(F^{-1}(u) - \epsilon) < u$

1
F(F^-1(u)+ε)
u
0
F^-1(u)
F^-1(u)+ε

B.1.6

$$P(y \mid X = k) = \binom{k}{y} p^y (1-p)^{k-y}$$

$$E(e^{tY} \mid X = k) = (1 - p + pe^t)^k$$

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

a) $$E(e^{tY}) = E(E(e^{tY} \mid X))$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda(1-p+pe^t))^k}{k!}$$

$$= e^{-\lambda} e^{\lambda(1-p+pe^t)}$$

$$= e^{p\lambda(e^t - 1)}$$

is mgf of $P(\lambda p)$

b) $$p(x,y) = p(y \mid x)\, p(x) = \binom{x}{y} p^y (1-p)^{x-y} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \underbrace{\frac{(\lambda(1-p))^{x-y} e^{-\lambda(1-p)}}{(x-y)!}}_{p(x \mid y)} \frac{(p\lambda)^y e^{-p\lambda}}{y!} \qquad x \geq y,\ y = 0, 1, \ldots$$

$\Rightarrow$ $X - Y$ and $Y$ independent

B.1.7 $\quad (X, Z) \sim N_2\left(\begin{pmatrix}\mu\\ \gamma\end{pmatrix}, \begin{pmatrix}\sigma^2 & 0\\ 0 & \tau^2\end{pmatrix}\right)$

$$\begin{pmatrix} X \\ Y\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}X\\ Z\end{pmatrix} \sim N_2\left(\begin{pmatrix}\mu\\ \mu+\gamma\end{pmatrix}, \begin{pmatrix}\sigma^2 & \sigma^2\\ \sigma^2 & \sigma^2+\tau^2\end{pmatrix}\right)$$

$$\begin{pmatrix} Y \\ X\end{pmatrix} \sim N_2\left(\begin{pmatrix}\mu+\gamma\\ \mu\end{pmatrix}, \begin{pmatrix}\sigma^2+\tau^2 & \sigma^2\\ \sigma^2 & \sigma^2\end{pmatrix}\right)$$

a) $\quad Y \mid X = x \sim N(x+\gamma, \tau^2)$

b) $\quad X \mid Y = y \sim N\left(\mu + \dfrac{\sigma^2}{\sigma^2+\tau^2}(y-(\mu+\gamma)),\right.$

$$\left.\sigma^2 - \frac{\sigma^2\sigma^2}{\sigma^2+\tau^2}\right)$$

or $\quad N\left(\dfrac{\mu\tau^2 + \sigma^2(y-\gamma)}{\sigma^2+\tau^2}, \dfrac{\sigma^2\tau^2}{\sigma^2+\tau^2}\right)$

Can use Bayes's rule as indicated but it's messy.

B.2.2

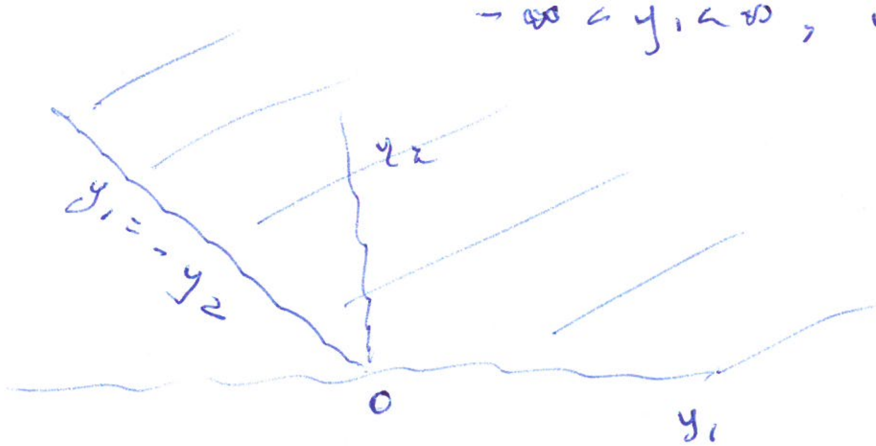
$$f(x_1, x_2) = \lambda^2 e^{-\lambda(x_1 + x_2)}, \quad x_1 > 0, \; x_2 > 0$$

$y_1 = x_1 - x_2 \qquad x_1 = y_1 + y_2 \qquad |J| = 1$

$y_2 = x_2 \qquad x_2 = y_2$

a) $$f_{\tilde{y}}(y_1, y_2) = \lambda^2 e^{-\lambda(y_1 + 2y_2)},$$

$$-\infty < y_1 < \infty, \quad y_2 > 0, \quad y_1 > -y_2$$




b) if $y_1 < 0$ $$f_{Y_1}(y_1) = \int_{y_2 = -y_1}^{\infty} \lambda^2 e^{-\lambda(y_1 + 2y_2)} \, dy_2$$

$$= \lambda e^{-\lambda y_1} \frac{e^{-2\lambda y_2}}{2} \Big|_{-y_1}^{\infty} = \lambda \frac{e^{-\lambda y_1} e^{2\lambda y_1}}{2} = \frac{\lambda e^{-\lambda |y_1|}}{2}$$

if $y_1 > 0$

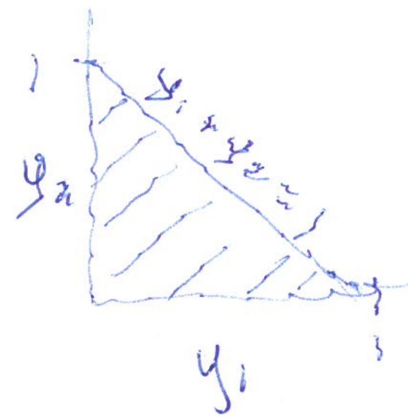
$$f_{Y_1}(y_1) = \lambda e^{-\lambda y_1} \int_0^{\infty} \lambda e^{-2\lambda y_2} \, dy_2$$

$$= \lambda e^{-\lambda y_1} \cdot \frac{1}{2} = \frac{\lambda}{2} e^{-\lambda |y_1|}$$

B.2.3  $Y_1 = X_1$  $Y_2 = X_2(1-X_1)$  $0 < y_1 < 1$

$0 < y_2 < 1 - y_1$

$$|J| = \frac{1}{1-y_1}$$




$f(y_1, y_2) \propto$

$$y_1^{r_1-1}(1-y_1)^{s_1-1}\left(\frac{y_2}{1-y_1}\right)^{r_2-1}\left(\frac{1-y_1-y_2}{1-y_1}\right)^{s_2-1}\frac{1}{1-y_1}$$

B.2.12

a)

for $0<u<1$, $P(F(X)\le u) = P(X \le F^{-1}(u)) = F(F^{-1}(u)) = u$

b) $P(F^{-1}(u) < x) = P(U \le F(x)) = F(x)$

c) $U_{(1)} < U_{(2)} < \dots < U_{(n)}$

$\Leftrightarrow F^{-1}(U_{(1)}) < F^{-1}(U_{(2)}) < \dots < F^{-1}(U_{(n)})$

for $x_1 < x_2 < \dots < x_n$

$P(F^{-1}(U_{(1)}) < x_1, F^{-1}(U_{(2)}) < x_2, \dots, F^{-1}(U_{(n)}) < x_n)$

$= P(U_{(1)} < F(x_1), U_{(2)} < F(x_2), \dots, U_{(n)} < F(x_n))$

the latter has joint density

$n!\, f_1(x_1) f_2(x_2) \dots f_n(x_n)$

Which is the joint density of the order statistics of a sample of size n from $f(\,)$

$$= \int_0^{F(x_1)} \dots \int_0^{F(x_n)} \frac{1}{n!}\, du_1 \dots du_n$$

$$= \prod_{i=1}^{n} F(x_i) \frac{1}{n!}$$

then take derivation

B.5.2

$$M_{\underset{\sim}{a}+B\underset{\sim}{u}}(\underset{\sim}{t}) = E\, e^{\underset{\sim}{t}'(\underset{\sim}{a}+B\underset{\sim}{u})}$$

$$= e^{\underset{\sim}{t}'\underset{\sim}{a}}\, E\, e^{(B'\underset{\sim}{t})'\underset{\sim}{u}}$$

$$= e^{\underset{\sim}{t}'\underset{\sim}{a}}\, M_{\underset{\sim}{u}}(B'\underset{\sim}{t})$$

$$K_{\underset{\sim}{a}+B\underset{\sim}{u}}(t) = \underset{\sim}{t}'\underset{\sim}{a} + K_{\underset{\sim}{u}}(B'\underset{\sim}{t})$$

B7.6

$P(|Z_{nj} - Z_j| > \epsilon) = P(|Z_{nj} - Z_j|^2 > \epsilon^2)$

$\leq P(|\underset{\sim}{Z}_n - \underset{\sim}{Z}|^2 > \epsilon^2) \rightarrow 0$

converse

if $P(|Z_{nj} - Z_j| > \epsilon) \rightarrow 0 \quad j = 1, 2, \ldots, d$

$P(|\underset{\sim}{Z}_n - \underset{\sim}{Z}|^2 > \epsilon^2) \leq P(\bigcup_{j=1}^{d} \{|Z_{nj} - Z_j|^2 > \epsilon^2\})$

$\leq \sum_{j=1}^{d} P(|Z_{nj} - Z_j|^2 > \epsilon^2) \rightarrow 0$

B.9.2

B.9.6 $\quad \mu = E(X_i)$

$|X_i - \mu| \leq c_i$

$P(|S_n - n\mu| \geq x) \leq 2\exp\{-\frac{1}{2}x^2 / \sum c_i^2\}$

Suppose $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$

$E(X_i) = p \qquad |X_i - p| \leq \max(p, 1-p) \leq 1$

let $x = n\epsilon$

$P(|S_n - np| \geq n\epsilon) \leq 2\exp\{-\frac{1}{2}(n\epsilon)^2/n\}$

$= 2\exp(-\frac{1}{2}n\epsilon^2)$

if $p = \frac{1}{2} \quad |X_i - p| = \frac{1}{2} = c_i \qquad \sum c_i^2 = \frac{n}{4}$

$\leq 2\exp\{-\frac{1}{2}(n\epsilon)^2/(n/4)\}$

$= 2\exp\{-2n\epsilon^2\} \leq 2\exp(-\frac{n\epsilon^2}{2})$