

ON NETWORK RELIABILITY

by

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To Mom and Dad,
for their love, support and for buying me an electronic typewriter for
the Christmas of 1994 instead of the Super Nintendo that I really
really wanted.

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Abstract

The *all terminal reliability* of a graph G is the probability that at least a spanning tree is operational, given that vertices are always operational and edges independently operate with probability $p \in [0, 1]$. In this thesis, an investigation of all terminal reliability is undertaken. An open problem regarding the non-existence of optimal graphs is settled and analytic properties, such as roots, thresholds, inflection points, fixed points and the average value of the all terminal reliability polynomial on $[0, 1]$ are studied.

A new reliability problem, the k -*clique reliability* for a graph G is introduced. The k -clique reliability is the probability that at least a clique of size k is operational, given that vertices operate independently with probability $p \in [0, 1]$. For k -clique reliability the existence of optimal networks, analytic properties, associated complexes and the roots are studied. Applications to problems regarding independence polynomials are developed as well.

List of Symbols Used

$E(G)$		Edge set of a graph G
$V(G)$		Vertex set of a graph G
$\Delta(G)$		Maximum degree of a vertex in G
$\delta(G)$		Minimum degree of a vertex in G
$N(v)$		Open neighbourhood of v
$N[v]$		Closed neighbourhood of v
$\overline{G}$		The complement of G
G^k		G with edges replaced by bundles of size k
P_n		Path on n vertices
C_n		Cycle on n vertices
K_n		Complete graph on n vertices
$K_{n_1, n_2, \dots, n_k}$		Complete multipartite graph
W_n		Wheel graph with a cycle of length n
$\Theta_{n_1, n_2, \dots, n_k}$		Theta graph
$C_{n, m}$		C_n with edge bundles of size $k = \lfloor \frac{m}{n} \rfloor$ and $k + 1$
$G \cdot e$		G with e contracted
$G - e$		G with e removed

$G[H]$	 G with edges replaced by copies of H
$G _S$	 Subgraph of G induced by S
$\mathcal{G}_{n,m}$	 Set of all graph on n vertices and m edges
$\mathcal{S}_{n,m}$	 Set of all simple graphs on n vertices and m edges
$\text{Rel}(G, p)$	 The reliability of G
$\text{Rel}_p(G, p)$	 The reliability of G in the variable p
$\text{Rel}_q(G, q)$	 The reliability of G in the variable q
$\text{Rel}_{p,q}(G, p, q)$	 The reliability of G in the variables p and q
$\text{ifp}(G)$	 Internal fixed point of G
$\text{avgRel}(G)$	 Average reliability of G
$\text{crel}_k(G, p)$	 k -clique reliability of G
$\text{cgen}_k(G, x)$	 k -clique generating polynomial of G
$\text{cfree}_k(G, x)$	 k -clique free polynomial of G
$\text{I}(G, x)$	 Independence polynomial of G
$I_k(G)$	 Complex on vertices of G whose faces don't contain a K_k
$\Delta_k(G)$	 Complex on vertices of G whose removal leaves at least a K_k
$H(I, d)$	 Hilbert function of the ideal I
$Q = \mathbf{k}[x_1 x_2 \dots x_n] / (x_1^2, x_2^2, \dots, x_n^2)$	 Kruskal-Katona Ring

Acknowledgements

I've been told that back in September 1990, the night before my first day at school, I look up at Mom and Dad with an excited face and say "Mom, Dad, I've been waiting for this day my whole life!" . . . 23 years later, I end my formal schooling with the same thoughts; "Mom, Dad, I've been waiting for this day my whole life!"

First, I need to thank my parents. If it were not for their love, support and great genes, I would not have been as successful as I have been, not just in my studies, but in all aspects of my life. This thesis is for you!

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Chapter 1

Introduction and Background

1.1 Introduction

We are a society that is surrounded by networks. Whether we are surfing the web, tagging friends in pictures on Facebook, or making a call on our cellphone, we use networks every day. Though society's dependency on networks is somewhat recent, the study of them is not. As far back as the 1950's, long before cellphones and Twitter existed, the structure and resilience of networks was an area of mathematical research.

When studying a network it is important to know how reliable it is. Depending on the structure and purpose of the network, there are different criteria for a network to be reliable. It may be that two specific vertices must have the ability to communicate (send and receive information to and from each other), or that all vertices can communicate. One of the more commonly studied models is that of *all terminal reliability*. In this model, vertices are assumed to be operational and edges operate independently with probability $p \in [0, 1]$. The all terminal reliability of a graph G , $\text{Rel}(G)$, is the probability that all the vertices can communicate with each other. This model has been well studied. Areas of research surrounding all terminal reliability include bounding [20, 23, 31], the existence of optimal networks [13, 15, 40, 67], the location of the roots [21, 70], and analytic properties [27, 38, 39].

In this thesis, we will disprove a conjecture regarding the existence of optimal networks. We will also look at the roots, thresholds, inflection points and fixed points of the all terminal reliability polynomial and answer some open problems regarding

these topics. We will also look at the average reliability, that is, the integral of the reliability polynomial on $[0, 1]$ and applications this has to optimal networks.

Besides an investigation of all terminal reliability, we will introduce a new reliability problem, which has vertex failures rather than edge failures, that requires for $k \geq 2$ a group of at least k vertices can communicate with each other. The simplicial complexes associated with this reliability problem, optimal networks and analytic properties will be studied.

1.2 Graph Theory Background

We will begin with the necessary graph theory terminology. The following definitions, unless otherwise indicated, are common terms in graph theory and can be found in any standard textbook, such as [35].

A *graph*, G , consists of a finite non-empty set, $V(G)$, which are the *vertices* of G and a multiset, $E(G)$, of subsets of size 2 of $V(G)$, which are the *edges* of G (in this thesis we will be assuming that graphs are *loopless*, that is, they have no edges of size 1). The set $V(G)$ is called the *vertex set* of G and the multiset $E(G)$ is the *edge set* of G . For a graph G , the cardinality of the vertex set, $|V(G)|$, is the order of the graph. The cardinality of the edge set, $|E(G)|$, is the size of G . If the edges of G are not all distinct, then we say that G has *multiedges*. If the edges are all distinct and of size 2, then G is a *simple* graph. Unless otherwise stated, we will assume that a graph can have multiedges.

Example 1.2.1 Refer to Figure 1.1. The graph G_1 is a simple graph of order 6 and size 10. The graph G_2 is a loopless multigraph and the graph G_3 is a multigraph with loops.

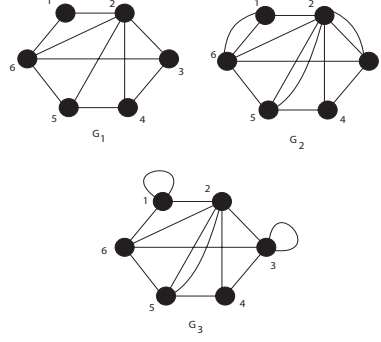


Figure 1.1: A simple graph, loopless multigraph and multigraph

We will let $\mathcal{S}_{n,m}$ denote the set of all simple graphs on n vertices and m edges and let $\mathcal{G}_{n,m}$ denote the set of all graphs (simple and with multiedges) on n vertices and m edges.

A *directed graph* (or *digraph*), D is a graph with vertex set $V(D)$, edge set $E(D)$, along with the maps $init : E \rightarrow V$ and $term : E \rightarrow V$. That is each edge e has an initial and terminal vertex. The edge is directed from $init(e)$ to $term(e)$. A directed edge is called an *arc*. This thesis will focus on graphs, but some results regarding directed graphs will be discussed.

If we say that two vertices, u and v of G are *adjacent* that means that u and v are joined by an edge $e = \{u, v\}$ and we say that e is *incident* to both u and v . The *degree* of a vertex, v , denoted $\deg(v)$ is the number of edges that are incident to v . The *open neighbourhood* of a vertex v , $N(v)$, is the set of vertices adjacent to v . The *closed neighbourhood* of v , $N[v]$ the $N(v) \cup \{v\}$. The *maximum degree of a vertex* of G is denoted $\Delta(G)$ and the *minimum degree of a vertex* of G is denoted $\delta(G)$. If a graph has a vertex, v , of degree 1, then v is called a *leaf*.

We can delete an edge, $e = \{x, y\}$ of a graph to obtain the graph $G - e$ where $V(G - e) = V(G)$ and $E(G - e) = E(G) - \{e\}$ or we can contract the edge, e , to obtain the graph $G \cdot e$. When an edge is contracted it is removed from the graph and

the incident vertices, x and y , merge into a new vertex, v , where an edge is incident to v if it was incident to x or y .

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$ and $E(H) \subseteq E(G)$ then we call H a *spanning subgraph* of G . An *induced subgraph* H of G is such that for all the vertices u, v in $V(H) \subseteq V(G)$ we have that $e = \{u, v\} \in E(H)$ if and only if $e = \{u, v\} \in E(G)$. If S is a subset of vertices of G , $G|_S$ denotes the subgraph of G induced by S .

Example 1.2.2 Refer to Figure 1.2 and let $e \in E(G)$, $e = \{2, 3\}$. Then $G - e$ is the graph G with the edge e removed and $G \cdot e$ is the graph G with the edge e contracted. The graph H_1 is a subgraph of G and H_2 is a spanning subgraph of G . The graph H_3 is an induced subgraph. It is the subgraph induced by the vertices $S = \{2, 3, 4, 5\}$, that is $H_3 = G|_S$. We can see that G has a leaf, (so $\delta(G) = 1$) and $\Delta(G) = 4$. For the vertex 2 in G , $N(2) = \{1, 3, 4, 5\}$ and $N[2] = \{1, 2, 3, 4, 5\}$

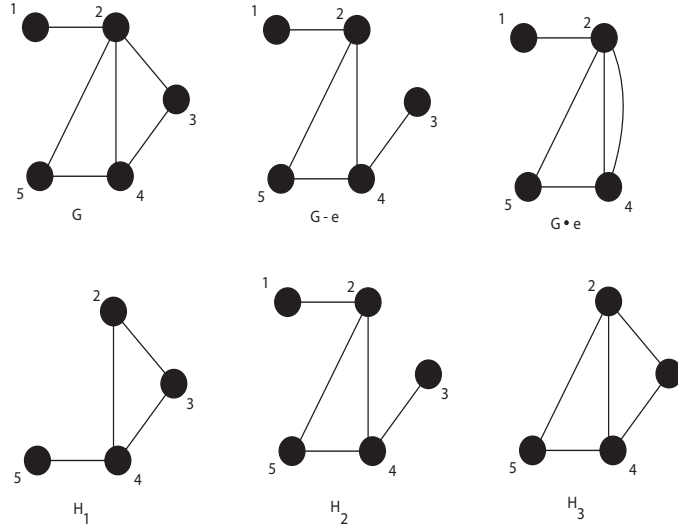


Figure 1.2: The graph in Example 1.2.2

If G is a simple graph, we call $\overline{G}$ the *complement* of G (see Figure 1.3). It is the graph on the same vertex set as G , so $V(G) = V(\overline{G})$ and an edge $e \in E(\overline{G})$ if and only if $e \notin E(G)$. The *union* of two graphs G and H is the graph $G \cup H$ where

$V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$ then we have the *disjoint union* $G \dot{\cup} H$ of G and H .

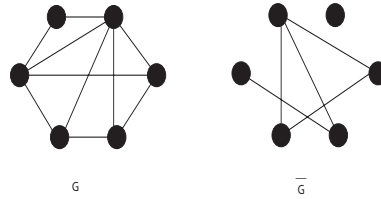


Figure 1.3: A graph G and its complement

A *path* of length n of a graph G is a finite sequence of vertices and edges $v_0 e_0 v_1 e_1 \dots e_{n-1} v_n$, where $e_i = \{v_i, v_{i+1}\}$ and no vertex is repeated (a *directed path* from v_0 to v_n is a path from v_0 to v_n , where each edge is directed from v_i to v_{i+1}). A graph is *connected* if every pair of vertices is connected by a path and *disconnected* otherwise. A subset of edges whose removal disconnects a graph is called a *cutset*. The size of the smallest cutset of a graph is called the *edge connectivity*. A maximally connected subgraph of a graph G is called a *connected component*.

A *cycle* of a graph G is a finite sequence of vertices and edges, $v_0 e_0 v_1 e_1 \dots e_k v_0$, where $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i < k$ and $e_k = \{v_k, v_0\}$, where no vertex is repeated, other than v_0 . The length of the smallest cycle in a graph is called the *girth* of the graph. If G does not contain a cycle, then we say it is *acyclic*.

Example 1.2.3 Refer to Figure 1.3. The edge connectivity of G is 2 and G has one connected component. The disconnected graph $\overline{G}$ has 2 connected components, one of order 5 and one of order 1.

If G has n vertices and consists of only a cycle on n vertices then we call G an n -cycle, or just a *cycle* and denote it C_n . If we remove exactly one edge from C_n , then the graph that remains is a *path* of length n and is denoted P_n .

If G is a simple graph on n vertices and every pair of vertices are connected by an edge then we have a *complete graph*, denoted K_n . A complete graph on n vertices has

$\binom{n}{2}$ edges. Every simple graph on n vertices is a subgraph of K_n . The complement of a complete graph is an *empty graph*, that is, a graph with no edges. The union of a graph on n vertices and its complement is K_n .

With respect to complete and empty graphs, we have two graph properties which will be discussing in this thesis. An *independent set* of a graph G is a subset of vertices whose induced subgraph is an empty graph. The *independence number* of a graph is the size of the largest independent set. If a graph G has an induced subgraph, H , which is a complete graph on k vertices, then we call H a *k-clique* of G . The *clique number* of a graph G is the size of the largest clique in G . The complement of a clique is an independent set. The graph G in Figure 1.2 has an independence number of 3 and a clique number of 3. The work in the latter half of this thesis will involve independent sets and cliques.

There are several other families of graphs which we will be used throughout the thesis. In the optimality section, the following families of graphs will play a large role. We call a graph, G a *d-regular graph* if $\delta(G) = \Delta(G) = d$. A graph is *d-semiregular* if $\delta(G) = d$ and $\Delta(G) \leq d + 1$ (see Figure 1.4). We should note that a *d-regular graph* is a *d-semiregular graph*.

Another graph which will be referred to is a *theta graph* $\Theta_{n_1, n_2, \dots, n_k}$. It is a graph that consists of two vertices u and v that are connected by k paths of lengths $n_1, n_2, \dots, n_k$ (see Figure 1.4).

A *tree* is a connected acyclic graph. A *spanning tree* of a graph G is a spanning subgraph that is a tree. As we will see, spanning trees play a big role in the study of all terminal reliability.

A *k-partite graph* is a graph whose vertex set can be partitioned into k independent subsets. If $k = 2$ we call G a *bipartite graph*. A *complete k-partite graph* $K_{n_1, n_2, \dots, n_k}$ is one which is *k-partite*, so the entire vertex set is partitioned into disjoint subsets

$S_1, S_2, \dots, S_k$ of sizes $n_1, n_2, \dots, n_k$ respectively, and for every pair of subset S_i, S_j , if $x \in S_i$ and $y \in S_j$ then there is an edge between x and y (see Figure 1.4).

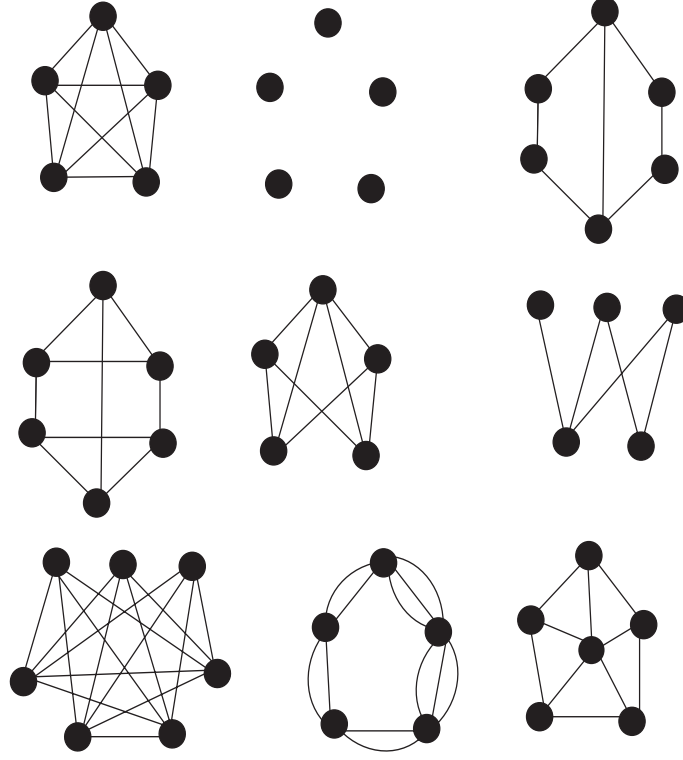


Figure 1.4: In order: K_5 , empty graph on 5 vertices, $\Theta_{4,2,4}$, a 3-regular graph, a 3-semiregular graph, bipartite graph, $K_{3,2,2}$, cycle bundle, W_5

Throughout the thesis we will refer to several graph operations. Let G be a graph, then the graph ℓG , $\ell \geq 1$ represents the disjoint union of ℓ copies of G . If we have two graphs G and H , the graph $G + H$ has vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = \{e = (x, y) \mid e \in E(G) \text{ or } e \in E(H) \text{ or } x \in V(G) \text{ and } y \in V(H)\}.$$

That is $G + H$ is the graph consisting of G , H and an edge between every pair of vertices x, y , $x \in V(G)$, $y \in V(H)$. For example, if $G = C_n$ and $H = K_1$, the graph which we obtain is a *wheel graph* $W_n = C_n + K_1$ (see Figure 1.4).

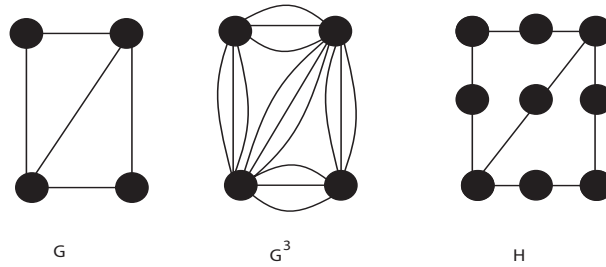


Figure 1.5: A graph G , the graph G^3 which is replacing each edge of G with a bundle of edges and the graph H which is the graph G with each edge subdivided into paths of length 3.

For a graph G , the graph G^k is G with each edge replaced with a bundle of k edges. To subdivide an edge k times is to replace the edge by a path of length k . An example can be found in Figure 1.5. When each edge of a graph is replaced by a bundle of edges, we call the resulting graph a *graph bundle*. For example, a tree with each edge replaced by bundles is a *tree bundle*, a cycle with edges replaced by bundles is a *cycle bundle* (see Figure 1.4). We will see that cycle bundles play a helpful role when studying the analytic properties of all terminal reliability polynomials.

1.3 Network Reliability Background

Now that we have the necessary graph theory terminology, we can begin to discuss the focus of this work, network reliability.

As mentioned in the introduction, the study of network reliability dates back to the 1950's. Some of the first papers regarding network reliability are by Shannon and Moore [65] and Birnbaum, Esary and Saunders [11]. The book "The Combinatorics of Network Reliability" by Charles Colbourn [31] is a great resource for someone who is interested in the topic and is the reference for the information remaining in this chapter.

A typical model for network reliability is that you have a graph or digraph and it is assumed that the vertices are always operational and that edges or arcs are

independently operational with probability $p \in [0, 1]$. To say an edge is operational means that it is in working condition and information can be passed along it. In the graph theoretical sense, to say that the edge e is *operational* means that it is present in a subgraph. An operational subgraph, H , is the subgraph induced by the vertices of H , that is $G|_{V(H)}$.

Generally, the vertices represent communication hubs, like computers, and the edges represent the communication links between the hubs. If the communication is bidirectional, then a graph is the best model, but if communication can be one way then a digraph is best. For instance, if a sewage treatment plant was the network of interest, a digraph would be best since sewage can only travel one way down a pipe. If the network being studied was a power grid, then a graph would be the best representation, since electricity can travel both ways along the power line.

When we ask if a network is *reliable*, we are asking whether a certain process can take place or if the graph has an operational subgraph with a certain property. If our graph represents a subway system, perhaps we desire that a commuter can reach all of the stations, no matter where located, so in the graph where the vertices are the stations and the edges are the rails, we want a path between all pairs of vertices. If the graph represents a mail processing plant then perhaps we desire that the sorting machine can send letters to all the other processing equipment. In the graph, where the equipment are the vertices and arcs are connecting one machine to another, we want a directed path from the vertex representing the sorting machine to all the other vertices.

Let P be some graph property, for example connectedness. The *reliability* of a (di)graph G is the probability that a subgraph of G with property P is operational. A subgraph of minimal size which has property P is called a *minpath*.

As mentioned, in most models the vertices are always operational and the edges that operate independently with probability $p \in [0, 1]$, but models where vertices

operate independently with probability $p \in [0, 1]$ have also been studied [37, 77]. In the vertex failure model, the operational vertices must induce a connected graph. The first half of this thesis will look at an edge failure model, the latter half, the vertex failure model.

Since there are many different properties that could be desired in a network, this has led to different network reliability models and problems. Below we will list a few of the well-known models. As mentioned, [31] is an excellent reference for these models and more.

Two Terminal Reliability and s, t -Connectedness:

In this model, it is assumed that the vertices are always operational and that the edges can fail independently with probability $p \in [0, 1]$. Suppose we have a graph G , a source vertex s and target vertex t . The desired property is that s and t can communicate, so the two terminal reliability of a graph is the probability that there is a path between two specified vertices, s and t . In the directed graph version it is required that a directed path exist between s and t . This reliability is called s, t -Connectedness.

k -Terminal Reliability and s, T -Connectedness:

Again, it is assumed that vertices always operate and that edges can fail independently with probability $p \in [0, 1]$. Suppose we have a graph G and a fixed subset of vertices, T , $|T| = k$. The desired property is that each pair of vertices $u, v \in T$ can communicate with each other. The k -terminal reliability of a graph is the probability that every pair of vertices, $u, v \in T$ are connected by a path. A minpath in this model is a tree containing just the vertices in T . The directed graph version is called s, T -connectedness and requires that there is a directed path from a vertex s to all vertices in T .

All Terminal Reliability and Strongly Connected Reliability:

This is equivalent to the k -terminal reliability, when $T = V(G)$. The directed version is called the *strongly connected reliability* and it requires that for every pair of vertices, x and y , there exists a directed path from x to y and from y to x .

Two-terminal, k -terminal, all terminal reliability and their directed counterparts are examples of a more general setting for reliability. Let X be a set of elements and $\mathcal{P}(X)$ the powerset of X . Let $\mathcal{W} \subseteq \mathcal{P}(X)$ be such that if $W \in \mathcal{W}$ and $W \subseteq Y$, then $Y \in \mathcal{W}$, that is, $\mathcal{W}$ is closed under supersets. Any such $\mathcal{W}$ is called a *coherent system*. Assume that the $x \in X$ operate independently with probability $p \in [0, 1]$. The probability that a subset of operational elements in X are in $\mathcal{W}$ is the *coherent reliability polynomial* of $\mathcal{W}$ [7, 38].

When we speak of the probability that a graph has an operational subgraph with property P , this probability can always be expressed as a polynomial in p . This is what has been referred to as the *reliability polynomial*. When looking at coherent systems, we require that property P be closed under supersets. Let $\mathcal{W}$ be the set of subgraphs of G with property P , that is closed under supersets. To obtain the reliability polynomial, we add up the probability of each $W \in \mathcal{W}$ being operational to obtain

$$\text{Rel}(\mathcal{W}, p) = \sum_{i=0}^m N_i p^i (1-p)^{m-i},$$

where m is the cardinality of the base set (either edge set or vertex set, depending on the model of interest). The N_i are the number of $W \in \mathcal{W}$ of size i .

We can also express the reliability polynomial for a graph G (or any coherent system for that fact) as

$$\text{Rel}(G, p) = \sum_{i=0}^m F_i p^{m-i} (1-p)^i,$$

where F_i is the number of ways to remove i edges such that the remaining subgraph has property P . Note that $F_i = N_{m-i}$.

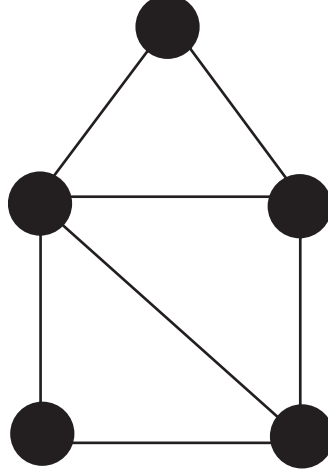


Figure 1.6: Graph for Example 1.3.1

Example 1.3.1 Consider the graph, G in Figure 1.6. Suppose the edges operate independently with probability $p \in [0, 1]$ and the desired property is that the operational subgraph be spanning connected. This is an example of a coherent system. Then $N_0 = N_1 = N_2 = N_3 = 0$, $N_4 = 21$, $N_5 = 19$, $N_6 = 7$, $N_7 = 1$ and $F_0 = 1$, $F_1 = 7$, $F_2 = 19$, $F_3 = 21$ and $F_4 = F_5 = F_6 = F_7 = 0$, so

$$\text{Rel}(G, p) = 21p^4(1-p)^3 + 19p^5(1-p)^2 + 7p^6(1-p) + p^7.$$

Now suppose the vertices operate independently with probability $p \in [0, 1]$ and we require that the subgraph induced by the operational vertices be connected. This is not an example of a coherent system. We have that $N_1 = 5$, $N_2 = 7$, $N_3 = 8$, $N_4 = 5$, $N_5 = 1$ and $F_0 = 1$, $F_1 = 5$, $F_2 = 7$, $F_3 = 8$, $F_4 = 5$, so

$$\text{Rel}(G, p) = 5p(1-p)^4 + 7p^2(1-p)^3 + 3p^3(1-p)^2 + 5p^4(1-p) + p^5$$

The F_i , for coherent systems, have additional combinatorial significance. Before we explain what it is, we need the following definition,

Definition 1.3.2 *A simplicial complex $\mathcal{C}$ is an ordered pair (E, I) consisting of a finite ground set E and a collection of subsets, I of E where $\emptyset \in I$ and if $X \in I$ and $Y \subseteq X$ then $Y \in I$. That is, the elements of I are closed under subsets. The elements of $\mathcal{C}$ are called faces and the maximal elements are called facets. A minimal non-face is called a circuit. An element found in every facet is called a coloop. The maximum size, d of a face is called the dimension of the complex. The associated F -vector for a simplicial complex $\mathcal{C}$ is the vector $\langle F_0, F_1, \dots, F_d \rangle$, where F_i is the number of faces of size i . Let $\sigma_1, \sigma_2, \dots, \sigma_r$ be the facets of $\mathcal{C}$. As the faces of $\mathcal{C}$ are closed under subsets, we will represent the complex by its set of facets, that is, $\mathcal{C} = \{\sigma_1, \sigma_2, \dots, \sigma_r\}$.*

For every subset, S of edges (or vertices) we remove from G which leaves a subgraph with property P (which is closed under supersets), the removal of any subset of S from G will also leave a subgraph with property P . This means that each graph has associated with it a simplicial complex whose faces are the subsets of edges (or vertices) whose removal leaves a subgraph with property P . The complements of the facets of the associated simplicial complex are the minpaths. As we will later see, knowing about the combinatorial structure of the associated complexes can assist in calculating the reliability polynomial.

Of the models described, all terminal reliability has garnered much attention. The first half of this thesis will focus on the all terminal reliability of graphs.

1.4 Introduction to All Terminal Reliability

Of the various models of network reliability one of the most popular and well-studied models is that of *all terminal reliability*, which as previously mentioned, is the probability that G has at least a spanning tree operational, given that vertices always operate and edges operate independently with probability $p \in [0, 1]$.

The requirement of having at least a spanning tree operational is a global structure requirement, since all the vertices need to be able to communicate with each other. For example, consider a transit system in a city centre. There are particular hubs (vertices) which the transit routes (edges) meet at. If there are outside influences, such as construction or accidents, which inhibit the use of some routes (so the edge fails) one still wants to be able to pick up the passengers at each hub. In the graph which represents the transit system, we need a path between each vertex, so we need at least a spanning tree operational.

The all terminal reliability for a graph G of order n and size m is an example of a coherent system. We can calculate the reliability by listing all spanning connected subgraphs of G and summing together the probability that each one will operate. A spanning subgraph of size i has probability $p^i(1-p)^{m-i}$ of being operational. Some areas of research surrounding the all terminal reliability of a graph G are finding explicit formulas for various families of graphs, bounding the polynomial, optimal graphs and the location of the roots of the polynomial [2–5, 11, 13, 21–23, 29, 31, 40, 65, 67, 69, 70, 74]

Example 1.4.1 *Let our graph be a tree, T on n vertices. Then $Rel(T, p) = p^{n-1}$. Suppose we replace every edge of our tree with a bundle of k edges to obtain the tree bundle, T^k . For the graph to have at least a spanning tree operational, we require that at least one edge in the bundle be operational, which occurs with probability $1 - (1-p)^k$, so $Rel(T^k, p) = (1 - (1-p)^k)^{n-1}$*

Example 1.4.2 *Let our graph be a cycle, C_n . We will calculate the all terminal reliability of C_n by considering all the possible spanning connected subgraphs of C_n and the probability that each occurs. There is one spanning connected subgraph of size n , which occurs with probability p^n and there are n spanning connected subgraphs of size $n - 1$, which are the spanning trees of C_n . They are operational with probability $np^{n-1}(1 - p)$. A subgraph of size less than $n - 1$ would not contain a spanning tree, so $\text{Rel}(C_n, p) = p^n + np^{n-1}(1 - p) = (1 - n)p^n + np^{n-1}$.*

We can calculate the reliability based on the inclusion\exclusion principle. Let T be the set of spanning trees of a graph G on n vertices. Since for a graph to be connected we need at least 1 spanning tree operational, we have that

$$\begin{aligned} \text{Rel}(G, p) = & \sum_{t \in T} p^{n-1} - \sum_{1 \leq i < j \leq |T|} p^{|t_i \cup t_j|} + \\ & \sum_{1 \leq i < j < k \leq |T|} p^{|t_i \cup t_j \cup t_k|} + \dots \\ & + (-1)^{|T|-1} p^{|t_1 \cup t_2 \cup \dots \cup t_{|T|}|} \end{aligned}$$

For example, with cycles we have

$$\text{Rel}(C_n, p) = np^{n-1} + \sum_{i=2}^n (-1)^{i+1} \binom{n}{i} p^n = np^{n-1} + (1 - n)p^n.$$

The calculation of the all terminal reliability polynomial for any graph G can also be expressed in terms of deleting and contracting an edge of G .

Theorem 1.4.3 [31] *Let G be a graph and e an edge of G . Then*

$$\text{Rel}(G, p) = p \cdot \text{Rel}(G \cdot e, p) + (1 - p)\text{Rel}(G - e, p)$$

This comes from the fact that a spanning connected subgraph either contains a particular edge or it doesn't. If it does not then $G - e$ must have a spanning tree

operational. If the subgraph does contain e , then we know that the end points of e can communicate, so we can contract the edge and then we need a spanning tree in the graph $G \cdot e$ to ensure that everyone can communicate. For example, for cycles we have $\text{Rel}(C_n, p) = (1 - p)\text{Rel}(P_n, p) + p\text{Rel}(C_{n-1}, p)$.

Sometimes, explicit formulas can not be calculated, there exist recursive formulas, like that for the complete graph on n vertices (for example [31], pg 33):

$$\text{Rel}(K_n, p) = 1 - \left(\sum_{i=1}^{n-1} \binom{n-1}{i-1} (1-p)^{i(n-i)} \text{Rel}(K_i, p) \right).$$

We have seen that $\text{Rel}(G, p)$ is a polynomial in p on the interval $[0, 1]$. If the edge probability is known, then the all terminal reliability is a real number between $[0, 1]$. Clearly, for any graph G , $\text{Rel}(G, 0) = 0$ since we will never have a spanning connected subgraph operational, and $\text{Rel}(G, 1) = 1$, if G is connected, since every edge is always operational. The all terminal reliability polynomial is an increasing function on the interval $[0, 1]$, so we know that the polynomial will not have any zeros in this interval. It was proved in [11, 65] that if for some $p_0 \in [0, 1]$ we have that $\text{Rel}(G, p_0) = p_0$ then $\text{Rel}(G, p) < p$ for $0 < p < p_0$ and $\text{Rel}(G, p) > p$ for $1 > p > p_0$, so if the all terminal reliability polynomial crosses the line $f(p) = p$, then it does so in what is called an *S-shape*. Other properties regarding the structure of the polynomial, such as inflection points [27, 39] and thresholds [53, 61] have also been studied.

There are several different ways to compute a reliability polynomial, which results in there being several different forms of the all terminal reliability polynomial for a graph G of order n and size m .

N-form:

$$\sum_{i=n-1}^m N_i p^i (1-p)^{m-i}$$

where N_i is the number of spanning connected subgraphs of size i .

F -form:

$$\sum_{i=0}^{m-n+1} F_i p^{m-i} (1-p)^i$$

where F_i is the number of ways to remove i edges and leave the graph connected.

C -form:

$$1 - \sum_{i=\lambda}^m C_i p^{m-i} (1-p)^i$$

where C_i is the number of cutsets of size i and λ is the edge connectivity of the graph.

H -form:

$$p^{n-1} \sum_{i=0}^{m-n+1} H_i (1-p)^i$$

where the H_i are positive integers and will be expanded upon shortly.

Example 1.4.4 Consider a cycle on n vertices:

N -form : $p^n + np^{n-1}(1-p)$, $N_n = 1, N_{n-1} = n$

F -form: $p^n + np^{n-1}(1-p)$, $F_0 = 1, F_1 = n$

C -form: $1 - \sum_{i=2}^n \binom{n}{i} p^{n-i} (1-p)^i$, $C_i = \binom{n}{i}$, $2 \leq i \leq n$

H -form: $p^{n-1}(1 + (n-1)(1-p))$, $H_0 = 1, H_1 = n-1$

For each different expression of the reliability polynomial, we obtain a sequence of coefficients. As mentioned above, the N_i count the number of spanning connected subgraphs of size i , the C_i the number of cutsets of size i , and the F_i the number of ways to remove i vertices and keep G connected. We have relationships between the different sequences, for instance, $N_{m-i} = F_i$ and $C_i + F_i = \binom{m}{i}$. It has been conjectured [31] that the sequences of N , C , H and F are always unimodal, that is non-decreasing then non-increasing. It has recently been proven that the F -sequence [56] and H -sequence are unimodal [48]. Of the different sequences associated with

the all terminal reliability of a graph G , the H - and F -sequences are particularly interesting since they have additional combinatorial significance.

First, we will look at the F -sequence. We know that the F -sequence enumerates the number of ways to remove edges and keep the graph connected. A subset S of edges whose removal does not disconnect G is closed under subsets. Since the sets have this property, the set of subsets whose removal leaves G connected forms a simplicial complex and the F -sequence is its F -vector. In fact, for all terminal reliability, there is even more structure and the simplicial complex associated with this reliability model is actually a *matroid*.

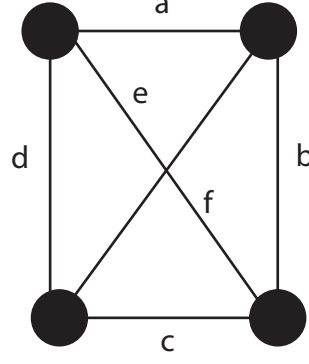
Definition 1.4.5 *A matroid $M = (E, I)$ is a simplicial complex with the exchange property, which is that for $X \in I$ and $Y \in I$ if $|X| > |Y|$ then there exists an $x \in X \setminus Y$ such that $Y \cup \{x\} \in I$.*

The matroid that is formed is the *cographic matroid* of G (for example see [31] page 60)

Definition 1.4.6 *Let G be a graph. The cographic matroid of G is the matroid whose faces are sets of edges whose removal leaves G connected. The facets of the cographic matroid are the complements of the spanning trees of G .*

Since the F -sequence is in fact the F -vector for the cographic matroid of G , then we have that F_{m-n+1} is the number of spanning trees in G , since the facets of the cographic matroid of G are complements of spanning trees. This term can be calculated in polynomial time via the Matrix Tree Theorem. If we know the edge connectivity, λ , of the graph, then we can quickly calculate F_i , $0 \leq i < \lambda$, since the removal of any subset of size $i < \lambda$ will not disconnect G .

Example 1.4.7 *Consider the graph in Figure 1.7. The set of facets of the cographic matroid for K_4 is*

Figure 1.7: K_4

$\mathcal{C} = \{\{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, c, e\}, \{a, c, f\}, \{a, d, f\}, \{a, e, f\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{b, d, f\}, \{b, e, f\}, \{c, d, e\}, \{c, e, f\}, \{d, e, f\}\}$. The F -vector for this matroid is $\langle 1, 6, 15, 16 \rangle$, so

$$Rel(K_4, p) = p^6 + 6p^5(1-p) + 15p^4(1-p)^2 + 16p^3(1-p)^3 = -6p^6 + 24p^5 - 33p^4 + 16p^3.$$

The calculation of the all terminal reliability of a graph G is $\sharp P$ -complete [31], so it is intractable. It is of interest to find efficient bounds for the all terminal reliability polynomial. Since simplicial complexes have nice structure, we are able to find bounds on the F -vector of the cographic matroid of a graph G and hence, obtain bounds for the all terminal reliability polynomial. For a simplicial complex, M , on a ground set of size m , we can use Sperner's Bound [71], $(m - i + 1)F_{i-1} \geq iF_i$, to estimate the reliability polynomial for various graphs if they can not be calculated explicitly.

There are other bounds which are better than Sperner's bound, namely the Kruskal-Katona bounds. Details can be found in [52, 55].

As mentioned previously, the H -sequence for a graph G on n vertices and m edges also has combinatorial significance. There is a relation between the F and H -sequences. For the coefficients $F_0, F_1, \dots, F_d$ and $H_0, H_1, \dots, H_d$,

$$H_i = \sum_{j=0}^i (-1)^{i-j} \binom{d-j}{d-i} F_j$$

and

$$\sum_{i=0}^d H_i = F_d.$$

If one can take the faces of a complex, $\mathcal{C}$ and put them into intervals

$$[L, U] = \{S : L \subseteq S \subseteq U\},$$

where every face is in exactly one interval and the U are facets, then $\mathcal{C}$ is *partitionable*.

A complex is called *shellable* if it is a pure d -dimensional complex, meaning each facet has size d and if the facets, $\sigma_1, \sigma_2, \dots, \sigma_s$ can be ordered such that for $2 \leq i \leq s$, $\mathcal{P}(\sigma_i) \cap \bigcup_{j=1}^{i-1} \mathcal{P}(\sigma_j)$, where $\mathcal{P}(\sigma_i)$ represents the set of all subsets of σ_i is purely $(d-1)$ -dimensional.

Shellable complexes are partitionable and since matroids are shellable, they are partitionable [31]. The H_i count the number of intervals in the partition of $\mathcal{C}$ with the lower face in the interval having size i .

They also have another combinatorial interpretation. Before we explain it, we need a few definitions. Let $\mathbf{k}$ be a field, then $\mathbf{k}[x_1, x_2, \dots, x_n]$ is the polynomial ring in n variables over $\mathbf{k}$. A monomial is a product of the variables of the form $x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}$, $d_i \geq 0$. The degree of the monomial is the sum of the d_i . There is a bijection between the faces in a shellable complex and a set of F_d monomials, which are closed under divisibility. This is an *order ideal of monomials*. The H_i represent the number of monomials of degree i . Another bijection that maps a monomial to a multiset, (where the exponent on the variable represents how many times the variable is in the multiset), also maps the faces in a shellable complex to a collection of multisets, that is closed under taking submultisets (this is *multicomplex*) [23]. In other words, the H_i also count monomials in degree i in a multicomplex [23, 62].

It is known that the Kruskal-Katona bounds are better than Sperner's bounds [71], but we are looking at a matroid, so there is more structure to take into consideration.

There are bounds by Ball and Provan [4–6], which exploit the structure of shellable complex to provide even better bounds. Details can be found in [4, 5]. The Ball and Provan bounds are better than the Kruskal-Katona bounds, so shellability is a desirable property for complexes when dealing with reliability.

Though there has been much research done regarding the all terminal reliability polynomial, there are still open problems to solve and properties of the polynomial to study. Now that we have a bit of background regarding all terminal reliability, our study of this reliability problem can begin. We will start by looking at the existence, or lack thereof, of most optimal networks. We will then proceed to the analytic properties of all terminal reliability polynomials on the interval $[0, 1]$. Afterwards, a new reliability problem, the k -clique reliability of a graph, will be introduced and investigated.

Chapter 2

Most and Least Optimal Graphs

In real-world applications of network reliability one wants to construct the network, given their particular constraints, that is most resilient to connection failures. That is, when constructing a network, one wants the best topology possible, so a natural question to ask is “If given n vertices and m edges, what is the best topology for the network?” The answer to this question of course depends on the definition of optimal. This might depend on the value of p , but perhaps not.

2.1 Introduction

In this Chapter will consider two families of graphs, $\mathcal{S}_{n,m}$ and $\mathcal{G}_{n,m}$. Traditionally, a graph, G , is called the most optimal graph if it is equally as reliable or more reliable than any other graph in the same family, for all values of $p \in (0, 1)$.

Example 2.1.1 *Consider the family of all connected graphs on $n \geq 2$ vertices and $m = n$ edges. The possible connected graphs are a cycle of size at least 2 with a tree on k vertices attached to it; this has reliability*

$$p^k(p^{n-k} + (n-k)p^{n-k-1}(1-p)) = p^n + (n-k)p^{n-1}(1-p).$$

This is clearly optimal for $k = 0$, so a cycle, and least optimal for $k = n - 2$, a tree with one edge replaced by a bundle of 2 edges.

In real-world applications of network reliability one might want to construct, given certain resources, a network that is most resilient no matter what the failure

rate. That is, given n vertices and m edges, is there an optimal network? Formally, optimality is defined in the following way.

Definition 2.1.2 *Let $\mathcal{F}$ be a family of graphs. We call $G \in \mathcal{F}$ an $\mathcal{F}$ -graph, and say that G is a most optimal $\mathcal{F}$ -graph if for any other $\mathcal{F}$ -graph H we have that $Rel(G, p) \geq Rel(H, p)$ for all $p \in [0, 1]$ (for $p_0 = 0$ or 1 we will also talk about G being a most optimal $\mathcal{F}$ -graph sufficiently close to p_0 when there is an $\varepsilon > 0$ such that $Rel(G, p) \geq Rel(H, p)$ for all $p \in [0, 1] \cap (p_0 - \varepsilon, p_0 + \varepsilon)$ and for all other $\mathcal{F}$ -graphs H).*

It was conjectured in [12, 13, 16] that, given n and m , there always exist most optimal $\mathcal{S}_{n,m}$ -graphs and most optimal $\mathcal{G}_{n,m}$ -graphs. Clearly, trees are the unique family of most optimal $\mathcal{S}_{n,n-1}$ -graphs and cycles are the unique most optimal $\mathcal{S}_{n,n}$ -graphs. There are unique most optimal $\mathcal{S}_{n,n+1}$ -graphs [13] and they are theta graphs with path lengths that differ by at most 1.

For $m = n + 2$ and $m = n + 3$, it is known that there also exists most optimal $\mathcal{S}_{n,m}$ -graphs, which turn out to be particular subdivisions of K_4 [13] and $K_{3,3}$ [75], respectively. It is conjectured that for $m = n + 4$, $m = n + 5$, $m = n + 6$ and $m = n + 7$ there are most optimal $\mathcal{S}_{n,m}$ -graphs [3]. These graphs are all fairly sparse. At the other end of the spectrum, for dense graphs in [54] it was shown that for n and $m \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$, removing a matching from K_n will result in a most optimal $\mathcal{S}_{n,m}$ -graph. The above results were for simple graphs, but when $m = n$, $m = n + 1$ and $m = n + 2$, Gross and Saccoman [40] extended the results to show that the graphs mentioned above are indeed most optimal $\mathcal{G}_{n,m}$ -graphs.

So there was positive evidence to support both conjectures, but it turns out that the conjecture for simple graphs failed [54, 67]. There exists an infinite family of simple graphs such that there does not exist a most optimal $\mathcal{S}_{n,m}$ -graph for a specific pair of n and m .

Theorem 2.1.3 [54, 67] *For $n \geq 6$ even and $m = \binom{n}{2} - \frac{n+2}{2}$ and for $n \geq 7$ odd and $m = \binom{n}{2} - \frac{n+5}{2}$ a most optimal $\mathcal{S}_{n,m}$ -graph does not exist.*

Computationally one can find that for $n = 4$ and $n = 5$ there are always most optimal $\mathcal{S}_{n,m}$ -graphs. When $n = 6$ and $n = 7$, computations show that the only situations where a most optimal $\mathcal{S}_{n,m}$ -graph does not exist is when the conditions of Theorem 2.1.3 hold. So for each $n \geq 6$ there is a single known value of m such that a most optimal $\mathcal{S}_{n,m}$ -graph does not exist, but are these the only cases where we do not have most optimal simple graphs? Moreover, if we extend our family to include all graphs, does a most optimal $\mathcal{G}_{n,m}$ -graph exist? We will show here that given n there are in fact several m such that a most optimal $\mathcal{S}_{n,m}$ -graph does not exist, and if we extend our family to include all graphs (that is, including multiple edges), we will find that there is still not a most optimal $\mathcal{G}_{n,m}$ -graph. The latter provides the first such counterexamples to the conjecture of the existence of most optimal $\mathcal{G}_{n,m}$ -graphs. This is in contrast to the directed version of all terminal reliability, strongly connected reliability, where allowing multiple edges always produced a most optimal digraph [28].

2.2 New $\mathcal{S}_{n,m}$ Classes with No Most Optimal Graphs

We will show that for a given $n \geq 8$ there is more than one value of m such that a most optimal $\mathcal{S}_{n,m}$ does not exist. To do so we will show that for values of p near 0 there is a most optimal graph and for values of p near 1 there is a most optimal graph, but it differs from the graph that is most optimal near 0.

Looking at the F -form of the all terminal reliability polynomial we can obtain the following useful observation. In the following, $F(G)$ denotes the coefficients in the F -form for the graph G .

Observation 2.2.1 *Let $G, H \in \mathcal{G}_{n,m}$. Consider the F -form of the all terminal reliability polynomials:*

$$\begin{aligned} \text{Rel}(G, p) &= \sum_{i=0}^{m-n+1} F_i(G) p^{m-i} (1-p)^i \\ \text{Rel}(H, p) &= \sum_{i=0}^{m-n+1} F_i(H) p^{m-i} (1-p)^i. \end{aligned}$$

Suppose that $F_i(G) = F_i(H)$ for $1 \leq i < \ell$ and $F_i(G) = F_i(H)$ for $k < i \leq m$. Then

- 1. $F_\ell(G) > F_\ell(H)$ implies that $\text{Rel}(G, p) > \text{Rel}(H, p)$ for values of p sufficiently close to 1, and*
- 2. $F_k(G) > F_k(H)$ implies that $\text{Rel}(G, p) > \text{Rel}(H, p)$ for values of p sufficiently close to 0.*

Proof. Let G and H be graphs on n vertices and m edges and consider

$$\begin{aligned} \text{Rel}(G, p) - \text{Rel}(H, p) &= \sum_{i=0}^{m-n+1} (F_i(G) - F_i(H)) p^{m-i} (1-p)^i \\ &= p^{m-k} (1-p)^\ell \sum_{i=\ell}^k (F_i(G) - F_i(H)) p^{k-i} (1-p)^{i-\ell} \\ &= p^{m-k} (1-p)^\ell ((F_\ell(G) - F_\ell(H)) p^{k-\ell} + \dots \\ &\quad + (F_k(G) - F_k(H)) (1-p)^{k-\ell}). \end{aligned}$$

From this we can see that if $F_\ell(G) > F_\ell(H)$, then for p near 1, $\text{Rel}(G, p) - \text{Rel}(H, p) > 0$, that is, $\text{Rel}(G, p) > \text{Rel}(H, p)$. Likewise, if $F_k(G) > F_k(H)$, then for p near 0, $\text{Rel}(G, p) > \text{Rel}(H, p)$. □

It follows from this observation that there are indeed always graphs (in *any* family) that are most optimal and least optimal near 0 and graphs that are most optimal and least optimal near 1. This is since we can order the F -vectors for the graphs in

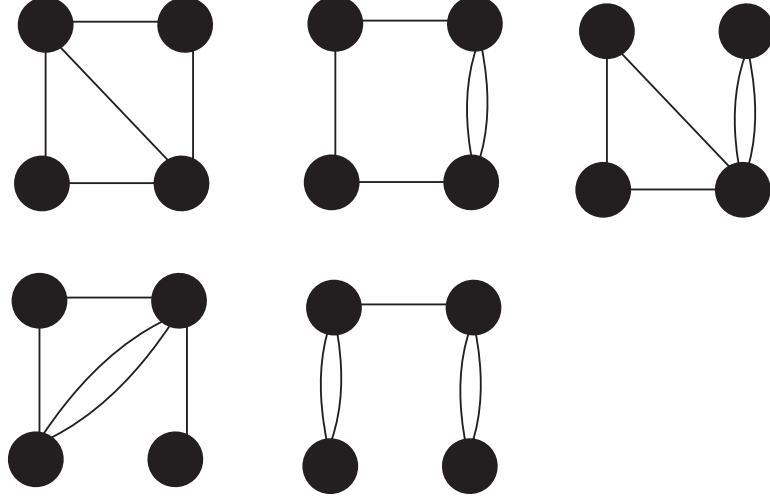


Figure 2.1: Figure for Example 2.2.2

the family of interest so that there will be some graph, or subfamily of graphs, whose F -vector is the largest, or smallest.

Example 2.2.2 Let $S_1 = \langle F_1, F_2, \dots, F_n \rangle$ and $S_2 = \langle E_1, E_2, \dots, E_n \rangle$ be F -vectors. We say that $S_1 > S_2$ if $F_i = E_i$ for $i < k$ but $F_k > E_k$. For example, consider Figure 2.1. The F -vectors can be ordered in the following way,

$$\langle 1, 5, 8 \rangle > \langle 1, 5, 7 \rangle > \langle 1, 5, 6 \rangle > \langle 1, 4, 5 \rangle > \langle 1, 4, 4 \rangle,$$

so the first graph is the most optimal, not only for p near 1, but overall and the last graph is the least optimal.

From Observation 2.2.1 we can see that when comparing the all terminal reliability of two graphs, and considering values of p sufficiently close to 0, it will be a graph with the most number of spanning trees which is most optimal. When considering values of p sufficiently close to 1, it is a graph with the largest edge connectivity, λ , and among all graphs that have the edge connectivity λ , the graphs that have the least number of cutsets of size λ will contain a most optimal graph near 1. We should note that if G has C_i cutsets of cardinality i , then $F_i = \binom{m}{i} - C_i$, so maximizing F_i

is equivalent to minimizing C_i . The key fact is that, for any family of graphs, if the graph that is most optimal for values of p near 0 differs from the graph that is most optimal for values of p near 1, then a most optimal graph for the family does not exist.

Example 2.2.3 *When looking at $\mathcal{S}_{6,11}$ using Maple, we can calculate the reliabilities of all simple graphs on $n = 6$ vertices and $m = 11$ edges and order their F -vectors. Refer to Figure 2.2. We find that G is most optimal for values of p near 0 and H is most optimal for values of p near 1, so different topologies are most optimal depending on the value of p .*

Another way to look at this, again using Maple and Observation 2.2.1, is that G is most optimal for values of p near 0 since G is the unique graph with the most number of spanning trees (225 of them) and H is the unique graph with the least number of cutsets of size 3. Since H has 224 spanning trees, G is more optimal than H near $p = 0$, but the graphs H and G have the same edge connectivity, 3. The F -vector of G is $\langle 1, 11, 55, 163, 309, 368, 225 \rangle$ and the F -vector for H is $\langle 1, 11, 55, 163, 310, 370, 224 \rangle$. We can see that $F_3(G) = F_3(H)$, so we need to look at the next F_i , $i > 3$ where they differ in the number of cutsets of size i . We can see that H has 20 cutsets of size 4 (since $\binom{11}{4} - 310 = 20$) and that G has 21 cutsets of size 4 (since $\binom{11}{4} - 309 = 21$). This makes H the most optimal graph for values of p close to 1, and since this graph differs from G , which is most optimal near $p = 0$, no most $\mathcal{S}_{6,11}$ optimal graph exists.

Since for values of p sufficiently close to 0, the graph with the most number of spanning trees will be most optimal, we will find the following result by Petingi, Boesch and Suffel [14] regarding simple graphs and the maximum number of spanning trees to be very useful.

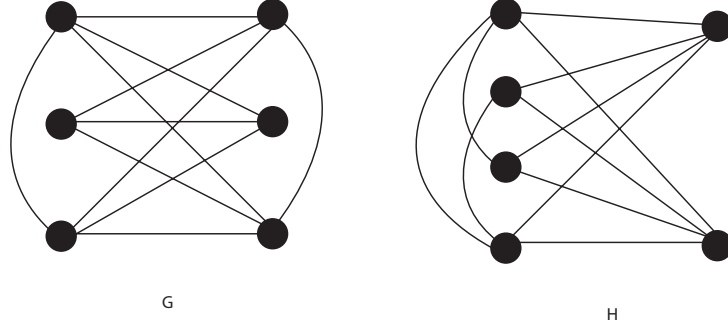


Figure 2.2: G is the most optimal $\mathcal{S}_{6,11}$ -graph for values of p near 0, and H is the most optimal $\mathcal{S}_{6,11}$ -graph for values of p near 1.

Theorem 2.2.4 [14] *Suppose that $m = \binom{n}{2} - (n - k)$ where $n \geq k \geq 2$, or $k \leq 1$ and $n - k \equiv 0 \pmod{3}$. Consider the graph $G \in \mathcal{S}_{n,m}$ where the complement of G is one of the following (see Figure 2.3):*

1. *a matching with $n - k$ edges, whenever $n > k \geq \frac{n}{2}$,*
2. *$\left(\frac{n-2k-2}{3}\right)C_3 \cup 2P_3 \cup (k-2)K_2$, whenever $n - 2k \equiv 2 \pmod{3}$ and $2 \leq k < \frac{n}{2}$,*
3. *$\left(\frac{n-2k-1}{3}\right)C_3 \cup P_3 \cup (k-1)K_2$, whenever $n - 2k \equiv 1 \pmod{3}$ and $2 \leq k < \frac{n}{2}$,*
4. *$\left(\frac{n-2k}{3}\right)C_3 \cup kK_2$, whenever $n - 2k \equiv 0 \pmod{3}$ and $k < \frac{n}{2}$, or*
5. *$\left(\frac{n-1}{3} - 1\right)C_3 \cup P_4$, whenever $k = 1$ and $n \equiv 1 \pmod{3}$.*

Then G is the unique simple graph on n vertices and m edges such that it has the maximum number of spanning trees when compared to any other graph $H \in \mathcal{S}_{n,m}$.

This theorem will be useful since for n and m satisfying the theorem above, we know the unique graph that is a most optimal $\mathcal{S}_{n,m}$ -graph for values of p sufficiently close to 0, and thus the unique candidate for a most optimal $\mathcal{S}_{n,m}$ -graph. In this section we will show that there are cases where the unique $\mathcal{S}_{n,m}$ -graph on n vertices and m edges with the most number of spanning trees is not the most optimal $\mathcal{S}_{n,m}$ -graph for values of p near 1 and hence, there is no most optimal $\mathcal{S}_{n,m}$ -graph.

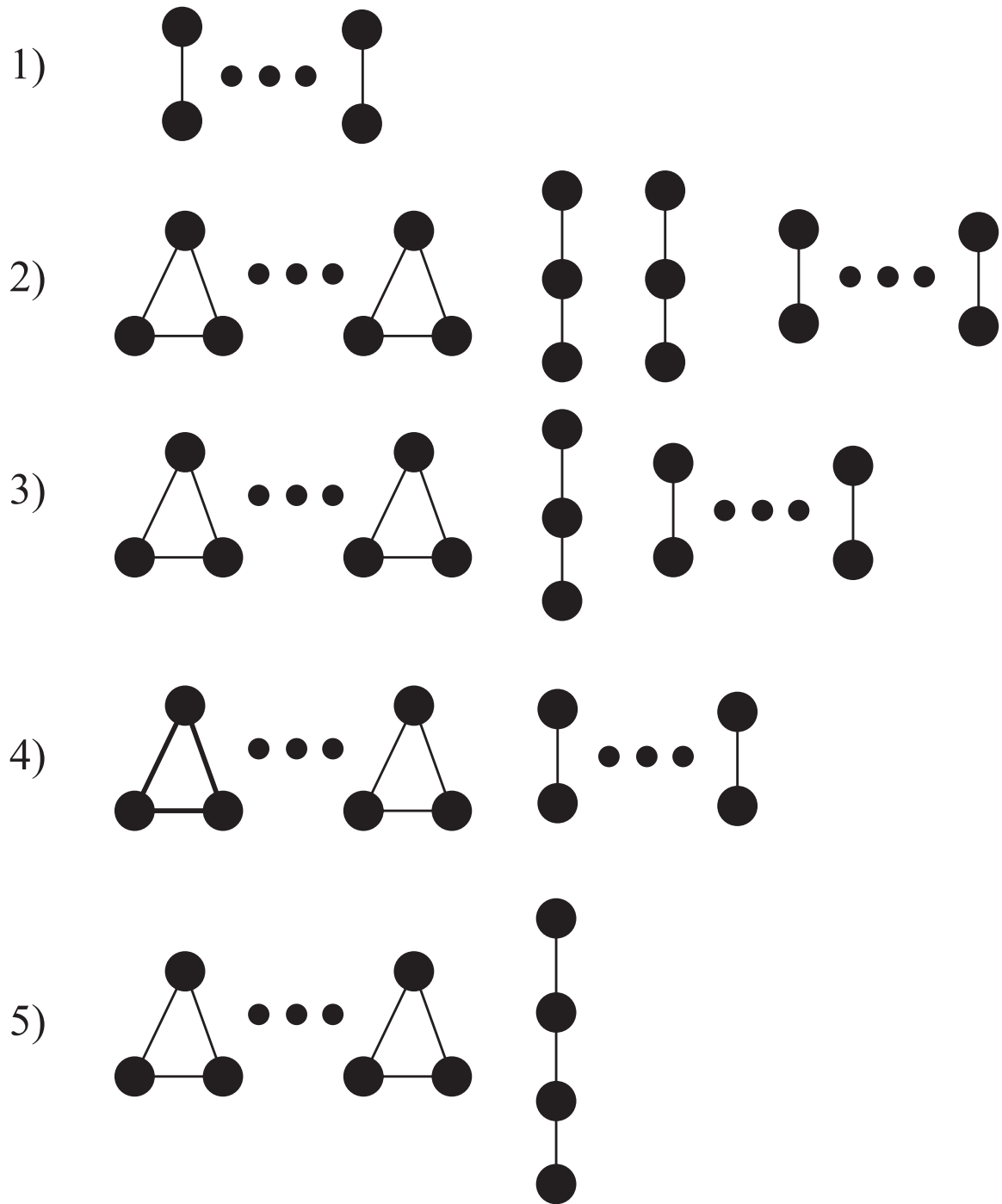


Figure 2.3: The complements of the graphs in Theorem 2.2.4

Recall that the minimum and maximum degrees of a graph are denoted by δ and Δ , respectively. Also recall that a graph is d -semiregular if $\delta = d$ and $\Delta \leq d + 1$ and it called d -regular if $\delta = \Delta = d$ (the class of d -semiregular graphs include the d -regular graphs). We will be looking at semiregular simple graphs since we will show that for our choices of n and m , if a most optimal simple graph exists, then it is semiregular. In this section we will be focusing on $(n - 3)$ -semiregular simple graphs. It is important to note that a simple graph G is $(n - 3)$ -semiregular if and only if $\overline{G}$, the complement of G , has no isolated vertices and $\Delta = 2$. This condition is equivalent to $\overline{G}$ being the disjoint union of nontrivial paths and cycles, with at least one cycle or at least one path of length 3. We will now prove some lemmas which will be needed to prove the non-existence of optimal graphs for several values of m .

Lemma 2.2.5 *Let G be a simple $(n - 3)$ -semiregular graph on $n \geq 6$ vertices and m edges, with x vertices of degree $n - 2$ and y vertices of degree $n - 3$, and let a_G denote the number of pairs of vertices of degree $n - 3$ that are adjacent in G . Let C_j be the number of cutsets of G of size j . Then $C_j = 0$ for $j < n - 3$,*

$$C_j(G) = x \binom{m - (n - 2)}{j - (n - 2)} + y \binom{m - (n - 3)}{j - (n - 3)} \text{ for } n - 3 \leq j \leq 2n - 9,$$

and

$$C_{2n-8}(G) = x \binom{m - (n - 2)}{n - 6} + y \binom{m - (n - 3)}{n - 5} + a_G.$$

Proof. The graph G has edge connectivity at most $n - 3$, as it has a vertex of degree $n - 3$. On the other hand, if S is a cutset of G , then there is a partition of the vertex set of $G - S$ into two parts, X and Y , where X is a component of $G - S$ of smallest size. Let $i = |X|$, so that $i \leq |Y|$ and hence $1 \leq i \leq n/2$. As S is a cutset, it contains all edges of G between X and Y . The number of edges between X and Y is at least $i(n - 3 - (i - 1)) = i(n - 2 - i)$. The function $f(t) = t(n - 2 - t)$ is a parabola opening downwards that is symmetric about its peak at $(n - 2)/2$. It

follows that $f(t)$ attains its minimum, $n - 3$, on $[1, n/2]$ at 1, so we conclude that $|S| \geq n - 3$. That is, G has edge connectivity exactly $n - 3$, so clearly $C_j = 0$ for $j < n - 3$. Moreover, as $n \geq 6$, i can attain the value of 2, which yields the next smallest value for f . However, $f(2) = 2n - 8$, so if $|S| \leq 2n - 9$ then $i = 1$. On the other hand, if $|S| = 2n - 8$ then it is possible for $i = 2$. This occurs if and only if X consists of a single edge joining two vertices of degree $n - 3$ and S consists of the $2n - 8$ edges joining X to the remainder of G .

Suppose first that $|S| \leq 2n - 9$ (and $n \geq 6$). The argument in the previous paragraph shows that $G - S$ has a component of size 1, say $\{u\}$ with degree $z = n - 3$ or $n - 2$; $|S|$ consists of all the edges incident with u , along with $|S| - z \leq 2n - 9 - (n - 3) = n - 6$ other edges. Note that u is the only isolated vertex of $G - S$, as otherwise S would have to contain at least $(n - 3) + (n - 3) - 1 = 2n - 7$ edges. As removing any set of edges around a single vertex, together with other edges, is a cutset of G , we find that

$$C_j(G) = x \binom{m - (n - 2)}{j - (n - 2)} + y \binom{m - (n - 3)}{j - (n - 3)}.$$

Finally, if $|S| = 2n - 8$, then again from above, $G - S$ either has a component of size 1 or 2. If it has a component of size 1, then as in the previous paragraph, the component is unique, and there are $x \binom{m - (n - 2)}{n - 6} + y \binom{m - (n - 3)}{n - 5}$ many such cuts sets. If there is no component of size 1 but a component X of size 2, then X must contain two adjacent vertices of degree $n - 3$, and hence S contains all edges between X and the other vertices (there are exactly $(n - 4) + (n - 4) = 2n - 8$ such edges). On the other hand, if u and v are two adjacent vertices of degree $n - 3$, then the edges between $\{u, v\}$ and the rest of the graph form a cutset of size $2n - 8$. It follows that

$$C_{2n-8}(G) = x \binom{m - (n - 2)}{n - 6} + y \binom{m - (n - 3)}{n - 5} + a_G$$

and we are done. □

Lemma 2.2.6 *Let G be an $(n - 3)$ -semiregular graph (simple or otherwise) with m edges. Then the number of degree $n - 3$ vertices is $n(n - 2) - 2m$ and the number of degree $n - 2$ vertices is $2m - n(n - 3)$ (and hence are only functions of n and m).*

Proof. Let x and y be the number of degree $n - 2$ and $n - 3$ vertices, respectively. Then $m = \frac{x(n-2)+y(n-3)}{2}$ and $x + y = n$. This gives us that $x = n(n - 2) - 2m$ and $y = 2m - n(n - 3)$. $\square$

Lemma 2.2.7 *For $n \geq 6$ and $m = \binom{n}{2} - (n - k)$, $k < \frac{n}{2}$, the simple graph that is most optimal for values of p near 1 is an $(n - 3)$ -semiregular simple graph.*

Proof. We show first that there is an $(n - 3)$ -semiregular simple graph G with n vertices and m edges. One can start with $\overline{G} = C_n$, a cycle of length n , and delete alternating $k < n/2$ edges so that there are no isolated vertices. By the proof of Lemma 2.2.5 this graph has edge connectivity $n - 3$.

Suppose H is a most optimal $\mathcal{S}_{n,m}$ -graph near 1 with a vertex of degree $n - 1$. Then clearly $\delta(H) = n - 3$ since if $\delta < n - 3$, the edge connectivity is too low for it to be optimal near 1 (we have seen that G is a graph with higher edge connectivity) and if $\delta > n - 3$, then by consideration of degrees, we have $2m \geq (n - 2)n$, which implies that $m = \binom{n}{2} - (n - k)$ where $k \geq n/2$, a contradiction.

We want to argue that H is $(n - 3)$ -semiregular. If not, H has $a \geq 1$ vertices of degree $n - 1$, b vertices of degree $n - 2$ and c vertices of degree $n - 3$. Let G have x vertices of degree $n - 2$ and y vertices of degree $n - 3$. Then we have that

$$2m = (a(n - 1) + b(n - 2) + c(n - 3)) = (x(n - 2) + y(n - 3))$$

where $a + b + c = n$ and $x + y = n$. This implies the following sequence of equations holds:

$$a(n - 1) + b(n - 2) + c(n - 3) = x(n - 2) + y(n - 3)$$

$$n^2 - n - b - 2c = n^2 - n - x - 2y$$

$$x + 2y = b + 2c$$

$$n + y = (n - c - a) + 2c$$

$$y = -a + c$$

This means that $c > y$, and therefore by Lemma 2.2.5 and Lemma 2.2.6 H has more cutsets of size $n - 3$ than G . It follows that H is not most optimal for values of p near 1, a contradiction.

We conclude H has vertices of only degrees $n - 2$ and $n - 3$, with at least one of the latter, which means that H indeed is an $(n - 3)$ -semiregular graph. $\square$

From Lemmas 2.2.5 and 2.2.6, for $n \geq 6$, $m = \binom{n}{2} - (n - k)$ and $k < \frac{n}{2}$, among all simple $(n - 3)$ -semiregular graphs of order n and size m , the number of cutsets of size at most $2n - 9$ will be independent of the graph chosen, and the graph with the fewer number of pairs of adjacent degree $n - 3$ vertices will have fewer cutsets of size $2n - 8$. So if we are looking at which simple graph is most optimal $\mathcal{S}_{n,m}$ -graph for values of p near 1, by Observation 2.2.1 and Lemma 2.2.7, a simple $(n - 3)$ -semiregular graph with the fewest number of adjacent degree $n - 3$ vertices will be most optimal.

Theorem 2.2.8 *Let $n \geq 6$, $m = \binom{n}{2} - (n - k)$ and $k < \frac{n}{2}$ (so $n(n - 3) \leq 2m < n(n - 2)$).*

1. If $n(n-2) - 2m = 1$ then the graph that is most $\mathcal{S}_{n,m}$ -optimal for values of p near 1 has as its complement $P_3 \cup \frac{n-3}{2}K_2$. Moreover, this graph is unique.
2. If $n(n-2) - 2m = 2$, then the graph that is most $\mathcal{S}_{n,m}$ -optimal for values of p near 1 has as its complement $P_4 \cup \frac{n-4}{2}K_2$. Moreover, this graph is unique.
3. If $n(n-2) - 2m \geq 3$, then a graph that is most $\mathcal{S}_{n,m}$ -optimal for values of p near 1 has as its complement cycles whose cycle lengths sum to $n(n-2) - 2m$ together with $\left(\frac{2m-n(n-3)}{2}\right)K_2$. Moreover, this family is unique (that is, graphs not belonging to this family are less reliable near 1).

Proof. By Lemma 2.2.7 we know that the most optimal $\mathcal{S}_{n,m}$ -graph for values of p near 1 will be $(n-3)$ -semiregular. Let G be an $(n-3)$ -semiregular simple graph on m edges; $\overline{G}$ consists of degree 1 and degree 2 vertices. We know from Lemma 2.2.5 that $C_i(G) = C_i(H)$ for $i \leq 2n-9$ for any other $(n-3)$ -semiregular graph H , so by Observation 2.2.1, to be most optimal for values of p near 1, we want to minimize the number of cutsets of size $2n-8$. By Lemma 2.2.6 there are $y = n(n-2) - 2m$ vertices of degree $n-3$ and $x = 2m - n(n-3)$ vertices of degree $n-2$. Partition the vertices into two sets, X and Y , which consist, respectively, of the degree $n-2$ and degree $n-3$ vertices (so $|X| = x$ and $|Y| = y$). The subgraph of $\overline{G}$ induced by X has maximum degree at most 1, and the vertices of Y in $\overline{G}$ all have degree 2. By Lemma 2.2.5, to minimize the number of cutsets of size $2n-8$, we want to minimize the number of edges of G in the subgraph induced by the vertices of Y , that is we want to maximize the number of edges in the subgraph of $\overline{G}$ induced by Y .

If $y = 1$, that is, we have only one vertex of degree $n-3$, then the most optimal $\mathcal{S}_{n,m}$ -graph for values of p near 1 has as its complement $P_3 \cup \left(\frac{n-3}{2}\right)K_2$, since it has the minimum number of adjacent degree $n-3$ vertices (note that n is odd since $n(n-2) - 2m = 1$). This graph is unique.

If $y = 2$ we have two vertices of degree $n - 3$, and the most optimal $\mathcal{S}_{n,m}$ -graph for values of p near 1 has as its complement $P_4 \cup \left(\frac{n-4}{2}\right)K_2$ (note that n is even since $n(n-2) - 2m = 2$). This is unique since any other graph would have adjacent degree $n - 3$ vertices.

Now consider when $y \geq 3$. To minimize the number of adjacent degree $n - 3$ vertices in G , if possible we want the vertices in Y to be adjacent to all the vertices in X , so that the two vertices a degree $n - 3$ vertex is not adjacent to are degree $n - 3$ vertices. This means that in $\overline{G}$, the subgraph induced by Y should be a 2-regular graph. We can do so, making this subgraph a disjoint union of cycles and the subgraph induced by X a matching in $\overline{G}$, provided that $x = 2m - n(n - 3)$ is even, which it is. So in this case a simple graph that is most optimal for values of p near 1 is the graph whose complement are cycles whose lengths sum to y together with $\frac{x}{2}K_2$. This graph is not unique, since the complement of the graph induced by Y could be several small cycles or one large cycle of size y , but the family of $(n - 3)$ -semiregular graphs whose complements are cycles whose lengths sum to y disjoint with $\frac{x}{2}K_2$ is unique, as any other graph would have more adjacent degree $n - 3$ vertices, and hence be less reliable near 1. $\square$

Now we have the tools to provide some new results involving the non-existence of most optimal $\mathcal{S}_{n,m}$ networks. As mentioned earlier, the only known result on non-optimality [54, 67] was for $m = \binom{n}{2} - \frac{n+2}{2}$ and $m = \binom{n}{2} - \frac{n+5}{2}$, which when expressed as $m = \binom{n}{2} - (n - k)$ gives $k = \frac{n-2}{2}$ and $k = \frac{n-5}{2}$ respectively. These occur when $n - 2k = 2 + 3j$ for $j = 0, 1$. We now extend this result to $j > 1$ using Theorem 2.2.4.

Theorem 2.2.9 *Let $G \in \mathcal{S}_{n,m}$, $n \geq 6$ where $m = \binom{n}{2} - (n - k)$, with $2 \leq k < \frac{n}{2}$. If $n - 2k \equiv 2 \pmod{3}$ then there does not exist a most optimal $\mathcal{S}_{n,m}$ -graph.*

Proof. Let $n \geq 6$. Suppose that $n - 2k \equiv 2 \pmod{3}$, so $n - 2k = 2 + 3j$, $j \geq 0$. This implies that $k = \frac{n-2-3j}{2}$. By Theorem 2.2.4 (part 2) we have that the

unique simple graph with the most number of spanning trees is the graph whose complement is $\left(\frac{n-2k-2}{3}\right)C_3 \cup 2P_3 \cup (k-2)K_2$ and hence it is the unique graph that is the most optimal $\mathcal{S}_{n,m}$ -graph for values of p sufficiently close to 0. Let H be the graph whose complement is $C_{n-2k} \cup kK_2$. Since $n-2k \equiv 2 \pmod{3}$, $n-2k \geq 3$, by Theorem 2.2.8, $\text{Rel}(G, p) < \text{Rel}(H, p)$ for values of p near 1, so no most optimal $\mathcal{S}_{n,m}$ -graph exists. $\square$

In a similar fashion, we can prove the nonexistence of most optimal $\mathcal{S}_{n,m}$ -graphs for other choices of n and m .

Theorem 2.2.10 *Let $G \in \mathcal{S}_{n,m}$, $n \geq 6$ where $m = \binom{n}{2} - (n-k)$, with $2 \leq k < \frac{n}{2}$ and $k \neq \frac{n-1}{2}$. If $n-2k \equiv 1 \pmod{3}$ then there does not exist a most optimal $\mathcal{S}_{n,m}$ -graph.*

Proof. As $n-2k \equiv 1 \pmod{3}$, $n-2k = 1 + 3j$ for some nonnegative integer j , so $k = \frac{n-1-3j}{2}$. Let G be the graph whose complement is $\left(\frac{n-2k-1}{3}\right)C_3 \cup P_3 \cup (k-1)K_2$. By Theorem 2.2.4 (part 3), this is the unique graph in $\mathcal{S}_{n,m}$ with the maximal number of spanning trees, i.e., G is the unique simple graph that is most optimal for values of p sufficiently close to 0. Let H be the graph whose complement is $C_{n-2k} \cup kK_2$. This graph exists since $n-2k \equiv 1 \pmod{3}$, so $n-2k \geq 4$ as $k \neq \frac{n-1}{2}$. By Theorem 2.2.8, $\text{Rel}(H, p) > \text{Rel}(G, p)$ for p sufficiently close to 1, so no most optimal simple graph exists. $\square$

We should note that the above theorem has the condition that $k \neq \frac{n-1}{2}$. It has not been proved that if $k = \frac{n-1}{2}$ most optimal $\mathcal{S}_{n,m}$ -graphs always exist, but calculations show that for $n = 5$ and $n = 7$, most optimal $\mathcal{S}_{n,m}$ -graphs do exist and it is the graph G whose complement is $P_3 \cup ((n-3)/2)K_2$. It can be argued, for general $n \geq 7$, that this graph is most optimal for values of p close to 0, since by Theorem 2.2.4 (part 3), it has the maximum number of spanning trees, but also near 1 by Theorem 2.2.8.

Our final result on the non-existence of most optimal $\mathcal{S}_{n,m}$ -graphs is the following.

Theorem 2.2.11 *For $n \geq 7$, such that $n \equiv 1 \pmod{3}$ and $m = \binom{n}{2} - (n - 1)$, there is no most optimal $\mathcal{S}_{n,m}$ -graph.*

Proof. Let $n \geq 7$, $n \equiv 1 \pmod{3}$ and $m = \binom{n}{2} - (n - 1)$. Let G be the graph whose complement is $(\frac{n-1}{3} - 1)C_3 \cup P_4$. By Theorem 2.2.4 (part 5) this is the unique graph in $\mathcal{S}_{n,m}$ with the maximal number of spanning trees and hence is the unique simple graph that is most optimal for values of p sufficiently close to 0. Let H be the graph whose complement is $C_{n-2} \cup K_2$. Then by Theorem 2.2.8(3) this implies that $\text{Rel}(G, p) < \text{Rel}(H, p)$ for values of p sufficiently close to 1, and therefore we do not have a most optimal $\mathcal{S}_{n,m}$ -graph. $\square$

So for a given $n \geq 8$ there is more than one m such that a most optimal $\mathcal{S}_{n,m}$ -graph does not exist. It would be satisfying to be able to say that, given n , the only m for which a most optimal $\mathcal{S}_{n,m}$ -graph does not exist is when m satisfies the conditions of one of the previous theorems, but this is not the case. For example, when $n = 8$ and $m = 19$, calculations show that we do not have a most optimal $\mathcal{S}_{n,m}$ -graph, and this pair of n and m does not fall into one of the cases considered above.

2.3 The Nonexistence of Most Optimal $\mathcal{G}_{n,m}$ -Graphs

Though for a given n there are m such that no most optimal $\mathcal{S}_{n,m}$ -graph exists, one may think that extending the family of simple graphs to include graphs with multiple edges may then introduce a most optimal graph.

If considering when $m = \binom{n}{2}$, we can show that if a most optimal graph exists, then it is a simple graph.

Theorem 2.3.1 *For a graph on n vertices and $m = \binom{n}{2}$ edges, if a most optimal graph exists, then it is the complete graph.*

Proof. Consider the graph K_n . Partition the vertex set of K_n into sets, A and B , $|A| = i \leq |B|$. The number of edges between A and B is $i(n - i)$, since each of the i vertices in A are adjacent to the $n - i$ vertices in B . Let $f(t) = t(n - t) = -t^2 + nt$, the graph of this function is opening downwards with roots at 0 and n and achieving a maximum value at $t = n/2$. For $n/2 \geq t \geq 2$, we find that $f(t) \geq 2n - 4$, so to remove $2n - 5$ or fewer edges disconnects at most 1 vertex.

Let G be a graph on $m = \binom{n}{2}$ edges that is $(n - 1)$ -semiregular. Suppose that G has a multiedge, then there exists at least one pair of vertices with a multiedge of size at least 2 between them, then those vertices can be disconnected by removing the edges not in the multiedge, which would be the removal of at most $2(n - 1 - 2) = 2(n - 3) = 2n - 6$ edges, therefore by Observation 2.2.1, the complete graph is more optimal for values of p near 1 and thus if a most optimal graph exists, then it must be the complete graph. $\square$

In [66], it was shown that for planar graphs with $n = 7$, $m = 14$ the most reliable graph for values of p near 1 was a graph that was not simple, suggesting that it is possible that graphs with multiedges may be more optimal than simple graphs. We will now show that the non-optimality results for simple graphs proved in the previous section can be extended to include graphs with multiple edges, proving for the first time that the conjecture for the existence of optimal $\mathcal{G}_{n,m}$ -graphs fails as well. To do so, we will need the following result of Harada, Sun and Nagamochi regarding a lower bound on the number of cutsets of graphs that may have multiple edges.

Theorem 2.3.2 [42] *Let $\mathbf{M}_i^*(n, m) \subseteq \mathcal{G}_{n,m}$ denote the set of graphs which have the minimum number of cutsets of size i . Set $\alpha = \lfloor \frac{2m}{n} \rfloor$ and $\gamma = \lfloor \frac{2m}{n(n-1)} \rfloor$. Then if $\frac{2m}{3} \geq n \geq 5$, $\alpha \geq 3$ and $\alpha \leq i \leq 2(\alpha - \gamma) - 3$, we have $G \in \mathbf{M}_i^*(n, m)$ if and only if G is α -semiregular and every cutset of size i yields exactly one isolated vertex.*

For the rest of this section, we assume that $m = \binom{n}{2} - (n - k)$ where $k < n/2$, as was true in the previous section; we want to show that for the cases of Theorems 2.2.9, 2.2.10 and 2.2.11 there are no most optimal $\mathcal{G}_{n,m}$ -graphs. We know that if there is one, then it must be at least as reliable as the most reliable simple graph for values of p near 1. The edge connectivity of a most optimal $\mathcal{S}_{n,m}$ -graph in these cases is $n - 3$, so if a most optimal $\mathcal{G}_{n,m}$ -graph, H , exists, it must have edge connectivity at least $n - 3$, and hence minimum degree at least $n - 3$. If the minimum degree were at least $n - 2$, then the number of edges would be at least $n(n - 2)/2 = \binom{n}{2} - n/2$, a contradiction to the restriction of m . Thus both the edge connectivity and minimum degree of a most optimal $\mathcal{G}_{n,m}$ -graph, should it exist, is $n - 3$. In proving that such a most optimal graph does not exist, the following lemma will also be of use.

Lemma 2.3.3 *Let G be a simple graph on $n \geq 6$ vertices, m edges and that is $(n - 3)$ -semiregular with the minimum number of adjacent degree $n - 3$ vertices. Furthermore, let M be a graph on n vertices and m edges, with edge connectivity $n - 3$ that is $(n - 3)$ -semiregular, but with at least one multiple edge between a pair of vertices. Then M is less reliable than G for values of p close to 1.*

Proof. We know from Lemma 2.2.5 that G has edge connectivity $n - 3$ and that any cutset of G with $2n - 8$ edges either disconnects a single vertex or a pair of adjacent degree $n - 3$ vertices. Now consider the graph M , with multiedges, and suppose that it is at least as reliable as G . Recall from Lemma 2.2.6 that M and G have the same number of vertices of each degree, say x of degree $n - 2$ and y of degree $n - 3$. Thus both G and M have $x \binom{m - (n - 2)}{j - (n - 2)} + y \binom{m - (n - 3)}{j - (n - 3)}$ cutsets of size $n - 1 \leq j \leq 2n - 9$ that necessarily isolate a vertex.

Now if X denotes the set of vertices with degree $n - 2$, Y the set of degree $n - 3$ vertices, and u and v denote a pair of vertices with multiple edges between them, then both vertices belong to X as per the following argument. If $u \in X$ and $v \in Y$

then the removal of all the edges incident to u and v , with the exception of the bundle between u and v , yields a cutset of size at most $2n - 9$, not accounted for previously. This contradicts the assumption that M is as reliable as G .

Likewise, if u and v both lie in Y then the cutset alluded to above, having size at most $2n - 10$, produces a contradiction as well. Moreover, the bundle between u and v must consist of two edges because more than two yields a cutset of order at most $2n - 10$. Next, realize that both G and M have that same number of cutsets of size $2n - 8$ that isolate a vertex, namely $x \binom{m-(n-2)}{n-6} + y \binom{m-(n-3)}{n-5}$. Since G has the minimum number a_G of adjacent vertices of degree $n - 3$, the same must be true of M , as the subgraph induced by Y is simple in M and $|Y| = y$ in both G and M . However, the cutset determined by a bundle of two edges between two vertices of degree $n - 2$ (consisting of all the edge incident to u and v , save the two in the bundle) has size $2n - 8$ and is not accounted for in the above count, thereby providing the final contradiction. $\square$

We will use these results to show that for some n and m there do not exist most optimal $\mathcal{G}_{n,m}$ -graphs.

Theorem 2.3.4 *Let $n \geq 6$.*

- *If $m = \binom{n}{2} - (n - k)$ where $2 \leq k < \frac{n}{2}$, there does not exist a most optimal $\mathcal{G}_{n,m}$ -graph provided either (i) $n - 2k \equiv 1 \pmod{3}$ and $k \neq \frac{n-1}{2}$, or (ii) $n - 2k \equiv 2 \pmod{3}$.*
- *If $m = \binom{n}{2} - (n - 1)$, where $n \equiv 1 \pmod{3}$ there does not exist a most optimal $\mathcal{G}_{n,m}$ -graph.*

Proof. First let $m = \binom{n}{2} - (n - k)$ where $2 \leq k < \frac{n}{2}$. Since we will be using Theorem 2.3.2 which requires that $\frac{2m}{3} \geq n \geq 5$, we should note that

$$\frac{2m}{3} - n = \frac{n(n-6)}{3} + \frac{2k}{3} \geq 0$$

for $n \geq 6$. Now we can find α and γ :

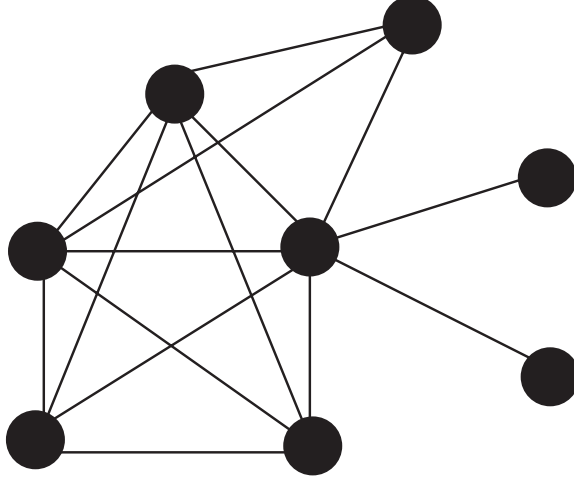
$$\begin{aligned} \alpha &= \left\lfloor \frac{2m}{n} \right\rfloor = \left\lfloor \frac{2(n(n-1) - 2(n-k))}{2n} \right\rfloor \\ &= \left\lfloor \frac{n(n-1) - 2(n-k)}{n} \right\rfloor = \left\lfloor (n-1) - 2 + \frac{2k}{n} \right\rfloor \\ &= \left\lfloor n-3 + \frac{2k}{n} \right\rfloor = n-3 \end{aligned}$$

and

$$\gamma = \left\lfloor \frac{2m}{n(n-1)} \right\rfloor = \left\lfloor \frac{m}{\binom{n}{2}} \right\rfloor = 0$$

Also, if $m = \binom{n}{2} - (n-1)$ it is not hard to verify that we get $\alpha = n-3$ and $\gamma = 0$ as well.

Suppose that M were a most optimal $\mathcal{G}_{n,m}$ -graph. It must then be optimal for all p close to 1, and hence, as we have seen, must have edge connectivity $n-3$. It must also minimize the number of cutsets of size $n-3$, and by Theorem 2.3.2, M must be $(n-3)$ -semiregular. From Lemma 2.3.3 it follows that M must be a simple graph. But we know from Theorems 2.2.9, 2.2.10 and 2.2.11 that M cannot be a most optimal $\mathcal{S}_{n,m}$ -graph, and hence not a most optimal $\mathcal{G}_{n,m}$ -graph. Thus there are no most optimal $\mathcal{G}_{n,m}$ -graphs in these cases as well. $\square$

Figure 2.4: $L_{8,13}$

2.4 Existence of Least Optimal $\mathcal{G}_{n,m}$ -Graphs

The results of the previous section provided examples of n and m for which a most optimal $\mathcal{G}_{n,m}$ -graph need not exist. At the other end of spectrum, one could ask whether, for each n and $m \geq n - 1$, *least* optimal graphs exist (a connected graph in $\mathcal{G}_{n,m}$ is a least optimal graph if for any other connected $\mathcal{G}_{n,m}$ -graph H we have that $\text{Rel}(G, p) \leq \text{Rel}(H, p)$ for all $p \in [0, 1]$). Previous focus has been exclusively on least optimal *simple* graphs. It was conjectured by Boesch et al. in [15] that a least optimal simple graph has the maximum possible number of 2-connected components, and Bogdanowicz [17] proved that the simple graph with the least number of spanning trees is $L_{n,m}$, which consists of an $(n - k)$ -clique, joined to $k - 1$ leaves plus one other vertex of degree $m - \binom{n-k}{2} - (k - 1)$, where k is the least integer such that $m \geq \frac{(n-k)(n-k-1)}{2} + k$. So, if there is a least optimal simple graph on n vertices and m edges, then $L_{n,m}$ is a candidate (see Figure 2.4). It is yet unknown as to whether $L_{n,m}$ is a least optimal simple graph. In contrast, we shall show that for all n and $m \geq n - 1$, there is indeed always a unique graph that is least optimal. To show that such a graph exists, we need the following lemma.

Lemma 2.4.1 [21] *Let M be the cographic matroid. If M has exactly r coloops, then the last r terms in the H -vector are 0 and all other terms are positive (integers).*

As the facets of the cographic matroid of any graph G are the complements of spanning trees, and an edge is in a spanning tree if and only if it is not a loop, we see that for a loopless graph G , the H -vector of its cographic matroid has all positive integer terms. As we are assuming that our graphs are loopless, we are now ready to determine the graphs with the minimum reliability.

Theorem 2.4.2 *Given n vertices and m edges, the family of graphs $\mathcal{T}_{n,m}$ which arise from tree bundles of order n , with one edge replaced by a bundle of $m - n + 2$ edges, are the least optimal $\mathcal{G}_{n,m}$ -graphs. Moreover, this family of graphs is unique.*

Proof. Let $T \in \mathcal{T}_{n,m}$. We have that

$$\begin{aligned} \text{Rel}(T, p) &= p^{n-2}(1 - (1 - p)^{m-n+2}) \\ &= p^{n-1}(1 + (1 - p) + (1 - p)^2 + \dots + (1 - p)^{m-n+1}) \end{aligned}$$

It follows that the H -form of the all terminal reliability polynomial for any $T \in \mathcal{T}_{n,m}$ has the H -vector $\langle 1, 1, \dots, 1 \rangle$, this is true as $\{1, (1-p), (1-p)^2, \dots, (1-p)^{m-n+1}\}$ is a basis for the polynomials of degree $m - n + 1$, so $\text{Rel}(T, p)/p^{n-1}$ is uniquely expressible as a polynomial in $1 - p$.

Let $G \in \mathcal{G}_{n,m}$. As G has no loops, the H -vector of its cographic matroid, $\langle H_0, H_1, \dots, H_{m-n+1} \rangle$, satisfies $H_i \geq 1$ for all $0 \leq i \leq m - n + 1$. Thus we have

$$\begin{aligned} \text{Rel}(G, p) &= p^{n-1} \sum_{i=0}^{m-n+1} H_i (1 - p)^i \\ &\geq p^{n-1} \sum_{i=0}^{m-n+1} (1 - p)^i \\ &= \text{Rel}(T, p) \end{aligned}$$

and hence T is a least optimal $\mathcal{G}_{n,m}$ -graph. Equality holds if and only if all $H_i = 1$.

To show the uniqueness of this family of graphs, let G be a graph with n vertices and m edges such that for the H -vector of the cographic matroid of G , $H_i = 1$ for $i = 0, \dots, m - n + 1$. We know that

$$H_i = \sum_{j=0}^i (-1)^{i-j} \binom{m-n+1-j}{m-n+1-i} F_j.$$

As $F_0 = 1$, we find that

$$1 = H_1 = (-1)^1 \binom{m-n+1}{m-n} F_0 + (-1)^0 \binom{m-n}{m-n} F_1 = -(m-n+1) + F_1,$$

therefore $F_1 = m - n + 2$. So there are precisely $m - n + 2$ ways to remove one edge and leave a spanning connected subgraph, which implies that the graph G has $n - 2$ bridges. As G is connected, it contains a spanning tree T , so G is T plus $m - n + 2$ other edges (none of which are bridges). If G contained a cycle of size at least 3 then we would have at most $n - 3$ bridges, and if the graph had more than one bundle of multiple edges of size larger than 1, then again we would have fewer than $n - 2$ bridges. It follows that G must arise from T by replacing one edge by a bundle of $m - n + 2$ edges. $\square$

So when considering graphs, there always exists a least optimal graph. But if the focus is most optimal graphs, for a given n there are positive integers m such that a most optimal graph does not exist. In situations when a most optimal graph does not exist, if it is known that the edge probabilities are close to 0, then we can look for the graph with the most spanning trees. If we know that the edge probabilities are near 1, we can look for graphs with the largest edge connectivity. If the edge probability is $p = \frac{1}{2}$ then we are interested in the graph with the highest number of spanning connected subgraphs.

Observation 2.4.3 *The graph which is most optimal for $p = \frac{1}{2}$ is the graph with the highest number of spanning connected subgraphs.*

Proof. Let G be a graph and consider the F -form of the all terminal reliability polynomial. Then

$$\begin{aligned}
 \text{Rel}(G, 1/2) &= \sum_{i=0}^{m-n+1} F_i(1/2)^{m-i} (1 - 1/2)^i \\
 &= \sum_{i=0}^{m-n+1} F_i(1/2)^{m-i} (1/2)^i \\
 &= \sum_{i=0}^{m-n+1} F_i(1/2)^m \\
 &= (1/2)^m \sum_{i=0}^{m-n+1} F_i.
 \end{aligned}$$

Now, $\sum_{i=0}^{m-n+1} F_i$ counts the number of spanning connected subgraphs of G . Since $(1/2)^m$ is a constant it follows that the graph that has the highest number of spanning connected subgraphs is most optimal for $p = \frac{1}{2}$. $\square$

If a most optimal graph exists, then it is best for all values of $p \in [0, 1]$. If we do not know if a most optimal graph exists, we could look for graphs that are most optimal for specific values of p , such as near 0, 1 or at $1/2$, as we will always have a most optimal graph in these situations, but these are a local definition of optimality. It may be of interest to investigate a new global definition of optimality to compare to the traditional notion.

Chapter 3

Analytic Properties of All Terminal Reliability Polynomials

At their most basic level, reliability polynomials are functions and an aspect of reliability polynomials that has been studied, but still has many questions left to be answered, is that of their analytic properties. With regards to the behavior of reliability polynomials on the interval $[0, 1]$, they are increasing functions and are S -shaped, meaning if they cross $y = p$, they do so exactly once [65]. It is also known [61] that graphs with large edge connectivity go from close to 0 to close to 1 over a small interval. Inflection points of reliability polynomials have also been studied, they may or may not occur in $(0, 1)$ and it has been shown that reliability polynomials can have more than one inflection point [27]. The roots of reliability polynomials have also been studied and are known to be dense in $|z - 1| \leq 1$, [21], but this disk is not their closure as roots can be found slightly outside this disk [70].

In this section, we will look at thresholds, fixed points, inflection points and the roots of reliability polynomials. We will also investigate another analytic property which has not been well studied, the integral of the reliability polynomial over $[0, 1]$, as it is the average value of the polynomial over $[0, 1]$.

We will begin by looking at thresholds, as many of the results in this section will be used throughout the chapter. We will be looking at the reliability polynomial in the variable p and in q , where $q = 1 - p$. Throughout this chapter, we will use the following notation. For a graph G , $\text{Rel}_p(G, p)$ will denote the reliability polynomial in the variable p . If we are considering the reliability polynomial in the variable q , we will denote this with $\text{Rel}_q(G, q)$. This means that $\text{Rel}_p(G, p) = \text{Rel}_q(G, 1 - p)$,

and $\text{Rel}_q(G, q) = \text{Rel}_p(G, 1 - q)$. If it is clear that we are looking at the polynomial in the variable p , then we will use $\text{Rel}(G, p)$ or $\text{Rel}(G)$. If looking at the reliability polynomial in both p and q , we have

$$\text{Rel}_{p,q}(G, p, q) = \sum_{i=0}^m F_i p^{m-i} q^i.$$

3.1 Thresholds

Suppose you are in charge of a network and you are wondering whether money should be spent to slightly increase the reliability of the existing connections. If you know there is a value, p_0 such that for values of $p < p_0$ your network is highly unreliable, but for values of $p > p_0$ your network is highly reliable, then investing a little money to improve the reliability of the connections in your network could go a long way. In this section we will look at some families of graphs called t -threshold families of graphs and introduce a new such family of graphs. These t -threshold families of graphs will reappear throughout this chapter in order to provide results about other analytic properties.

3.1.1 Thresholds for Families of Graphs

Margulis [61] looked at families of graphs with large edge connectivity, and how the reliability goes from close to 0 to close to 1 over a small interval. One question that can be asked is where can this jump occur? For example, Figure 3.1 shows a plot of the reliability polynomials for cycles and as n increases, the interval where the polynomial goes from close to 0 to close to 1 decreases, and moves toward 1. This suggests that cycles have a *threshold* at $p = 1$.

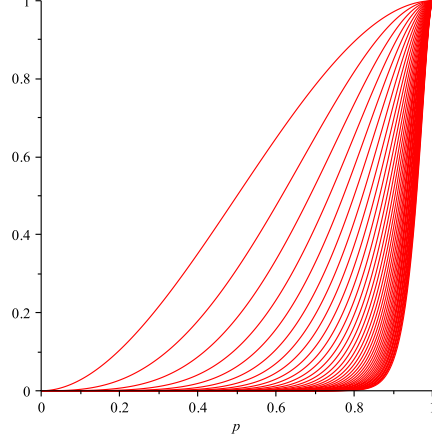


Figure 3.1: Plot of reliability polynomials for cycles on 3 to 40 vertices

Definition 3.1.1 Let $\mathcal{G} = \{G_n : n \geq 1\}$ be a family of connected graphs such that

$$\lim_{n \rightarrow \infty} \text{Rel}(G_n, p) = \begin{cases} 0 & \text{if } 0 \leq p < t \\ 1 & \text{if } t < p \leq 1. \end{cases}$$

Then we call $\mathcal{G}$ a t -threshold family of graphs.

We remark that if we are considering the reliability polynomial in the variable q , then t is a threshold if

$$\lim_{n \rightarrow \infty} \text{Rel}_q(G_n, q) = \begin{cases} 1 & \text{if } 0 \leq q < t \\ 0 & \text{if } t < q \leq 1. \end{cases}$$

We will now look at a few examples of families of graphs and the location of their thresholds. These results will be used in later sections. Kelmans [53] showed that both the family of complete bipartite graphs, $\{K_{n_1, n_2} \mid n_2 = ca^{n_1}, c > 0, a > 1\}$ and the family of tree bundles, $\{T_n^{\log a(n)} \mid n \geq 1, a > 1\}$ are a $(1 - 1/a)$ -threshold family of graphs. We consider the family of complete bipartite graphs, $\{K_{n, n} \mid n \geq 2\}$ and show they are a 0-threshold family of graphs. Since no explicit formulas for the

reliability of complete or complete bipartite graphs are known, we will bound the polynomial from below and show that the lower bound has a threshold.

Proposition 3.1.2 *We have that*

$$\lim_{n \rightarrow \infty} \text{Rel}(K_{n,n}, p) = \begin{cases} 0 & \text{if } p = 0 \\ 1 & \text{if } p \in (0, 1], \end{cases}$$

so $\mathcal{G} = \{K_{n,n} : n \geq 2\}$ is a 0-threshold family of graphs.

Proof. Consider the graph $K_{n,n}$ with bipartitions (A, B) . Let E_1 denote the event that any pair of vertices in A or in B are connected by a path of length 2. First we show that if E_1 holds, then G is connected.

Suppose that E_1 holds, but G is not connected. For any pair of vertices, $x, y \in A$, x and y are in the same component since they are connected by a path of length 2 (similarly for the vertices of B). Now let $u \in A$ and $v \in B$ be in different components. Since E_1 holds, for any other vertex, z in B , there is some vertex $w \in A$ such that v and z are connected by a path of length 2, which contains w , so v is adjacent to w and since any pair of vertices in A are in the same connected component, then u and w are in the same connected component, and hence v and u are in the same connected component, which is a contradiction.

Note that $P(\overline{E_1}) \leq 2\binom{n}{2}(1-p^2)^n$ since if E_1 fails, there is a pair of vertices, $x, y \in A$ (or in B) such that there is no path of length 2 with x and y as end points. The number of ways to pick a pair of vertices, x, y in A (or in B) is $2\binom{n}{2}$. The probability that the other n vertices in the other partition are not incident to both x and y is $1 - p^2$. It follows that for $p \in (0, 1]$, $\text{Rel}(K_{n,n}, p) \geq P(E_1) = 1 - P(\overline{E_1}) \geq 1 - 2\binom{n}{2}(1 - p^2)^n$.

As n approaches infinity, $2\binom{n}{2}(1-p^2)^n$ approaches 0, so $P(\overline{E_1})$ approaches 0.

Thus,

$$\lim_{n \rightarrow \infty} \text{Rel}(K_{n,n}, p) = \begin{cases} 0 & \text{if } p = 0 \\ 1 & \text{if } p \in (0, 1], \end{cases}$$

so the family $\{K_{n,n} : n \geq 2\}$ is a 0-threshold family of graphs. $\square$

In fact, complete graphs also have a threshold approaching 0 since the reliability polynomial approaches the same step function as that of $K_{n,n}$. Before we look at that example, we will prove a lemma which will be referenced various times throughout this chapter.

Lemma 3.1.3 $\text{Rel}(K_n, p) \geq 1 - n^2(1-p^2)^{n-2}$.

Proof. For a graph G , $\text{Rel}(G, p) = 1 - NC(G)$ where $NC(G)$ is the probability that the operational subgraph G is not spanning connected. If every pair of vertices in a graph are connected by a path of length 2, then the graph is connected. Hence, if a graph is not connected, then it is the case that there are a pair of vertices, x, y , not joined by a path of length 2.

Consider the graph K_n . The probability that at least one pair of vertices are not connected is at most $\binom{n}{2}(1-p^2)^{n-2}$, so this gives us that $NC(G) \leq \binom{n}{2}(1-p^2)^{n-2}$, and it follows that $\text{Rel}(K_n, p) \geq 1 - \binom{n}{2}(1-p^2)^{n-2} > 1 - n^2(1-p^2)^{n-2}$. $\square$

Proposition 3.1.4 *We have that*

$$\lim_{n \rightarrow \infty} \text{Rel}(K_n, p) = \begin{cases} 0 & \text{if } p = 0 \\ 1 & \text{if } p \in (0, 1] \end{cases}$$

so the family of complete graphs is 0-threshold family of graphs.

Proof. Let $p \in (0, 1]$. By Lemma 3.1.3 we know that $\text{Rel}(K_n, p) \geq 1 - n^2(1 - p^2)^{n-2}$. As n approaches infinity $1 - n^2(1 - p^2)^{n-2}$ approaches 1, so

$$\lim_{n \rightarrow \infty} \text{Rel}(K_n, p) = \begin{cases} 0 & \text{if } p = 0 \\ 1 & \text{if } p \in (0, 1] \end{cases}$$

and thus we have that complete graphs are a 0-threshold family of graphs. $\square$

Of course, not all families of graphs are 0-threshold families. There are also families of graphs whose thresholds approach 1. As Figure 3.1 suggests, cycles have a threshold approaching 1, and we will now prove this to be true.

Proposition 3.1.5 *The family $\mathcal{G} = \{C_n : n \geq 3\}$ is a 1-threshold family of graphs.*

Proof. We know that $\text{Rel}(C_n, p) = (1 - n)p^n + np^{n-1}$, and when $p = 1$, the reliability is 1. For $p \in [0, 1)$, $\lim_{n \rightarrow \infty} \text{Rel}(C_n, p) = 0$, so

$$\lim_{n \rightarrow \infty} \text{Rel}(C_n, p) = \begin{cases} 0 & \text{if } p \in [0, 1) \\ 1 & \text{if } p = 1 \end{cases}$$

which means that cycles are a 1-threshold family of graphs. $\square$

The families we have looked at so far are simple graphs, but what can be said about graphs with multiple edges? If a family of graphs $\mathcal{G} = \{G_n : n \geq 1\}$ has a threshold at $t \in [0, 1]$, then replacing the edges of each graph, $G_n \in \mathcal{G}$ with a bundle of k edges, to obtain a new family of graphs $\mathcal{G}^k$, consisting of graphs G_n^k pulls the threshold toward 0. This is because for $G_n \in \mathcal{G}$, we have that

$\text{Rel}(G_n^k, p) = \text{Rel}(G_n, 1 - (1 - p)^k)$ and $t = 1 - (1 - p)^k$ when $p = 1 - (1 - t)^{1/k}$, thus

$$\lim_{n \rightarrow \infty} \text{Rel}(G_n^k, p) = \lim_{n \rightarrow \infty} \text{Rel}(G_n, 1 - (1 - p)^k)$$

$$= \begin{cases} 0 & \text{if } 0 \leq p < 1 - (1 - t)^{1/k} \\ 1 & \text{if } 1 - (1 - t)^{1/k} < p \leq 1. \end{cases}$$

For fixed $t \in (0, 1)$ the limit of $1 - (1 - t)^{1/k}$ is 0 as k goes to infinity, hence $\mathcal{G}^k$ is a 0-threshold family of graphs.

We will now proceed to present a family of graphs, with multiedges, whose thresholds will be shown to be dense in $[0, 1]$. This family of graphs will be used in other sections of this chapter to obtain results regarding other analytic properties of all terminal reliability polynomials.

3.1.2 A New Threshold Family of Graphs

In this section we will look at a family of graphs which turns out to be a threshold family of graphs. As mentioned earlier, Kelmans showed that tree bundles with bundles of size $\log_a(n)$ are a $(1 - \frac{1}{a})$ -threshold family of graphs. Unlike the tree bundles that Kelmans considered, the graphs we will look at are semiregular, have large edge connectivity, and for n and m will be shown to always be more reliable than tree bundles. For a later result, we will find it easier to consider the all terminal reliability polynomial in the variable q . The following lemma, which we state without proof, allows us to switch variables.

Lemma 3.1.6 *Let $\mathcal{G} = \{G_n : n \geq 1\}$ be a family of graphs. Then*

$$\lim_{n \rightarrow \infty} \text{Rel}_q(G_n, q) = \begin{cases} 1 & \text{if } 0 \leq q < q_0 \\ 0 & \text{if } q_0 < q \leq 1 \end{cases}$$

if and only if

$$\lim_{n \rightarrow \infty} \text{Rel}_p(G_n, p) = \begin{cases} 0 & \text{if } 0 \leq p < 1 - q_0 \\ 1 & \text{if } 1 - q_0 < p \leq 1 \end{cases}$$

Let C_n^k be a cycle on n vertices where each edge is replaced with bundles of k edges. This graph has reliability polynomial

$$\text{Rel}_q(C_n^k, q) = \text{Rel}_q(C_n, q^k) = (1 - q^k)^n + n(1 - q^k)^{n-1}q^k,$$

since either at least one edge in each bundle is operational, or exactly one bundle is down and at least one edge in the remaining bundles are operational. A nice property of this family is that we can choose n and k in such a way that they will be a threshold family of graphs.

Theorem 3.1.7 *Fix $m \geq 1$ and $j \geq 1$. Then the family of graphs*

$\mathcal{C}_{m,j} = \{C_{m^k}^{kj} : k \geq 1\}$ *is a t -threshold family of graphs, with $t = 1 - \left(\frac{1}{m}\right)^{1/j}$.*

Proof. Consider $\text{Rel}_q(C_{m^k}^{kj}, q)$.

$$\begin{aligned} \text{Rel}_q(C_{m^k}^{kj}, q) &= m^k(1 - q^{kj})^{m^k-1} + (1 - m^k)(1 - q^{kj})^{m^k} \\ &= (1 - q^{kj})^{m^k-1}(m^k + (1 - m^k)(1 - q^{kj})) \\ &= (1 - q^{kj})^{m^k-1}(1 - q^{kj} + m^k q^{kj}) \\ &= (1 - q^{kj})^{m^k} + m^k q^{kj}(1 - q^{kj})^{m^k-1} \\ &= y_1 + y_2 \end{aligned}$$

where $y_1 = (1 - q^{kj})^{m^k}$ and $y_2 = (mq^j)^k(1 - q^{kj})^{m^k-1}$.

We will first consider what happens to $y_1 + y_2$ for fixed values of $q > (\frac{1}{m})^{1/j}$ as k approaches infinity. To see what happens to y_1 we will look at $\ln(y_1)$. From Taylor's Theorem we have that $\ln(1 - x) = -x + O(x^2)$, so this gives us

$$\ln(y_1) = m^k \ln(1 - q^{kj}) = m^k(-q^{kj} + O(q^{2kj})).$$

Then

$$\lim_{k \rightarrow \infty} m^k(-q^{kj} + O(q^{2kj})) = \lim_{k \rightarrow \infty} (mq^j)^k(-1 + O(q^{kj})) = -\infty.$$

As $q > (\frac{1}{m})^{1/j}$, this implies that $m(q^j) > 1$, so y_1 tends to 0.

To see what happens to y_2 , consider $\ln(y_2)$.

$$\begin{aligned} \ln(y_2) &= k \ln(mq^j) + (m^k - 1) \ln(1 - q^{kj}) \\ &= k(\ln(m) + j \ln(q)) + (m^k - 1) \ln(1 - q^{kj}) \\ &= k(\ln(m) + j \ln(q)) + (m^k - 1)(-q^{kj} + O(q^{2kj})). \end{aligned}$$

Now, by similar reasoning to that of y_1 ,

$$\lim_{k \rightarrow \infty} (k(\ln(m) + j \ln(q)) + (m^k - 1)(-q^{kj} + O(q^{2kj}))) = -\infty,$$

since $q > (\frac{1}{m})^{1/j}$. So for $q > (\frac{1}{m})^{1/j}$ as k approaches infinity, $y_1 + y_2 = \text{Rel}(C_{m^k}^{kj}, q)$ tends to 0.

We will now consider what happens to $y_1 + y_2$ for values of $q < (\frac{1}{m})^{1/j}$ as k approaches infinity. Looking at when $q < (\frac{1}{m})^{1/j}$ we have that $m(q^j) < 1$ and so

$$\lim_{k \rightarrow \infty} m^k(-q^{kj} + O(q^{2kj})) = \lim_{k \rightarrow \infty} (mq^j)^k(-1 + O(q^{kj})) = 0.$$

Since $\ln(y_1)$ tends to 0, y_1 will approach 1. To see what happens to y_2 , as before let us consider $\ln(y_2)$:

$$\lim_{k \rightarrow \infty} (k(\ln(m) + j \ln(q)) + (m^k - 1)(-q^{kj} + O(q^{2kj}))) = -\infty,$$

and since $q < (\frac{1}{m})^{1/j}$, $(m^k - 1)(-q^{kj} + O(q^{2kj}))$ approaches 0 and $k(\ln(m) + j \ln(q))$ approaches $-\infty$. Therefore y_2 tends to 0, so for $q < (\frac{1}{m})^{1/j}$ as k approaches infinity, $y_1 + y_2 = \text{Rel}(C_{m^k}^{kj}, q)$ tends to 1.

We have seen, looking at the reliability polynomial in the variable q that $t = (\frac{1}{m})^{1/j}$ is a threshold for the family of graphs of $\mathcal{C}_{m,j}$, so expressing the reliability polynomials in the variable p , it has a threshold at $p = 1 - \left(\frac{1}{m}\right)^{1/j}$. $\square$

3.1.3 Closure of Thresholds for $\mathcal{C}_{m,j}$

We have seen that our family of graphs has a threshold at $p = 1 - (1/m)^{1/j}$. We will now show that these thresholds are dense in $[0, 1]$.

Theorem 3.1.8 *The set $\{t \in [0, 1] : \mathcal{C}_{m,j} \text{ is a } t\text{-threshold family of graphs}\}$ is dense in $[0, 1]$.*

Proof. Let $r \in (0, 1)$ and $\epsilon > 0$; without loss of generality we may assume $0 < r - \epsilon < r + \epsilon < 1$. We have thresholds at $(\frac{1}{m})^{1/j}$, so if we can find m and j such that $|r - (\frac{1}{m})^{1/j}| < \epsilon$, then the family $\mathcal{C}_{m,j}$ will be a t -threshold family of graphs for $t \in (r - \epsilon, r + \epsilon)$.

We want m and j such that

$$\begin{aligned} r - \epsilon &< \left(\frac{1}{m}\right)^{1/j} < r + \epsilon, \\ \text{that is } (r - \epsilon)^j &< \frac{1}{m} < (r + \epsilon)^j. \end{aligned}$$

So we want positive integers, m and j , such that (for the given r and ϵ) we have

$$\frac{-\ln(m)}{\ln(r - \epsilon)} < j < \frac{-\ln(m)}{\ln(r + \epsilon)}.$$

Since we can pick m and j , the question is, given r and ϵ , for what values of m is there a positive integer in $\left(\frac{-\ln(m)}{\ln(r-\epsilon)}, \frac{-\ln(m)}{\ln(r+\epsilon)}\right)$? The inequality $\frac{-\ln(m)}{\ln(r+\epsilon)} - \frac{-\ln(m)}{\ln(r-\epsilon)} > 1$ would guarantee this.

One can see that $\frac{-\ln(m)}{\ln(r+\epsilon)} - \frac{-\ln(m)}{\ln(r-\epsilon)} > 1$ when $\ln(m) > \frac{\ln(r-\epsilon)\ln(r+\epsilon)}{\ln(r+\epsilon)-\ln(r-\epsilon)}$. Let $S = \frac{\ln(r-\epsilon)\ln(r+\epsilon)}{\ln(r+\epsilon)-\ln(r-\epsilon)}$, then we want to know when $\ln(m) > S$, which occurs when $m > e^S$. So given r and ϵ , if we pick $m > e^S$ then there is a j such that $|r - (\frac{1}{m})^{1/j}| < \epsilon$ and we are done. $\square$

In fact, we can obtain, from our family of graphs, a family of simple graphs whose thresholds are dense in $[0, 1]$. Before we begin, we will describe a particular graph construction, which will be used several times throughout this chapter.

Consider a graph G on n vertices and m edges and another graph, H , which we will call a *gadget*. For H , we specify two distinct vertices, x and y (which may or may not be adjacent). We replace each edge $e = \{z, w\} \in G$ with a copy of H , where x and y identify with z and w . Call this new graph $G[H]$ (see Figure 3.2 and Figure 3.3).

We can find the reliability of $G[H]$ using the reliabilities of G and our gadget H . Let $p_{\text{new}}(H)$ be the probability that H is spanning connected, so $p_{\text{new}}(H) = \text{Rel}(H)$. This is since if the original edge in G is operational, then in $G[H]$, H must be spanning connected.

For $G[H]$ to be spanning connected, if an edge of G is down, this corresponds to H being disconnected in $G[H]$, but every vertex of H needs to be able to reach the other vertices of $G[H]$. This means the vertices on G need to be able to reach x or y , but not both (else H is connected). Let $q_{\text{new}}(H)$ be the probability that each vertex of H can reach x or y , but not both (which is denoted $\text{Rel}_{\{x,y\}}(H)$). This corresponds to the original edge of G being non-operational.

Consider Figure 3.2. We have that $G = P_3$ and our gadget is C_3 . This gives us that $p_{\text{new}}(C_3) = -2p^3 + 3p^2$ and $q_{\text{new}}(C_3) = (1 - p)(2p(1 - p))$.

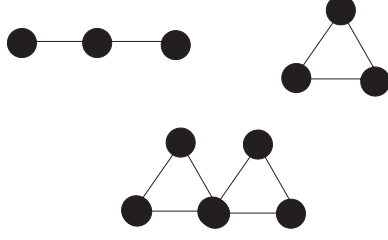


Figure 3.2: P_3 , C_3 and the graph $P_3[C_3]$

For a graph G and gadget H , we can calculate the reliability of $G[H]$ using $p_{\text{new}}(H)$, $q_{\text{new}}(H)$ and $\text{Rel}(G, p)$. The probability that a spanning connected subgraph S of G of size i is operational is $p^i q^{m-i}$. Consider the corresponding subgraph in $G[H]$. If an edge of G , with end points x and y , is operational, then in $G[H]$ this corresponds to H being spanning connected. This occurs with probability $p_{\text{new}}(H)$. If an edge of G is non-operational then in $G[H]$ this corresponds to each vertex of H being able to communicate with x or y , but not both and this occurs with probability $q_{\text{new}}(H)$.

So from an operational spanning subgraph S of G , we obtain a family $\mathcal{S}$ of spanning subgraphs in $G[H]$. Suppose that we remove i edges from G to obtain S . The probability that $\mathcal{S}$ is operational in $G[H]$ is $p_{\text{new}}(H)^{m-i} q_{\text{new}}(H)^i$. This gives us that

$$\text{Rel}_p(G[H], p) = \sum_{i=0}^{m-n+1} F_i(G) p_{\text{new}}(H)^{m-i} q_{\text{new}}(H)^i,$$

where $F_i(G)$ is the i -th term in the F -vector of the cographic matroid of G .

We can use this construction to obtain a threshold family of simple graphs, whose thresholds are dense in $[0, 1]$. Let G be the cycle bundles, $C_{m^k}^{kj}$ and let our gadget be a complete graph, K_n , $n \geq 2$, with a leaf attached. Call this graph G_n . Identify the leaf and a vertex (not adjacent to the leaf) in K_n as the endpoints, x, y of the

edge e in $C_{m^k}^{kj}$, so that we obtain the graph $C_{m^k}^{kj}[G_n]$ (see Figure 3.3). We will let $M = m^k k j$, which is the number of edges in $C_{m^k}^{kj}$.

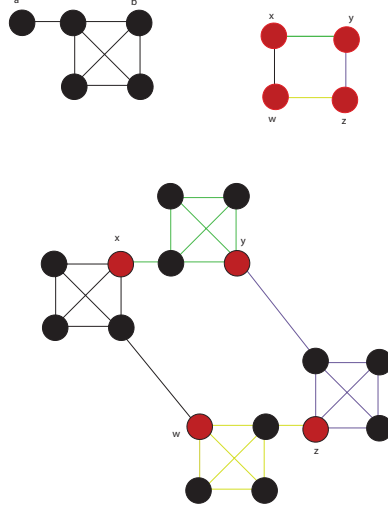


Figure 3.3: G_4 , C_4 and the graph $C_4[G_4]$, where the vertex a identifies with x and b with y .

We have that

$$\text{Rel}_p(C_{m^k}^{kj}[G_n], p) = (p_{\text{new}}(G_n) + q_{\text{new}}(G_n))^M \text{Rel}_p\left(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right).$$

This comes from the fact that

$$\begin{aligned} & \left(\frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right)^{M-i} \left(1 - \left(\frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right)\right)^i \\ &= \left(\frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right)^{M-i} \left(\frac{q_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right)^i \\ &= \frac{p_{\text{new}}(G_n)^{M-i} q_{\text{new}}(G_n)^i}{(p_{\text{new}}(G_n) + q_{\text{new}}(G_n))^M} \end{aligned}$$

so

$$\text{Rel}_p(C_{m^k}^{kj}[G_n], p) = (p_{\text{new}}(G_n) + q_{\text{new}}(G_n))^M \text{Rel}_p\left(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right)$$

$$\begin{aligned}
&= (p_{\text{new}}(G_n) + q_{\text{new}}(G_n))^M \sum_{i=0}^M F_i(C_{m^k}^{kj}) \frac{p_{\text{new}}(G_n)^{M-i} q_{\text{new}}(G_n)^i}{(p_{\text{new}}(G_n) + q_{\text{new}}(G_n))^M} \\
&= \sum_{i=0}^M F_i(C_{m^k}^{kj}) p_{\text{new}}(G_n)^{M-i} q_{\text{new}}(G_n)^i
\end{aligned}$$

as desired.

We have that

$$p_{\text{new}}(G_n) = p(\text{Rel}(K_n, p))$$

and

$$q_{\text{new}}(G_n) = (1 - p)\text{Rel}(K_n, p) + p(\text{Rel}_{x,y}(K_n)),$$

where x is adjacent to the leaf and y is as previously described. From Lemma 3.1.3 we have that

$$0 \leq \text{Rel}_{\{x,y\}}(K_n) \leq n^2(1 - p^2)^{n-2},$$

since we require that K_n be disconnected.

Looking at

$$p_{\text{new}}(G_n) + q_{\text{new}}(G_n) = \text{Rel}(K_n, p) + p(\text{Rel}_{\{x,y\}}(K_n)),$$

we can see that as n approaches infinity, this approaches 1, since complete graphs are a 0-threshold family of graphs and $p(\text{Rel}_{\{x,y\}}(K_n))$ goes to 0. From similar reasoning, we also have that

$$\begin{aligned}
\lim_{n \rightarrow \infty} q_{\text{new}}(G_n) &= \lim_{n \rightarrow \infty} ((1 - p)\text{Rel}(K_n, p) + p(\text{Rel}_{x,y}(K_n))) \\
&= 1 - p = q.
\end{aligned}$$

This gives us that

$$\lim_{n \rightarrow \infty} \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)} = p. \quad (3.1)$$

We will now consider what happens to $(p_{\text{new}}(G_n) + q_{\text{new}}(G_n))^M$, as n approaches infinity. Let $A = (p_{\text{new}}(G_n) + q_{\text{new}}(G_n))^M$ and consider $\ln(A)$.

$$\begin{aligned} \ln(A) &= M \ln(p_{\text{new}}(G_n) + q_{\text{new}}(G_n)) \\ &= M \ln(\text{Rel}(K_n, p) + p \text{Rel}_{x,y}(H)) \\ &\geq M \ln(1 - n^2(1 - p^2)^{n-2}) \\ &= M(-n^2(1 - p^2)^{n-2} + O((-n^2(1 - p^2)^{n-2})^2)) \end{aligned}$$

by Taylor's theorem. Since $M = m^k k j$, we have that as long as n grows at least as fast as M (so pick $n = M$), $M(-n^2(1 - p^2)^{n-2} + O((-M^2(1 - p^2)^{n-2})^2))$ approaches 0 as n approaches infinity, so $\ln(A) \geq 0$ and A approaches 1. This means that as n approaches infinity, $(p_{\text{new}}(G_n) + q_{\text{new}}(G_n))^M = 0$ for $p = 0$ and is 1 for $p \in (0, 1]$.

We know that

$$\text{Rel}_p(C_{m^k}^{kj}[G_n], p) = 0$$

for $p = 0$ and for $p \in (0, 1]$,

$$\text{Rel}_p(C_{m^k}^{kj}[G_n], p) = \text{Rel}_p\left(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right).$$

We will now look at

$$\text{Rel}_p\left(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right)$$

as n approaches infinity. We know that the family of cycle bundles,

$$\mathcal{C}_{m,j} = \{C_{m^k}^{kj} : k \geq 1\}$$

is a t -threshold family of graphs, with $t = 1 - \left(\frac{1}{m}\right)^{1/j}$. So for values of p such that

$$\frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)} < t$$

we have that $\text{Rel}_p(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)})$ approaches 0 and for values of p such that

$$\frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)} > t$$

we have that $\text{Rel}_p(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)})$ approaches 1. This means that

$$\text{Rel}_p(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)})$$

has a threshold at

$$\frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)} = t.$$

Let $p < t$ be such that for some $\epsilon > 0$, $p + \epsilon < t$. Thus, for n large enough,

$$\frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)} < p + \epsilon < t,$$

so by Equation 3.1 this means that

$$\text{Rel}_p\left(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right) < \text{Rel}_p(C_{m^k}^{kj}, p + \epsilon)$$

and $\text{Rel}_p(C_{m^k}^{kj}, p + \epsilon)$ approaches 0, since $p + \epsilon < t$ and $\mathcal{C}_{m,j}$ is a t -threshold family of graphs.

Now, let $p > t$ be such that for some $\epsilon > 0$, $p - \epsilon > t$. Thus, for n large enough,

$$\frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)} > p - \epsilon > t,$$

so by Equation 3.1 this means that

$$\text{Rel}_p\left(C_{m^k}^{kj}, \frac{p_{\text{new}}(G_n)}{p_{\text{new}}(G_n) + q_{\text{new}}(G_n)}\right) > \text{Rel}_p(C_{m^k}^{kj}, p - \epsilon)$$

and $\text{Rel}_p(C_{m^k}^{kj}, p - \epsilon)$ approaches 1, since $p - \epsilon > t$ and $\mathcal{C}_{m,j}$ is a t -threshold family of graphs. Thus, we have t -threshold families of simple graphs,

$$\mathcal{F}_{m,j} = \{C_{m^k}^{kj}[G_n] \mid n = m^k k j, k \geq 1\}$$

whose thresholds are dense in $[0, 1]$.

As mentioned earlier, Kelmans looked at a threshold family of graphs that are tree bundles. Let $T^{k_1, k_2, \dots, k_{n-1}}$ denote a tree where edge e_i is replaced by a bundle of size k_i . The reliability of this family of graphs is

$$\text{Rel}_q(T^{k_1, k_2, \dots, k_{n-1}}, q) = \prod_{i=1}^{n-1} (1 - q^i).$$

We will show that our cycle bundles are uniformly more reliable than tree bundles and hence are better in terms of reliability.

Theorem 3.1.9 *The most optimal cycle bundle is uniformly more reliable than any tree bundle on the same number of edges and vertices.*

Proof. First we will show that a tree whose edges are bundles of size $\lfloor \frac{m}{n} \rfloor$ and $\lceil \frac{m}{n} \rceil$ is most optimal over all other trees. We will do so by showing that if there are a pair of edges whose bundle sizes differ by more than 1, then it is more optimal to shift an edge from the bigger bundle to the smaller bundle.

Consider a tree with a bundle of size a and another bundle of size b , $a > b + 1$. The reliability of this tree is

$$(1 - q^a)(1 - q^b) \prod_{i=3}^{n-1} (1 - q^{k_i}).$$

Consider the tree where an edge from the bundle of size a is moved to the bundle of size b . This has a reliability polynomial of

$$(1 - q^{a-1})(1 - q^{b+1}) \prod_{i=3}^{n-1} (1 - q^{k_i}).$$

We can see that

$$(1 - q^{a-1})(1 - q^{b+1}) \prod_{i=3}^{n-1} (1 - q^{k_i}) > (1 - q^a)(1 - q^b) \prod_{i=3}^{n-1} (1 - q^{k_i})$$

when

$$1 - q^a - q^b + q^{a+b} < 1 - q^{a-1} - q^{b+1} + q^{a+b}.$$

This inequality holds when $q^{a-1} + q^{b+1} < q^a + q^b$, so when $q^b(q - 1) < q^{a-1}(q - 1)$. This holds since $q^{a-1} < q^b$, as $a > b + 1$. So, if there are a pair of bundles whose size differ by more than 1, a more reliable tree can be obtained by shifting an edge from the larger bundle to the smaller one. Thus, the most optimal tree bundle on n vertices and m edges has bundles of size $\lfloor \frac{m}{n} \rfloor$ and $\lceil \frac{m}{n} \rceil$.

Now that we know the most optimal tree bundles, we will show that the cycle bundles are more reliable. Since all tree bundles have the same reliability, for convenience, we will look at path bundles. Consider a path bundle, $P_{n,m}$ ($m \geq n \geq 2$) which is a path with bundles of size $k = \lceil \frac{m}{n} \rceil$ and $k + 1$. Also, consider the cycle bundle, C , obtained from shifting one edge from the path to form a cycle (see Figure 3.4). We will show that $\text{Rel}(P_{n,m}, p) < \text{Rel}(C, p)$ for $p \in (0, 1)$.

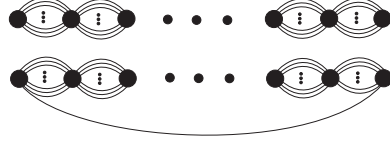


Figure 3.4: The graph $P_{n,m}$ and the cycle bundle C obtained from moving one edge from $P_{n,m}$.

Let e be the edge that we moved to form a cycle. Using the deletion and contraction formula we have that $\text{Rel}(P_{n,m}, p) = (1 - p)\text{Rel}(P_{n,m} - e, p) + p(\text{Rel}(P_{n,m} \cdot e, p))$ and

$$\begin{aligned} \text{Rel}(C, p) &= (1 - p)\text{Rel}(C - e, p) + p(\text{Rel}(C \cdot e, p)) \\ &= (1 - p)\text{Rel}(P_{n,m} - e, p) + p(\text{Rel}(C \cdot e, p)) \end{aligned}$$

and since $P_{n,n} \cdot e$ is a proper subgraph of $C \cdot e$, (ignoring the loops obtained from contracting e in $P_{n,m}$, since they do not affect the reliability) we have that $\text{Rel}(C, p) > \text{Rel}(P_{n,m}, p)$.

As with the paths, if we have a cycle bundle with two bundles of edges, whose sizes differ by more than 1, it is more optimal to shift an edge from the larger bundle to the smaller bundle. We have that a cycle bundle $C_n^{k_1, k_2, \dots, k_n}$ with bundles of size $k_1, k_2, \dots, k_n$ has reliability polynomial

$$\text{Rel}_q(C_n^{k_1, k_2, \dots, k_n}, q) = \prod_{i=1}^n (1 - q^{k_i}) + \sum_{i=1}^n q^{k_i} \prod_{j \neq i} (1 - q^{k_j}).$$

Let G_1 be a cycle bundle with a bundle of size a and another of size b , $a > b + 1$. Let $P(A) = \prod_{i \geq 3} (1 - q^{k_i})$ and let $P(B) = \sum_{i=3}^n q^{k_i} \prod_{j \neq i} (1 - q^{k_j})$. The reliability of G_1 is

$$\begin{aligned} \text{Rel}_q(G_1, q) &= (1 - q^a)(1 - q^b)(P(A) + P(B)) \\ &\quad + q^a(1 - q^b)P(A) + q^b(1 - q^a)P(A), \end{aligned}$$

since for G_1 to be connected, either you can have at least one edge in the bundle of size a operational and at least one edge in the bundle of size b operational, and in the remaining bundles, you can have at least one edge up or you can have exactly one bundle down and the rest with at least one edge operational. You could also have the bundle of size a non-operational and the other bundles have at least one edge up or you can have the bundle of size b non-operational and the other bundles with at least one edge operational.

Let G_2 be the cycle obtained from G_1 by shifting an edge from the bundle of size a to the bundle of size b . The reliability of G_2 is

$$\begin{aligned} \text{Rel}_q(G_2, q) &= (1 - q^{a-1})(1 - q^{b+1})(P(A) + P(B)) \\ &\quad + q^{a-1}(1 - q^{b+1})P(A) + q^{b+1}(1 - q^{a-1})P(A). \end{aligned}$$

Simplifying the reliability polynomial for G_1 , we get that

$$\text{Rel}_q(G_1, q) = P(B)(1 - q^a - q^b + q^{a+b}) + P(A)(1 - q^{a+b})$$

and simplifying the reliability polynomial for G_2 we have that

$$\text{Rel}_q(G_2, q) = P(B)(1 - q^{a-1} - q^{b+1} + q^{a+b}) + P(A)(1 - q^{a+b}).$$

From this, we can see that $\text{Rel}(G_2, q) > \text{Rel}(G_1, q)$ when $q^a + q^b > q^{a-1} + q^{b+1}$, which is true since $a > b + 1$. So we have that the best tree bundles are uniformly less reliable than the most reliable cycle bundle. $\square$

We will return to this family of graphs, the cycle bundles, later in this chapter.

3.2 Internal Fixed Points

Threshold families of graphs, like those discussed in the previous section, go from close to 0 to close to 1, so it should be clear that these reliability polynomials would have a fixed point at a value other than 0 and 1. Fixed points for reliability polynomials have been studied by Shannon and Moore [65] and by Birnbaum, Esary and Saunders [11]. They both show that if a reliability polynomial has a fixed point, p_0 in $(0, 1)$ then it is *S-shaped*, meaning $\text{Rel}(G, p) < p$ for $p \in [0, p_0)$ and $\text{Rel}(G, p) > p$ for $p \in (p_0, 1]$. This means that for $p < p_0$ the graph as a whole is less reliable than a single edge, and for $p > p_0$ the graph as a whole is more reliable than a single edge. Knowing when and if these polynomials cross $y = p$ provides further insight as to the behaviour of the reliability polynomial on $[0, 1]$. So we turn our attention in this section to the fixed points of the reliability polynomial. To begin, we need the following definitions.

Definition 3.2.1 [34] *The fixed point of a function $f(x)$ is a point x_0 for which $f(x_0) = x_0$. Let $f^{\circ k}(x) = f^{\circ k-1}(f(x))$, with $f^{\circ 1}(x) = f(x)$. A fixed point x_0 is said to be an attractive fixed point if there is a neighborhood D of x_0 such that if $z \in D$, then $f^{\circ n}(z) \rightarrow x_0$ for all $n > 0$. A fixed point x_0 is said to be a repelling fixed point if there exists a neighborhood D which contains x_0 and if $z \in D - \{x_0\}$, then there exists $n > 0$ such that $f^{\circ n}(z) \notin D$.*

Fixed points can also be classified as attractive or repelling by looking the derivative of the function at the fixed point [34].

- If $f(x)$ has a fixed point at x_0 and $|f'(x_0)| < 1$, then x_0 is attractive.
- If $|f'(x_0)| > 1$, then x_0 is repelling.
- If $|f'(x_0)| = 1$, then x_0 is called neutral.

When specifically looking at the reliability polynomials, since we are not interested in the fixed points at $p = 0$ and $p = 1$, we will call the fixed point of reliability polynomials in $(0, 1)$ the *internal fixed point* of the graph.

Definition 3.2.2 *The internal fixed point, $ifp(G)$ of a graph G , should it exist, is the unique point, $p_0 \in (0, 1)$ such that $\text{Rel}(G, p_0) = p_0$*

Example 3.2.3 *Consider the graph T_n , a tree on n vertices. We have $\text{Rel}(T_n, p) = p^n$ and $p^n = p$ when $p = 0$ or $p = 1$, so trees do not have an internal fixed point. (See Figure 3.5, the red plot).*

Example 3.2.4 *Consider a cycle on 4 vertices. The reliability polynomial for C_4 is $\text{Rel}(C_4, p) = -3p^4 + 4p^3$. To find when $\text{Rel}(C_4, p) = p$ we need to find the roots of $\text{Rel}(C_4, p) - p = -3p^4 + 4p^3 - p = -p(p-1)(3p^2 - p - 1)$. The roots in $[0, 1]$ for this function are $p = 1, p = 0$ and $p = (1/6)(1 + \sqrt{13})$, so $ifp(C_4) = (1/6)(1 + \sqrt{13})$. (See Figure 3.5, the blue plot)*

Example 3.2.5 *Consider P_2^k , then the all terminal reliability polynomial is $1 - (1 - p)^k$ and this has fixed points only at $p = 0, 1$, so this graph does not have an internal fixed point. (See Figure 3.5, the green plot)*

Recall the N -form of the reliability polynomial, $\sum_{i=0}^m N_i p^i (1 - p)^{m-i}$. Birnbaum, Esary and Saunders [11] proved that for certain classes of coherent systems the reliability polynomial crosses $f(p) = p$ if and only if $N_1 = 0$ and $N_{n-1} = n$. Below is our own proof of this result, specific to all terminal reliability, with some restrictions on the values of n .

Theorem 3.2.6 *Let G be a connected graph of order $n \geq 3$. Then $\text{Rel}(G, p)$ has an internal fixed point if and only if G has no bridges.*

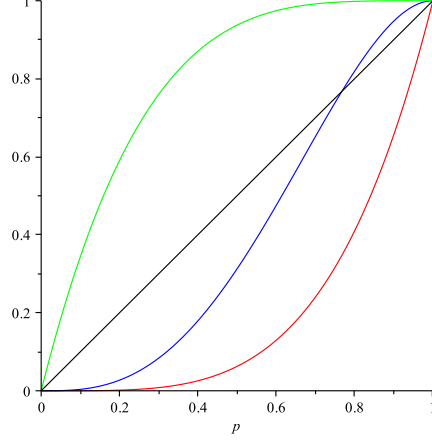


Figure 3.5: Plot of the reliability polynomials for P_4 (red), C_4 (blue), P_2^4 (green) and the line $y = p$.

Proof. Let G be a connected graph of order at least 3. If G has a bridge e then $\text{Rel}_p(G, p) = p(\text{Rel}_p(G \cdot e, p)) < p$ since $0 < \text{Rel}_p(G \cdot e, p) < 1$ for $p \in (0, 1)$, so G does not have an internal fixed point.

Now let $G \in \mathcal{G}_{n,m}$ be a bridgeless graph on $n \geq 3$ vertices. Since G is bridgeless, $N_m = 1$ and $N_{m-1} = m$, so

$$\text{Rel}(G, p) = \sum_{i=n-1}^m N_i p^i (1-p)^{m-i} \geq p^m + mp^{m-1}(1-p).$$

When is $mp^{m-1}(1-p) + p^m > p$ for $p > 0$? This occurs when $mp^{m-2}(1-p) + p^{m-1} > 1$.

Let $f_m(p) = mp^{m-2}(1-p) + p^{m-1} = mp^{m-2} + (1-m)p^{m-1}$. We know that $f_m(1) = 1$ and

$$\begin{aligned} f'_m(p) &= m(m-2)p^{m-3} + (1-m)(m-1)p^{m-2} \\ &= p^{m-3}(m^2 - 2m + (-m^2 + 2m - 1)p) \\ &= p^{m-3}(m^2(1-p) - 2m(1-p) - p) \end{aligned}$$

so this gives us that

$$\lim_{p \rightarrow 1^-} f'_m(p) = -1$$

so as we approach $p = 1$ from the left, we are above the line $y = p$. So for values of $p < 1$ sufficiently close to 1, we are above the line $y = p$.

It is also the case that for values of p sufficiently close to 0 that we are below the line $y = p$. Consider

$$\begin{aligned} \text{Rel}(G, p) &= \sum_{i=n-1}^m N_i p^i (1-p)^{m-i} \\ &\leq \sum_{i=n-1}^m \binom{m}{i} p^i (1-p)^{m-i} \\ &\leq \sum_{i=n-1}^m \binom{m}{i} p^{n-1} \end{aligned}$$

Let

$$S = \sum_{i=n-1}^m \binom{m}{i}.$$

If $Sp^{n-1} < p$ then $\text{Rel}(G, p) < p$, and $Sp^{n-1} < p$ when $p < S^{-\frac{1}{n-2}}$. So for values of $p > 0$ sufficiently close to 0, we have that $\text{Rel}(G, p) < p$. By the continuity of $\text{Rel}(G, p)$, it must cross $y = p$ for some $p \in (0, 1)$, so we have an internal fixed point.

□

We now know when an all terminal reliability polynomial has an internal fixed point, so we can ask questions like, is it a repelling, attractive or neutral fixed point? For families of graphs, $\mathcal{F}_n$, what happens to the internal fixed point as n approaches infinity? And for what $r \in [0, 1]$ can we find graphs with internal fixed points close to r ?

We know at the internal fixed point that the reliability polynomial crosses the line $y = p$, since the reliability polynomial is S -shaped, thus the derivative at $\text{ifp}(G)$ is greater than 1, so it is a repelling fixed point.

One may conjecture that the reliability polynomial's derivative achieves a maximum value at the internal fixed point, that is, the maximum increase in reliability occurs at the internal fixed point, but that is not always the case. For example, consider C_4 , which has reliability polynomial $\text{Rel}(C_4, p) = -3p^4 + 4p^3$. We know that this has an internal fixed point at $\frac{1+\sqrt{13}}{6}$, and the derivative of the reliability polynomial evaluated at the internal fixed point is $\frac{22-2\sqrt{13}}{9} \approx 1.643$. However the derivative attains a maximum value of $16/9 \approx 1.778$ at $p = 2/3$, which is close to, but not equal to the internal fixed point.

3.2.1 Bounding Internal Fixed Points

One question that we would like to answer is, what values can the internal fixed points take on? Before we answer this problem, we will look at bounding the internal fixed point for a graph G . One bound can be obtained if we know something about the proportion of spanning subgraphs. Since $\text{Rel}(G, 1/2) = (1/2^m) \sum (F_i)$, then for a graph on n vertices and m edges has the following property: if the proportion of spanning subgraphs exceeds $1/2$, then the internal fixed point is in the interval $(0, 1/2)$; otherwise it is in $(1/2, 1)$.

We will bound the internal fixed points for a graph by comparing the reliability polynomial with another function that has a known internal fixed point. To use this method, the following observation and lemma will be of use. The first observation will provide some insight as to how the internal fixed point of a graph G behaves in comparison to an upper or lower bound on the reliability polynomial.

Observation 3.2.7 *Let G be a bridgeless graph on $n \geq 3$ vertices and $f(p)$ a function on $[0, 1]$.*

- If $\text{Rel}(G, p) \geq f(p)$ for all $p \in [0, 1]$ and $f(p') > p'$ then the $\text{ifp}(G)$ lies to the left of p' .
- If $\text{Rel}(G, p) \leq f(p)$ for all $p \in [0, 1]$ and $f(p') < p'$ then the $\text{ifp}(G)$ lies to the right of p' .

Proof. Let $f(p)$ be a function on $[0, 1]$. Suppose that for some p' , $f(p') > p'$. If $\text{Rel}(G, p) \geq f(p)$, then $\text{Rel}(G, p') \geq f(p') > p'$, so the internal fixed point of G is to the left of p' .

Similarly, if we suppose that for some p' , $f(p') < p'$. If $\text{Rel}(G, p) \leq f(p)$, then $\text{Rel}(G, p') \leq f(p') < p'$, so the internal fixed point of G is to the right of p' . $\square$

The next lemma provides an upper bound on the reliability polynomial in terms of the number of spanning trees the graph has.

Lemma 3.2.8 *Let G be a graph, $\text{Rel}(G, p) \leq F_{m-n+1}p^{n-1}$.*

Proof. Let T be the set of spanning trees of G . Since for a graph to be connected we need at least one spanning tree operational, using the Inclusion\Exclusion formula, we have that

$$\begin{aligned}
 \text{Rel}(G, p) &= \sum_{t \in T} p^{n-1} - \sum_{1 \leq i < j \leq |T|} p^{|t_i \cup t_j|} \\
 &\quad + \sum_{1 \leq i < j < k \leq |T|} p^{|t_i \cup t_j \cup t_k|} + \dots + (-1)^{|T|-1} p^{|t_1 \cup t_2 \cup \dots \cup t_{|T|}|} \\
 &\leq \sum_{t \in T} p^{n-1},
 \end{aligned}$$

thus, $\text{Rel}(G, p) \leq F_{m-n+1}p^{n-1}$. $\square$

Using the lemma and observation we can find a general lower bound for the internal fixed point.

Theorem 3.2.9 *If G is a simple, bridgeless graph on $n \geq 3$ vertices, then $\text{ifp}(G) \geq 1/n$.*

Proof. For any simple graph of order $n \geq 3$, we know that $\text{Rel}(G, p) \leq \text{Rel}(K_n, p)$. Thus, by Observation 3.2.7, the internal fixed point of G is to the right of the internal fixed point of K_n .

Since $\text{Rel}(K_n, p) < n^{n-2}p^{n-1}$, by Lemma 3.2.8, the internal fixed point of K_n is to the right of that for $n^{n-2}p^{n-1}$. Since $n^{n-2}p^{n-1} = p$ when $(np)^{n-2} = 1$, which occurs when $p = 1/n$, so the internal fixed points of any simple, bridgeless graph on $n \geq 3$ vertices is to the right of $1/n$. $\square$

For a simple graph G we have that $\text{Rel}(G, p) < \text{Rel}(K_n, p)$, so the internal fixed point of G is to the right of the internal fixed point of K_n . Therefore, if the $\text{ifp}(K_n) \rightarrow 1$ as $n \rightarrow \infty$, then by Observation 3.2.7 the internal fixed point of G would tend to 1. This means that if the internal fixed point of K_n tends to 1, then the internal fixed point of any simple graph tends to 1. The following will show that this is not the case, as the internal fixed point of K_n goes to 0, though perhaps slowly.

Theorem 3.2.10 *For n sufficiently large, the internal fixed point, $\text{ifp}(K_n)$, of the complete graph on n vertices is to the left of $\frac{1}{\ln(n)}$.*

Proof. From Lemma 3.1.3 we have that

$$\text{Rel}(K_n, p) > 1 - n^2(1 - p^2)^{n-2}.$$

If we can find a value of p for which $1 - n^2(1 - p^2)^{n-2} - p > 0$, then by Observation 3.2.7, $\text{ifp}(K_n)$ is to the left of this value. Let $p = \frac{1}{\ln(n)}$. We will show that $1 - n^2(1 - \frac{1}{\ln^2(n)})^{n-2} - \frac{1}{\ln(n)} > 0$ for values of n sufficiently large.

Now,

$$\lim_{n \rightarrow \infty} n^2 \left(1 - \frac{1}{\ln^2(n)}\right)^{n-2} = 0$$

so we have that

$$\lim_{n \rightarrow \infty} \left(1 - n^2 \left(1 - \frac{1}{\ln^2(n)}\right)^{n-2} - \frac{1}{\ln(n)}\right) = 1.$$

Thus $1 - n^2(1 - p^2)^{n-2} > p$ for $p = \frac{1}{\ln(n)}$ and n sufficiently large.

□

n	$\text{ifp}(K_n)$	$1/\ln(n)$
3	0.5	0.920
4	0.399	0.721
5	0.335	0.621
6	0.290	0.558
7	0.257	0.514
8	0.231	0.481
9	0.210	0.455
10	0.194	0.434

Table 3.1: The internal fixed points of K_n and $1/\ln(n)$

Table 3.1 shows the internal fixed point for complete graphs on $n \leq 10$ vertices and the upper bound of $1/\ln(n)$. Since the internal fixed points of K_n tend to 0, this does not force the internal fixed points of any simple graph on n vertices to go to 0. But this is no problem, since it is more interesting to look for families of simple graphs whose internal fixed points have limits other than 0. While the internal fixed points of complete graphs tend to 0, we can have families of graphs whose internal fixed points tend to 1.

Theorem 3.2.11 *For cycles C_n we have $1 - \frac{1}{n} < \text{ifp}(C_n) < 1 - \frac{1}{n^2}$, for n sufficiently large.*

Proof. We know that $\text{Rel}(C_n, p) = np^{n-1} + (1-n)p^n$. Let

$$f(n, p) = np^{n-1} + (1-n)p^n - p.$$

We will show first that $f(n, 1 - \frac{1}{n}) < 0$. Note that

$$\begin{aligned} f(n, 1 - 1/n) &= n(1 - 1/n)^{n-1} + (1-n)(1 - 1/n)^n - (1 - 1/n) \\ &= (1 - 1/n)^{n-1}(n + (1-n)(1 - 1/n)) - (1 - 1/n) \\ &= (1 - 1/n)^{n-1}(2 - 1/n) - (1 - 1/n). \end{aligned}$$

Thus $f(n, 1 - 1/n) < 0$ when $(1 - 1/n)^{n-1}(2 - 1/n) < (1 - 1/n)$, which occurs when

$$2 < \frac{1}{n} + \left(\frac{n}{n-1}\right)^{n-2}.$$

Since $(\frac{n}{n-1})^{n-2}$ is an increasing function whose limit is e , we know that for n sufficiently large $2 < \frac{1}{n} + (\frac{n}{n-1})^{n-2}$ and so $f(n, 1 - 1/n) < 0$.

We will now show that $f(n, 1 - 1/n^2) > 0$.

$$\begin{aligned} f(n, 1 - 1/n^2) &= n(1 - 1/n^2)^{n-1} + (1-n)(1 - 1/n^2)^n - (1 - 1/n^2) \\ &= (1 - 1/n^2)^{n-1}(n + (1-n)(1 - 1/n^2)) - (1 - 1/n^2) \\ &= (1 - 1/n^2)^n + (1 - 1/n^2)^{n-1}(1/n) - (1 - 1/n^2) \\ &= (1 - 1/n^2)((1 - 1/n^2)^{n-1} + (1 - 1/n^2)^{n-2}(1/n) - 1) \end{aligned}$$

thus $f(n, 1 - 1/n^2) > 0$ when $(1 - 1/n^2)^{n-1} + (1 - 1/n^2)^{n-2}(1/n) - 1 > 0$. So we need

$$(1 - 1/n^2)^{n-1} + (1 - 1/n^2)^{n-2}(1/n) > 1,$$

which occurs when

$$(1 - 1/n^2)^{n-2}(1 - 1/n^2 + 1/n) > 1$$

Since $(1 - 1/n^2)^{n-2}$ approaches 1 and $1 - 1/n^2 + (1/n) > 1$ for $n \geq 2$, we have that $f(n, 1 - 1/n^2) > 0$ for n sufficiently large. $\square$

Since $1 - 1/n < \text{ifp}(C_n) < 1 - 1/n^2$ for n sufficiently large, $\text{ifp}(C_n)$ approaches 1.

n	$1 - 1/n$	$\text{ifp}(C_n)$	$1 - 1/n^2$
3	0.667	0.50	0.889
4	0.750	0.767	0.938
5	0.800	0.869	0.960
6	0.833	0.916	0.972
7	0.857	0.942	0.979
8	0.875	0.957	0.984
9	0.889	0.967	0.987
10	0.900	0.974	0.990

Table 3.2: The internal fixed point of C_n and upper and lower bounds

Table 3.2 shows the internal fixed point for cycles on $n \leq 10$ vertices and the upper and lower bounds of $1 - 1/n$ and $1 - 1/n^2$. Since we saw in the section on thresholds that cycles were a 1-threshold family of graphs, it makes sense that the internal fixed point should tend to 1. In fact, we can say something about when a family of graphs has an internal fixed point that tends to 1.

Proposition 3.2.12 *Let $\mathcal{G}$ be a family of bridgeless graphs on at least 3 vertices. If*

$$\lim_{n \rightarrow \infty} \left(\frac{1}{F_{m-n+1}} \right)^{1/(n-2)} = 1$$

then the internal fixed point tends to 1.

Proof. By Lemma 3.2.8, we know that for $G \in \mathcal{G}$, $\text{Rel}(G, p) \leq F_{m-n+1}p^{n-1}$. We know that $F_{m-n+1}p^{n-1} = p$ when $p = \left(\frac{1}{F_{m-n+1}} \right)^{1/(n-2)}$, and to the left of this value we

have $F_{m-n+1}p^{n-1} < p$, so by Observation 3.2.7, the internal fixed point of G lies to the right of where $F_{m-n+1}p^{n-1} = p$. So when

$$\lim_{n \rightarrow \infty} \left(\frac{1}{F_{m-n+1}} \right)^{1/(n-2)} = 1$$

then we have that the internal fixed point of G tends to 1. $\square$

If $\ln(F_{m-n+1})$ grows at a slower rate than $n-2$, then $\ln(F_{m-n+1})/(n-2)$ approaches 0, so the family of graphs will have an internal fixed point tending to 1. So for a family of bridgeless graphs on at least 3 vertices, if we know that the number of spanning trees is polynomial in n , then the internal fixed point will approach 1. For example, cycles on n vertices have n spanning trees so the internal fixed points of these graphs approach 1 (see Figure 3.6). Another example is the theta graphs with k paths of length $n_1, n_2, \dots, n_k$, so the number of vertices is $n = 2 + \sum_{i=1}^k (n_i - 1)$. The number of spanning trees for this graph is $\sum_{i=1}^k \prod_{j \neq i} n_j$, which is polynomial in n , so the graphs in this family also have internal fixed points that tend to 1 (see Figure 3.7).

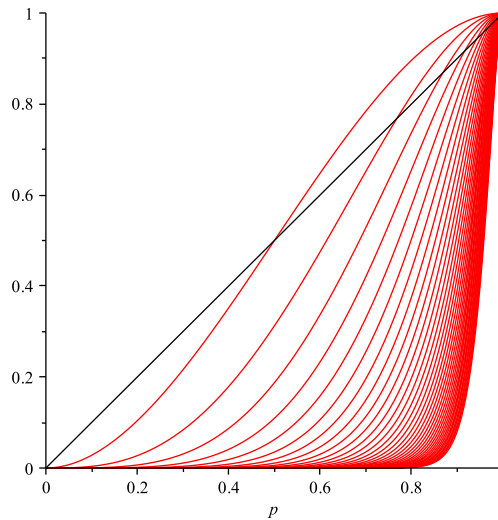


Figure 3.6: Plot of the reliability polynomials for cycles.

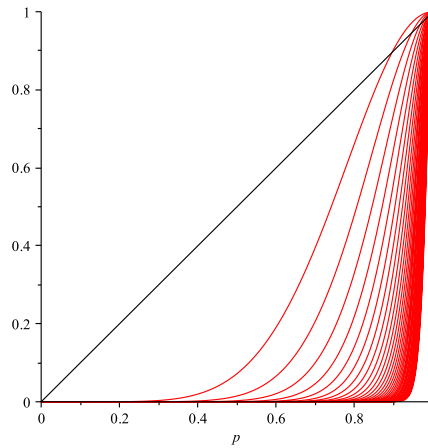


Figure 3.7: Plot of the reliability polynomials for a theta graph with 3 paths of equal size.

3.2.2 Closure of Internal Fixed Points

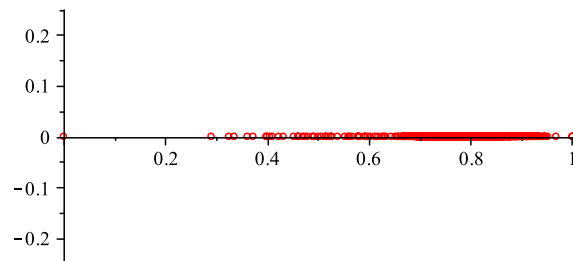


Figure 3.8: Plot of the internal fixed points for simple graphs on $n \leq 8$ vertices.

We have seen families of graphs whose internal fixed points tend to 0 and to 1, but can we have them tend to other values in $(0, 1)$? What is the closure of the values the internal fixed points can take on? We will show it to be $[0, 1]$. Figure 3.8 shows a plot of the internal fixed points for simple graphs on $n \leq 8$ vertices.

We will begin by looking at simple graphs, G with known internal fixed points, $\text{ifp}(G)$ and show that there are graphs with multiedges whose internal fixed points tend to $\text{ifp}(G)$.

Theorem 3.2.13 *For any $r \in (0, 1)$ such that there exists a simple graph G with an internal fixed point at r , there exists a family of graphs with multiple edges whose internal fixed points approach r .*

Proof. For every simple graph, G , with an internal fixed point at r we can find a family of graphs with multiedges, with an internal fixed point approaching r by attaching a leaf via a bundle of k edges to G to obtain the graph H . We can see that

$$\text{Rel}(H, p) = (1 - (1 - p)^k) \text{Rel}(G, p).$$

We know that $\text{Rel}(G, r) = r$, so clearly for $p < r$, $(1 - (1 - p)^k) \text{Rel}(G, p) < p$. Consider $p > r$. Let $\epsilon > 0$, $r + \epsilon < 1$ and let $\alpha > 0$. Then for $p = r + \epsilon$, we have that $\text{Rel}(G, r + \epsilon) > r + \epsilon$, so let $\text{Rel}(G, r + \epsilon) = (r + \epsilon + \alpha)$, thus

$$\begin{aligned} \text{Rel}(H, p) &= (1 - (1 - p)^k) \text{Rel}(G, p) \\ &= (1 - (1 - (r + \epsilon))^k) \text{Rel}(G, r + \epsilon) \\ &= (1 - (1 - (r + \epsilon))^k) (r + \epsilon + \alpha). \end{aligned}$$

We want to show that there is a k such that $\text{Rel}(H, r + \epsilon) > r + \epsilon$. That is, we want to find a k such that $(1 - (1 - (r + \epsilon))^k) (r + \epsilon + \alpha) > r + \epsilon$, and

$$\begin{aligned} (1 - (1 - (r + \epsilon))^k) (r + \epsilon + \alpha) > r + \epsilon &\Leftrightarrow 1 - (1 - (r + \epsilon))^k > \frac{r + \epsilon}{r + \epsilon + \alpha} \\ &\Leftrightarrow (1 - (1 - (r + \epsilon))^k) > 1 - \frac{\alpha}{r + \epsilon + \alpha} \\ &\Leftrightarrow (1 - (r + \epsilon))^k < \frac{\alpha}{r + \epsilon + \alpha}. \end{aligned}$$

k	Internal Fixed Point
10	0.402111
11	0.401118
12	0.400513
13	0.400147
14	0.399925
15	0.399792
16	0.399710
17	0.399662
18	0.399633
19	0.399615
20	0.399605

Table 3.3: The internal fixed point of K_4 with a leaf attached by a bundle of k edges. $\text{ifp}(K_4) \approx 0.399589$

Since we assumed that $r + \epsilon < 1$, for a k sufficiently large, it is the case that

$$(1 - (r + \epsilon))^k < \frac{\alpha}{r + \epsilon + \alpha},$$

so $\text{Rel}(H, p)$ has an internal fixed point approaching r . □

For an example of the above result, look at Table 3.3, which consists of values of the internal fixed point for K_4 with a leaf attached by a bundle of k edges. You can see that when $k = 20$, the internal fixed point of this graph is very close to $\text{ifp}(K_4)$.

So, we can have internal fixed points approaching 0, 1, so the closure of the internal fixed points for all terminal reliability include these values, but what about any $r \in (0, 1)$? We can prove that the closure of the set of internal fixed points for all terminal reliability polynomials is in fact $[0, 1]$. We know from Theorem 3.1.8 that there is a t -threshold family of graphs for any $t \in (0, 1)$, so given any $\epsilon > 0$, there is a graph such that $\text{Rel}(G, t - \epsilon) < t - \epsilon$ and $\text{Rel}(G, t + \epsilon) > t + \epsilon$, thus the internal fixed point of G is found in the interval $(t - \epsilon, t + \epsilon)$. This gives us the following result.

Theorem 3.2.14 *The internal fixed points of all terminal reliability polynomials are dense in $[0, 1]$.*

3.3 Inflection Points

We have just finished looking at the internal fixed points of all terminal reliability polynomials and in that section we saw an example which showed the derivative of the reliability polynomial may not achieve the maximum value at the internal fixed point. So, this then draws our attention to the problem of finding where a reliability polynomial's derivative can achieve a maximum value, which in turn leads to the broader study of inflection points. The number of inflection points in $(0, 1)$ and when an all terminal reliability polynomial has an inflection point has been an active area of study [27, 38, 39]. Brown et al showed in [27] that

Theorem 3.3.1 [27] *If G has edge connectivity $\lambda \geq 2$, then the all terminal reliability polynomial of G is concave down near $p = 1$.*

They also showed that

Theorem 3.3.2 [27] *If G is a connected graph on at least 3 vertices then $Rel(G, p)$ is concave up near $p = 0$.*

With regards to the study of inflection points, we know that if a graph has order at least 3, then when $\lambda \geq 2$, we have an inflection point. What about when $\lambda = 1$? In [27] it was conjectured that the closure of the values that the inflection points of all terminal reliability polynomials can take on is $[0, 1]$. We will provide some results regarding these problems.

Theorem 3.3.3 *Let G be a graph of order at least 3 with edge connectivity, 1. If G has exactly 1 bridge, then it must have an inflection point.*

Proof. Let G be a graph of order at least 3 with exactly one bridge. Then $F_0 = 1, F_1 = m - 1$ and $F_i \leq \binom{m-1}{i}$ for $2 \leq i \leq m - n + 1$, since to remove the bridge disconnects the graph. In [27] it was shown that

$$\text{Rel}''(G, p) = \sum_{i=0}^{m-n+1} D_i p^{m-i-2} (1-p)^i \quad (3.2)$$

where, $D_i = (i+2)(i+1)F_{i+2} - 2(m-i-1)(i+1)F_{i+1} + (m-i)(m-i-1)F_i$, $F_i = 0$ for values of $i > m - n + 1$. We know by Theorem 3.3.2 that the reliability polynomial is concave up near $p = 0$. If we can show that near $p = 1$ it is concave down, then we have an inflection point.

The dominant term of (3.2) for values of p near 1 is

$$(2F_2 - 2(m-1)F_1 + m(m-1)F_0)p^{m-2}$$

and since $F_0 = 1$ and $F_1 = m - 1$, if $F_2 < \binom{m-1}{2}$ then

$$(2F_2 - 2(m-1)F_1 + m(m-1)F_0) < 0$$

therefore the polynomial is concave down near 1. If $F_2 = \binom{m-1}{2}$ then $(2F_2 - 2(m-1)F_1 + m(m-1)F_0)p^{m-2} = 0$ and we will need to look at the first term which has a non-zero coefficient.

Let λ be the edge connectivity of $G \cdot e$, where e is the bridge of G . For $i < \lambda$ we have that $F_i = \binom{m-1}{i}$ since the only way to disconnect the graph would be to remove the bridge and $F_\lambda < \binom{m-1}{\lambda}$. Consider

$$((i+2)(i+1)F_{i+2} - 2(m-i-1)(i+1)F_{i+1} + (m-i)(m-i-1)F_i$$

for $i < \lambda - 2$. We find that

$$(i+2)(i+1)\binom{m-1}{i+2} - 2(m-i-1)(i+1)\binom{m-1}{i+1} + (m-i)(m-i-1)\binom{m-1}{i} = 0.$$

The first non-zero coefficient occurs when $i = \lambda - 2$ and we find that the coefficient is

$$\begin{aligned} & (\lambda)(\lambda-1)F_\lambda - 2(m-\lambda+1)(\lambda-1)F_{\lambda-1} + (m-\lambda+1)(m-\lambda+2)F_{\lambda-2} \\ < & (\lambda)(\lambda-1)\binom{m-1}{\lambda} - 2(m-\lambda+1)(\lambda-1)\binom{m-1}{\lambda-1} \\ & + (m-\lambda+1)(m-\lambda+2)\binom{m-1}{\lambda-2} \\ = & 0. \end{aligned}$$

Therefore the reliability polynomial is concave down for values of p near 1 and thus we have an inflection point. $\square$

In this section we will be looking at the reliability polynomial in terms of q , and since $q = 1 - p$ the following holds.

Lemma 3.3.4 *Let G be a graph, then $Rel_q(G, q)$ has an inflection point at q_0 if and only if $Rel_p(G, p)$ has an inflection point at $p_0 = 1 - q_0$.*

It has been conjectured that all terminal reliability polynomials can have at most one inflection point [32], but this was proven false in [27], as there are infinite families of graphs, both simple and graphs with multiple edges, were found to have 2 inflection points. These families, as well as all simple graphs on 8 or fewer vertices which have 2 inflection points, have exactly 2 bridges. This may lead one to think that if a graph has 2 inflection points, then it has 2 leaves, but for any given number of bridges, $\ell \geq 3$, there exists a graph with multiple edges with at least 2 inflection points.

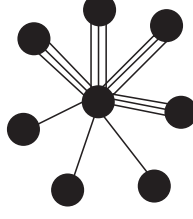


Figure 3.9: Figure for Example 3.3.5

Example 3.3.5 Let $\ell \geq 3$, $k \geq 2$. Consider the graph G whose underlying simple graph is $K_{1,\ell+k}$. The ℓ vertices are leaves and the other k are attached to the central vertex by a bundle of 3 edges (see Figure 3.9). Let $q = 1 - p$. The all terminal reliability of G in q is

$$\text{Rel}_q(G, q) = (1 - q)^\ell (1 - q^3)^k.$$

We show that for a fixed $\ell \geq 3$ we can find a k such that $\text{Rel}_q(G, q)$ has at least 2 inflection points. First we find

$\text{Rel}'_q(G, q) = (-3k)q^2(1 - q)^\ell(1 - q^3)^{k-1} - \ell(1 - q)^{\ell-1}(1 - q^3)^k$ so that we can calculate

$$\begin{aligned} \text{Rel}''_q(G, q) &= \ell(\ell - 1)(1 - q)^{\ell-2}(1 - q^3)^k \\ &\quad + 6kq(1 - q)^{\ell-1}(1 - q^3)^{k-1}(\ell q + q - 1) \\ &\quad + 9k(k - 1)(1 - q)^\ell q^4(1 - q^3)^{k-2} \end{aligned} \tag{3.3}$$

Since we have at least 3 vertices Theorem 3.3.2 says that for values of p near 0, (so values of q near 1) (3.3) is positive. We can also see that for values of q near 0, (3.3) is positive since the dominant term in this case will be $\ell(\ell - 1)(1 - q)^{\ell-2}(1 - q^3)^k$ which is positive. If, for some values of q , (3.3) is negative, then we know we have at least 2 inflection points.

Let $q = \frac{1}{\ell^2}$ and $k = \ell^5$. Then $\text{Rel}''(G, 1/\ell^2) = f(\ell)/g(\ell)$ where

$$f(\ell) = -(6\ell^{10} - 16\ell^9 + \ell^8 - 8\ell^7 + 2\ell^6 - 9\ell^5 + 6\ell^4 - 2\ell^3 + 2\ell^2 - \ell + 1)(\ell^5)(\ell^6 - 1)^{\ell^5}(\ell^2 - 1)^\ell$$

and

$$g(l) = (\ell^6 - 1)^2 \ell^{6\ell^5} \ell^{2\ell}.$$

The denominator, $g(l)$ is always positive since $\ell > 1$ and the numerator, $f(l)$ has real roots at approximately $\ell = 0.673, 1$ and 2.783 . Thus for values of $\ell \geq 3$, with $k = \ell^5$, (3.3) is less than 0, so we have at least 2 inflection points.

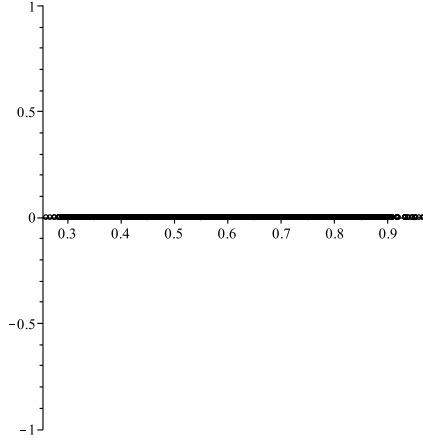


Figure 3.10: Plot of the inflection points for simple graphs on $n \leq 8$ vertices.

As mentioned earlier, another open problem regarding inflection points of all terminal reliability polynomials is their closure [27]. We now proceed to demonstrate that the inflection points of all terminal reliability polynomials are dense in $[0, 1]$. Figure 3.10 is a plot of the inflection points for simple graphs on $n \leq 8$ vertices.

Theorem 3.3.6 *The inflection points of all terminal reliability polynomials are dense in $[0, 1]$.*

Proof. Consider the family of cycle bundles, $\mathcal{C}_{m,j} = \{C_{m^k}^{kj} : m^k \geq 2, kj \geq 1\}$ which we had previously studied. We know that

$$\text{Rel}_q(C_{m^k}^{kj}, q) = (1 - q^{kj})^{m^k} + m^k q^{kj} (1 - q^{kj})^{m^k - 1}$$

Let $a = kj$ and let $b = m^k$. Then

$$\text{Rel}'_q(C_b^a, q) = -ab(b-1)q^{2a-1}(1-q^a)^{b-2},$$

and

$$\text{Rel}''(C_b^a, q) = -ab(b-1)(1-q^a)^{b-3}q^{2a-2}(2a-1+q^a(1-ab)).$$

This gives us real roots for $\text{Rel}''(C_b^a, q)$ at $q = 0, q = 1$ and $q = \pm(\frac{2a-1}{ab-1})^{1/a}$. The roots we are interested in are in the interval $(0, 1)$, so that leaves only $(\frac{2a-1}{ab-1})^{1/a}$. Since the edge connectivity is at least 2 and our graph has at least 3 vertices, by Theorem 3.3.1 and Theorem 3.3.2, we know that this graph has an inflection point, and as $(\frac{2a-1}{ab-1})^{1/a}$ is the only possible choice, it is an inflection point.

Consider

$$\lim_{k \rightarrow \infty} \left(\frac{2a-1}{ab-1} \right)^{1/a} = \lim_{k \rightarrow \infty} \left(\frac{2kj-1}{kjm^k-1} \right)^{1/kj}$$

We will look at $\ln \left(\frac{2kj-1}{kjm^k-1} \right)^{1/kj}$. Since $\ln(x) - 1 < \ln(x-1) < \ln(x)$ for $x > 2$, we have that

$$(1/kj)(\ln(2kj) - 1) < (1/kj) \ln(2kj - 1) < (1/kj) \ln(2kj).$$

As the limit of $(1/kj)(\ln(2kj) - 1)$ and $(1/kj) \ln(2kj)$ as k approaches infinity is 0, we have that $(1/kj) \ln(2kj - 1)$ also approaches 0 as k approaches infinity.

Now looking at $(kjm^k - 1)^{1/kj}$, we have that

$$(1/kj)(\ln(kjm^k) - 1) < (1/kj) \ln(kjm^k - 1) < (1/kj) \ln(kjm^k).$$

Since the limit of $(1/kj)(\ln(kjm^k) - 1)$ and $(1/kj) \ln(kjm^k)$ as k approaches infinity is $\frac{\ln(m)}{j}$, we have that $(1/kj) \ln(kjm^k - 1)$ also approaches $\frac{\ln(m)}{j}$ as k approaches infinity.

So

$$\lim_{k \rightarrow \infty} \ln \left(\frac{2kj - 1}{kjm^k - 1} \right)^{1/kj} = 0 - \frac{\ln(m)}{j}$$

and thus

$$\lim_{k \rightarrow \infty} \left(\frac{2kj - 1}{kjm^k - 1} \right)^{1/kj} = \left(\frac{1}{m} \right)^{1/j}.$$

This shows that as k tends to infinity that the inflection points of the family $\mathcal{C}_{m,j}$ tend to where the threshold of $\mathcal{C}_{m,j}$ occurs. We know the thresholds for these graphs are dense in $[0, 1]$. As the inflection points are dense for $q \in [0, 1]$, by Lemma 3.3.4 they are dense for $p \in [0, 1]$ as well. $\square$

3.4 Average Reliability

Thus far for the reliability polynomial, we have looked at thresholds, internal fixed points, and we just finished looking at inflection points. Another analytic property we could consider is the average value of the reliability polynomial over $[0, 1]$ (the *average value* of a continuous function on an interval $[a, b]$ is the definite integral over $[a, b]$ divided by $b - a$).

Integration of functions has been utilized in the study of reliability polynomials, but the functions being integrated were not the reliability polynomials themselves. In [23] the average value was used as a way to compare upper and lower bounds for a reliability polynomial. The smaller the difference in the average value of the upper and lower bounds, the better the pair of bounds were. In [64] the average value was used as a measure of how well a network had been improved. We will extend this use of the average value of a function to the study of reliability polynomials.

Definition 3.4.1 *The average reliability, $avgRel(G)$, of a graph G is defined as*

$$avgRel(G) = \int_0^1 Rel(G, p) \, dp$$

F_i	Kruskal-Katona	Ball-Provan	Actual
0	1	1	1
1	32	32	32
2	496	496	496
3	4940	4940	4940
4	14 949	30 265	35 342
5	65 778	123 559	192 196
6	230 207	370 230	819 228
7	657 568	8 88 401	2 278 207
8	1 560 738	1 826 687	7 517 243
9	3 118 311	332 480	16 079 317
10	5 286 089	5 406 235	26 517 778
11	7 655 362	76 993 46	32 039 959
12	9 501 848	9 510 882	25 500 420
13	10 122 705	10 122 705	10 122 705

Table 3.4: Table of Lower Bounds for Red Arpa

Example 3.4.2 Consider K_4 . We know that $Rel(K_4, p) = -6p^6 + 24p^5 - 33p^4 + 16p^3$ so the average reliability is $\int_0^1 (-6p^6 + 24p^5 - 33p^4 + 16p^3) dp = 19/35$

Before we begin our study of the average reliability, we will look at an example of how the average value of a function was previously used as a way to compare upper and lower bounds. In [31] a graph depicting a network called *Red Arpa* was considered, it has 32 edges and 20 vertices (see Figure 3.11) and bound the F -vector for that graph using the Ball-Provan bounds and the Kruskal-Katona bounds.

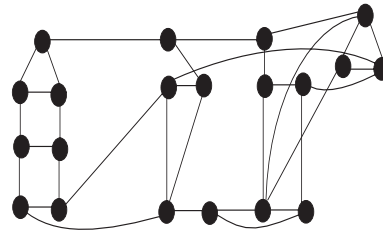


Figure 3.11: Red Arpa

From Table 3.6 we can see that the difference in average values between the Kruskal-Katona bounds is approximately 0.210165 and the difference between the Ball-Provan bounds is approximately 0.138045.

F_i	Kruskal-Katona	Ball-Provan	Actual
0	1	1	1
1	32	32	32
2	496	496	496
3	4940	4940	4940
4	35 570	35 552	35 342
5	197 436	197 103	192 196
6	878 997	875 841	819 228
7	3 233 602	3 203 337	2 278 207
8	9 924 736	9 826 707	7 517 243
9	260 13 120	25 636 590	16 079 317
10	58 659 360	50 268 909	265 17 778
11	114 717 340	61 160 797	32 039 959
12	195 747 826	39 317 998	25 500 420
13	101 22 705	10 122 705	10 122 705

Table 3.5: Table of Upper Bounds for Red Arpa

Bound	Average Value
Actual	0.295516
Kruskal-Katona LB	0.167462
Kruskal-Katona UB	0.377628
Ball-Provan LB	0.200936
Ball-Proval UB	0.338981

Table 3.6: Table of Average Values for the Bounds of Red Arpa

In this section, we will study the average reliability of a graph G , $\text{avgRel}(G)$. The average reliability will have two applications, one is to explore the behaviour of the polynomial on the interval $[0, 1]$ and also representing the reliability polynomial as a single, unique value will provide a new notion of optimality.

3.4.1 Calculating the Average Reliability of a Graph

For any graph G we have $0 \leq \text{avgRel}(G) \leq 1$, since $0 \leq \text{Rel}(G) \leq 1$. For a connected graph G , the average reliability is always positive and it is always rational, since the coefficients of the reliability polynomial are rational. The average reliability also provides a single measure which can be used to compare reliability polynomials for different graphs. If the value of p is unknown, then perhaps the average reliability is ‘the’ measure to use when comparing graphs. The following is an obvious, but important observation.

Observation 3.4.3 *Let H and G be graphs such that $\text{Rel}(G, p) \geq \text{Rel}(H, p)$ for $p \in [0, 1]$. Then $\text{avgRel}(G) \geq \text{avgRel}(H)$.*

This observation and some of its consequences will be expanded on later in this section.

Example 3.4.4 *For trees on n vertices, T_n , we have $\text{Rel}(T_n, p) = p^n$, so the average reliability for T_n is*

$$\int_0^1 p^n dp = \frac{p^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1}$$

In fact, for a given n this is a lower bound on the average reliability of order n and size m , regardless of m . For any other connected graph G of order n , we must have at least a spanning tree operational, so by Observation 3.4.3, $\frac{1}{n+1}$ is a lower bound on the average reliability.

Example 3.4.5 *The average reliability for cycles is*

$$\text{avgRel}(C_n) = \int_0^1 ((1-n)p^n + np^{n-1})dp = \frac{(1-n)p^{n+1}}{n+1} + p^n \Big|_0^1 = \frac{2}{n+1}$$

and since this family of graphs is most $\mathcal{G}_{n,n}$ optimal, by Observation 3.4.3 we know that for any other graph on n vertices and $m = n$ edges, their average reliability will be less than $\frac{2}{n+1}$.

Example 3.4.6 *The all terminal reliability of theta graphs with 3 paths of lengths n_1, n_2, n_3 is $p^N + Np^{N-1}(1-p) + Sp^{N-2}(1-p)^2$ where $N = n_1 + n_2 + n_3$ and $S = n_1n_2 + n_1n_3 + n_2n_3$. This has an indefinite integral of $\frac{p^{N+1}(1-N+S)}{N+1} + \frac{p^N(N-2S)}{N} + \frac{p^{N-1}S}{N-1}$, so the average reliability is $\frac{2N^2-2N+2S}{N(N^2-1)}$. When n_1, n_2, n_3 are such that they differ by at most 1 then this is the most $\mathcal{G}_{n,n+1}$ optimal graph so, again by Observation 3.4.3, for any other graph with $m = n + 1$ this average reliability will be an upper bound.*

We remark that it is possible for more than one graph to have the same average reliability. Of course if there are non-isomorphic graphs which have the same reliability polynomial, such as trees, they will have the same average reliability. But it is possible for two graphs with different reliability polynomials to have the same average reliability. Consider a tree on n_1 vertices and a cycle on $2n_1 + 1$ vertices. The average reliability of the tree is $\frac{1}{n_1+1}$ and the average reliability of the cycle is $\frac{2}{2n_1+2} = \frac{1}{n_1+1}$. More interestingly, we can even find graphs with the same number of vertices and edges whose reliability polynomials differ, but their average reliability is the same. For example, consider the graphs G and H in Figure 3.12. The graph G has reliability polynomial $-15p^8 + 56p^7 - 72p^6 + 32p^5$ and the reliability polynomial for H is $-12p^8 + 48p^7 - 65p^6 + 30p^5$, and $\text{avgRel}(G) = \text{avgRel}(H) = 8/21$.

We can calculate the average reliability of a graph by using the F -form of the reliability polynomial and Bernstein polynomials. A *Bernstein polynomial* is a positive linear combination of Bernstein basis polynomials, which are of the form $\binom{n}{i}x^i(1-x)^{n-i}$,

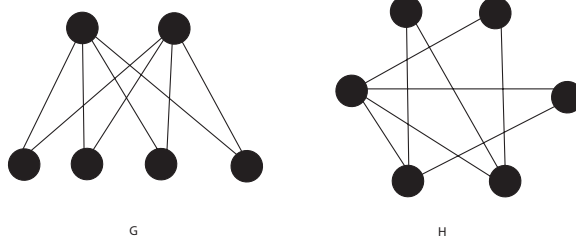


Figure 3.12: Graphs G and H have different reliability polynomials, but the same average reliability.

$0 \leq i \leq n$ [58]. The integral of a Bernstein basis polynomial on the interval $[0, 1]$ is

$$\int_0^1 \binom{n}{i} x^i (1-x)^{n-i} dx = \frac{1}{n+1}.$$

Since the F -form of the all terminal reliability polynomial is

$$\sum_{i=0}^{m-n+1} F_i p^{m-i} (1-p)^i,$$

the all terminal reliability is a positive linear combination of Bernstein basis polynomials,

$$\text{Rel}(G) = \sum_{i=1}^m f_i \binom{m}{i} p^i (1-p)^{m-i}$$

where $f_i = \frac{F_i}{\binom{m}{i}}$. Using this fact, we can find an interesting way of calculating the average reliability of a graph.

Lemma 3.4.7 *Let G be a graph on n vertices and m edges, whose F -form of the all terminal reliability polynomial is given by*

$$\sum_{i=0}^{m-n+1} F_i p^i (1-p)^{m-i}.$$

Then

$$\text{avgRel}(G) = \frac{1}{m+1} \sum_{i=0}^{m-n+1} \frac{F_i}{\binom{m}{i}}.$$

Proof. Let G be a graph of order n with m edges and let $f_i \binom{m}{i} = F_i$, $0 \leq i \leq m - n + 1$. We derive from the formula for the integral of Bernstein polynomials on $[0, 1]$ the following,

$$\begin{aligned} \text{avgRel}(G) &= \int_0^1 \sum_{i=0}^{m-n+1} F_i p^i (1-p)^{m-i} dp \\ &= \sum_{i=0}^{m-n+1} \int_0^1 F_i p^{m-i} (1-p)^i dp \\ &= \sum_{i=0}^{m-n+1} \int_0^1 f_i \binom{m}{i} p^{m-i} (1-p)^i dp \\ &= \sum_{i=0}^{m-n+1} f_i \int_0^1 \binom{m}{i} p^{m-i} (1-p)^i dp \\ &= \sum_{i=0}^{m-n+1} \frac{f_i}{(m+1)} \end{aligned}$$

which proves the result, with $f_i = \frac{F_i}{\binom{m}{i}}$. □

This is a nice formulation of the average reliability since it involves the sum of $\frac{F_i}{\binom{m}{i}}$, which represents what proportion of faces of size i are in the cographic matroid of G .

Example 3.4.8 *The f -vector of the cographic matroid of K_4 is $\langle 1, 6, 15, 16 \rangle$. The total possible number of faces of size i , $0 \leq i \leq 6$ is $\binom{6}{i}$. The proportion of faces of size 0 is $\frac{1}{1} = 1$, size 1 is $\frac{6}{6} = 1$, size 2 is $\frac{15}{15} = 1$ and size 3 is $\frac{16}{20} = \frac{4}{5}$, thus the average reliability will be $\frac{1+1+1+(4/5)}{7} = 19/35$.*

This definition of average reliability will be useful when looking at bounding the average reliability, since known bounds on the F -vector can be utilized.

We have seen some examples of average reliabilities and they seem to take on a variety of values across $[0, 1]$. What can we say about the *average* of the average reliability of all simple graphs of order n ? Perhaps surprisingly, while the values are all over $[0, 1]$, they are concentrated at one end.

Before we begin, we need the following proposition from [33].

Proposition 3.4.9 [33] *Let $\{f_n\}$ be a sequence of integrable functions defined on $[a, b]$. Assume that*

- *for each $x \in [a, b]$ that $f_n(x) \rightarrow f(x)$ (as $n \rightarrow \infty$), where f is integrable on $[a, b]$, and*
- *there exists a constant K such that $|f_n| \leq K$ for all n .*

Then $\int f_n \rightarrow \int f$.

Theorem 3.4.10 *The average of the average reliability of all simple graphs on n vertices approaches 1.*

Proof. Consider the set $\mathcal{S}_{n,m}$ and the reliability polynomials of all these graphs. What is the average of the average reliability of all these polynomials?

Let $\text{Avg}(\text{AvgRel}(n, p))$ denote the average of the average reliability of all the all terminal reliability polynomials of simple graphs on n vertices. Let H and S be spanning connected subgraphs.

$$\begin{aligned}
 \text{Avg}(\text{AvgRel}(n, p)) &= \int_0^1 \frac{1}{2^m} \sum_{H \subseteq K_n} \sum_{S \subseteq H} p^{|S|} (1-p)^{|H|-|S|} dp \\
 &= \frac{1}{2^m} \int_0^1 \sum_{H \subseteq K_n} p^{|H|} \sum_{j=0}^{m-|H|} \binom{m-|H|}{j} (1-p)^j dp
 \end{aligned}$$

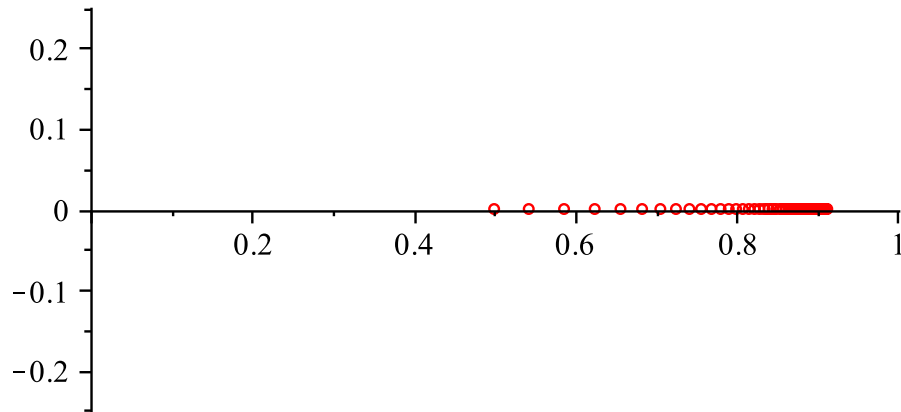


Figure 3.13: The average reliabilities of complete graphs for $n \leq 50$

$$\begin{aligned}
 &= \frac{1}{2^m} \int_0^1 \sum_{H \subseteq K_n} p^{|H|} (2-p)^{m-|H|} \, dp \\
 &= \frac{1}{2^m} \int_0^1 \sum_{H \subseteq K_n} p^{|H|} 2^{m-|H|} (1-p/2)^{m-|H|} \, dp \\
 &= \frac{1}{2^m} \int_0^1 \sum_{H \subseteq K_n} 2^m (p/2)^{|H|} (1-p/2)^{m-|H|} \, dp \\
 &= \int_0^1 \sum_{H \subseteq K_n} (p/2)^{|H|} (1-p/2)^{m-|H|} \, dp \\
 &= \int_0^1 \text{Rel}(K_n, p/2) \, dp
 \end{aligned}$$

Now consider

$$\int_0^1 \text{Rel}(K_n, p/2) \, dp.$$

Letting $t = p/2$ we get that

$$\text{Avg}(\text{AvgRel}(n, t)) = 2 \int_0^{1/2} \text{Rel}(K_n, t) \, dt.$$

Using Proposition [33], since as n approaches infinity we have that

$$\lim_{n \rightarrow \infty} \text{Rel}(K_n, p) = \begin{cases} 0 & \text{if } p = 0 \\ 1 & \text{if } p \in (0, 1] \end{cases}$$

so

$$\text{Avg}(\text{AvgRel}(n, t)) = 2 \int_0^{1/2} \text{Rel}(K_n, t) \, dt = 2(1/2) = 1,$$

which proves the theorem. □

See Figure 3.13 for a plot of the average reliabilities of complete graphs.

3.4.2 Bounds on the Average Reliability

Calculating average reliabilities appears to be difficult, as it seems to require knowledge of the precise reliability polynomial, and calculating the latter is #P-hard. In light of this, there may be situations where we cannot calculate the average reliability explicitly. We will now look at bounding the average reliability of a graph.

We should note that sometimes it is useful to look at the average of the reliability polynomial as a function of q , which proves no problem since

$$\begin{aligned} \int_0^1 \text{Rel}_p(G, p) \, dp &= \int_1^0 \text{Rel}_q(G, q)(-1) \, dq \\ &= - \int_0^1 \text{Rel}_q(G, q)(-1) \, dq \end{aligned}$$

$$= \int_0^1 \text{Rel}_q(G, q) \, dq.$$

So when calculating the average reliability, we can consider the graph in the variable p or q , as convenient and obtain the same results. This fact will be used several times throughout the rest of the section.

We know that for a graph G , that

$$\text{avgRel}(G) = \sum \frac{F_i}{(m+1)\binom{m}{i}},$$

so if we know how the sequence of $\frac{F_i}{\binom{m}{i}}$ behaves, we can bound the average reliability.

Lemma 3.4.11 *Let G be a graph on n vertices, m edges and with F -vector*

$\langle F_0, F_1, \dots, F_{m-n+1} \rangle$. Then, $\frac{F_i}{\binom{m}{i}} \geq \frac{F_{i+1}}{\binom{m}{i+1}}$, for $i = 0, \dots, m-n$. That is, setting $f_i = \frac{F_i}{\binom{m}{i}}$, the sequence

$$\langle f_0, f_1, \dots, f_{m-n+1} \rangle$$

is non-increasing.

Proof. Sperner's well known bound [71] states that

$$(m-i)F_i \geq (i+1)F_{i+1},$$

which implies that

$$\frac{i!(m-i)!F_i}{m!} \geq \frac{(i+1)!(m-i-1)!F_{i+1}}{m!}$$

and this gives us the desired result of

$$f_i = \frac{F_i}{\binom{m}{i}} \geq \frac{F_{i+1}}{\binom{m}{i+1}} = f_{i+1}.$$

□

Using Lemma 3.4.11 and Bernstein polynomials we will look at some bounds for the average reliability of a graph. We had mentioned earlier that the all terminal reliability is a linear combination of Bernstein polynomials, and using this we can obtain the following upper bound on average reliability.

Lemma 3.4.12 *Let G be a graph on n vertices and m edges. Then*

$$\int_0^1 \text{Rel}(G, p) \, dp \leq 1 - \frac{n-1}{m+1},$$

with equality if and only if G is a tree or a cycle.

Proof. Let G be a graph on n vertices and m edges. Noticing that $f_i = \frac{F_i}{\binom{m}{i}} \leq 1$ for all i , we find that

$$\begin{aligned} \int_0^1 \text{Rel}(G, p) \, dp &= \sum_{i=0}^{m-n+1} \int_0^1 F_i p^{m-i} (1-p)^i \, dp \\ &= \sum_{i=0}^{m-n+1} \int_0^1 f_i \binom{m}{i} p^{m-i} (1-p)^i \, dp \\ &\leq \sum_{i=0}^{m-n+1} \int_0^1 \binom{m}{i} p^{m-i} (1-p)^i \, dp \\ &= \sum_{i=0}^{m-n+1} \frac{1}{m+1} \\ &= \frac{m-n+2}{m+1} \\ &= 1 - \frac{n-1}{m+1}. \end{aligned}$$

The bound will be tight if and only if $\frac{F_i}{\binom{m}{i}} = 1$ for all $i = 0 \dots m-n+1$. We know from Lemma 3.4.11 as i increases, the ratio $\frac{F_i}{\binom{m}{i}}$ is non-increasing. Thus, all

the ratios being equal to 1 is equivalent to having $\frac{F_{m-n+1}}{\binom{m}{m-n+1}} = 1$, that is, every subset of $n - 1$ edges induces a spanning tree. This is true for trees and cycles. Suppose $m > n$. Then the resulting graph has girth less than n . In such a graph one can find a set of $n - 1$ edges that is not a spanning tree, since any $(n - 1)$ subset of edges containing the smallest cycle will clearly not be a spanning tree. Thus this bound is only tight for trees and cycles. $\square$

Another set of bounds is the following.

Lemma 3.4.13 *Let G be a graph on n vertices and m edges with edge connectivity λ . Then*

$$\frac{\lambda}{m+1} + \frac{(m-n+2-\lambda)F_{m-n+1}}{(m+1)\binom{m}{m-n+1}} \leq \text{avgRel}(G) \leq \frac{\lambda}{m+1} + \frac{(m-n+2-\lambda)F_\lambda}{(m+1)\binom{m}{\lambda}}$$

and

$$\text{avgRel}(G) \leq \frac{1}{n-1} - \frac{1}{(m+1)\binom{m}{n-2}}.$$

Proof. Let G be a graph on n vertices, m edges and edge connectivity, λ . We will start by proving the first set of bounds. For $i < \lambda$, $F_i = \binom{m}{i}$, so using the fact that

$$\left\langle \frac{F_i}{\binom{m}{i}} : i = 0, \dots, m - n + 1 \right\rangle$$

is non-increasing and letting $f_i = \frac{F_i}{\binom{m}{i}}$, $i = 0 \dots m - n + 1$, we have that $f_i = 1$ for $i \leq \lambda - 1$ and

$$\begin{aligned} \text{avgRel}(G) &= \int_0^1 \sum_{i=0}^{m-n+1} F_i p^{m-i} (1-p)^i \, dp \\ &= \sum_{i=0}^{m-n+1} \int_0^1 F_i p^{m-i} (1-p)^i \, dp \\ &= \sum_{i=0}^{m-n+1} \int_0^1 f_i \binom{m}{i} p^{m-i} (1-p)^i \, dp \end{aligned}$$

n	Lower Bound	avgRel(K_n)	Upper Bound
3	0.5	0.5	0.5
4	0.5428	0.5428	0.5438
5	0.5259	0.5866	0.6299
6	0.4743	0.6245	0.6868
7	0.4135	0.6564	0.7272
8	0.3559	0.6832	0.7586
9	0.3059	0.7058	0.7838
10	0.2643	0.7251	0.8043

Table 3.7: Upper and lower bounds for the average reliability of K_n

$$\begin{aligned}
&= \frac{\lambda}{m+1} + \sum_{i=\lambda}^{m-n+1} \frac{f_i}{m+1} \\
&= \frac{\lambda}{m+1} + \sum_{i=\lambda}^{m-n+1} \frac{F_i}{\binom{m}{i}(m+1)}.
\end{aligned}$$

Since the sequence of $\langle \frac{F_i}{\binom{m}{i}} \mid i = 0 \dots m-n+1 \rangle$, is non-increasing, we have that

$$\frac{\lambda}{m+1} + \frac{(m-n+2-\lambda)F_{m-n+1}}{(m+1)\binom{m}{m-n+1}} \leq \text{avgRel}(G) \leq \frac{\lambda}{m+1} + \frac{(m-n+2-\lambda)F_\lambda}{(m+1)\binom{m}{\lambda}}.$$

To prove the second bound, we will use Observation 3.4.3 and the fact that we know for a given n and m , there exists a least optimal graph and its structure is known. By Theorem 2.7.2, we have a least optimal graph on n vertices and m edges, which consists of a tree with one edge replaced by a bundle of size $m-n+2$. That graph has a reliability polynomial of $p^{n-2}(1-(1-p)^{m-n+2})$, so we have that

$$\int_0^1 p^{n-2}(1-(1-p)^{m-n+2}) \, dp = \frac{1}{n-1} - \frac{1}{(m+1)\binom{m}{n-2}},$$

and this is a lower bound for $\text{avgRel}(G)$. □

The first set of upper and lower bounds can be tight, for graphs where $m - n + 2 - \lambda = 0$, so $\lambda = m - n + 2$. These graphs will have to be such that

$$m \geq \frac{n(m - n + 2)}{2}$$

since each vertex must have degree at least $m - n + 2$. This gives us that $m \leq n$, so for $m = n - 1$ this bound is tight for trees and when $m = n$ we have that $\lambda = 2$, so these bounds are tight for cycles.

The bounds are also tight when

$$\frac{(m - n + 2 - \lambda)F_\lambda}{(m + 1)\binom{m}{\lambda}} = \frac{(m - n + 2 - \lambda)F_{m-n+1}}{(m + 1)\binom{m}{m-n+1}}.$$

This occurs when $\frac{F_\lambda}{\binom{m}{\lambda}} = \frac{F_{m-n+1}}{\binom{m}{m-n+1}}$. If $m - n + 1 = \lambda$ then these bounds are tight; for example, they will be tight for K_4 since $m = 6$, $n = 4$ and $\lambda = 3 = 6 - 4 + 1$. If $m - n + 1 \neq \lambda$, we require that Sperner's bounds are tight for $i = \lambda, \dots, m - n + 1$. This would mean that $F_i = \frac{(m-n+1)F_{i-1}}{i}$, $i > \lambda$. First, this may not always be an integer. Second, taking a face of the cographic matroid of size $i - 1$ and adding one of the $m - (i - 1)$ remaining edges does not guarantee that this new subset of edges is a face of the cographic matroid. Again, Sperner's bounds are tight for trees and cycles.

Table 3.7 gives the upper and lower bounds

$$\frac{\lambda}{m + 1} + \frac{F_{m-n+1}(m - n + 2 - \lambda)}{(m + 1)\binom{m}{m-n+1}} \leq \text{avgRel}(K_n) \leq \frac{F_\lambda}{(m + 1)\binom{m}{\lambda}} + \frac{\lambda}{m + 1}$$

for $n = 3, \dots, 10$. These bounds, in particular for the lower bounds are not that good for $n > 5$, but if we know more of the F_i , $i > \lambda$, then these bounds can be improved upon.

Example 3.4.14 Consider the graph we looked at in the beginning of this section, Figure 3.11, which has 20 vertices, 32 edges and edge connectivity 3. From Table 3.6, we have that the average reliability for this graph is approximately 0.295516. Using Table 3.5, we can find bounds on the average reliability. Assuming we know the edge connectivity λ , the number of cutsets of size λ and the number of spanning trees, we can use the previously described bounds to get a lower bound of

$$\frac{3}{33} + \frac{(32 - 20 + 2 - 3)(10122705)}{33 \binom{32}{13}} = 0.422898$$

and an upper bound of

$$\frac{3}{33} + \frac{(32 - 20 + 2 - 3)(4940)}{33 \binom{32}{3}} = 0.100623.$$

As mentioned, if we know more F_i then we can obtain better bounds. Suppose we know $F_{\lambda+1}$ for our graph (and we do, since from Table 3.5 $F_4 = 35342$). Then our new bounds are

$$\frac{\lambda}{m+1} + \frac{F_\lambda}{(m+1) \binom{m}{\lambda}} + \frac{(m-n+1-\lambda)F_{m-n+1}}{(m+1) \binom{m}{m-n+1}} \leq \text{avgRel}(G)$$

and

$$\text{avgRel}(G) \leq \frac{\lambda}{m+1} + \frac{F_\lambda}{(m+1) \binom{m}{\lambda}} + \frac{(m-n+1-\lambda)F_{\lambda+1}}{(m+1) \binom{m}{\lambda+1}}$$

which when substituting in the correct values, we get a lower bound of 0.129920 and upper bound of 0.418912 which are tighter than the previous ones. If we didn't know $F_{\lambda+1}$ we could have used the Kruskal-Katona bounds, which for us, from Table 3.5, is $F_4 = 35570$ to get an upper bound of 0.450641 for the average reliability.

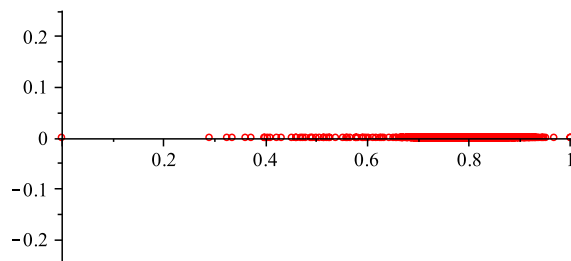


Figure 3.14: The average reliability for simple graphs on $n \leq 8$ vertices.

3.4.3 The Closure of the Average Reliability

Beyond bounding the average reliability or determining its average value over all graphs of order n , one might ask what are the possible values that it can take on. We know that $\{\text{avgRel}(G) \mid G \text{ is a graph}\}$ is a subset of the rationals on $[0, 1]$, but can the set fill out $[0, 1]$? That is, can we find a graph whose average reliability is arbitrarily close to any value $r \in [0, 1]$ or are there subintervals of $[0, 1]$ which are “free of” average reliabilities? We will see that in fact the closure of the average reliabilities of graphs is the entire interval, $[0, 1]$. Figure 3.14 shows a plot of the average reliabilities for all simple graphs on $n \leq 8$ vertices.

We know that 0 is in the closure since the average reliability for a tree on n vertices is $\frac{1}{n+1}$ and

$$\lim_{n \rightarrow \infty} \text{avgRel}(T_n) = 0.$$

We also know that 1 is in the closure since the graph P_2^k , a bundle of k edges, has reliability $1 - (1 - p)^k$, so the average reliability is $1 - \frac{1}{k+1}$, and as k approaches infinity, the average reliability approaches 1 (also, the reliability of K_1 is 1).

We are ready to show that the closure of the average reliabilities is the entire interval $[0, 1]$, using, yet again, the cycle bundles.

Theorem 3.4.15 *The average reliabilities for all terminal reliability polynomials are dense in $[0, 1]$.*

Proof. We will use Proposition 3.4.9. Let

$$\mathcal{F}_{m,j} = \{\text{Rel}_q(C_{m^k}^{kj}, q) = (1 - q^{kj})^{m^k} + m^k q^{kj} (1 - q^{kj})^{m^k - 1} \mid k \geq 1\},$$

which is the set all terminal reliability polynomials for $C_{m^k}^{kj}$, m, j fixed and all $k \geq 1$.

We have seen in Theorem 3.1.7 that as k approaches infinity, the reliability polynomials, $\mathcal{F}_{m,j}$ for the family of graph, $\mathcal{C} = \{C_{m^k}^{kj} : k \geq 1\}$ approaches the step function

$$f_{m,j}(q) = \begin{cases} 1 & : q < \left(\frac{1}{m}\right)^{1/j} \\ 0 & : q > \left(\frac{1}{m}\right)^{1/j} \end{cases}$$

which has an average value over $[0, 1]$ of $1 - \left(\frac{1}{m}\right)^{1/j}$. By Proposition 3.4.9, the average reliabilities of $\mathcal{C}$ approach $1 - \left(\frac{1}{m}\right)^{1/j}$. We have seen that the thresholds, which occur at $\left(\frac{1}{m}\right)^{1/j}$ are dense in $[0, 1]$, so $\{1 - \left(\frac{1}{m}\right)^{1/j} \mid m, j \geq 1\}$ is dense in $[0, 1]$. $\square$

While Proposition 3.4.9 implies that if a sequence of reliability polynomials has a limit f (which may, or may not, be a reliability function), then the average reliabilities for the functions tend to the average value of f . The converse need not be true, but here we show two cases where it does hold.

Theorem 3.4.16 *Let $\mathcal{G} = \{G_n : n \geq 1\}$ be a family of graphs. Then,*

$$\lim_{n \rightarrow \infty} \text{Rel}(G_n, p) = \begin{cases} 0 & \text{if } p = 0 \\ 1 & \text{if } p \in (0, 1] \end{cases}$$

if and only if $\text{avgRel}(G_n)$ tends to 1.

Proof. Suppose that for the family of graphs $\mathcal{G}$, the all terminal reliability of these graphs tends to 1 for $p \neq 0$, as n approaches infinity. Then, clearly the average reliability tends to 1 as n approaches infinity, since the polynomial approaches the constant function 1 for values of $p \neq 0$. Now, let us suppose that the average reliability of the family of graphs $\mathcal{G}_n$ tend to 1 as n approaches infinity. Let $p_0 \in (0, 1)$. Then, since $\text{Rel}(G)$ is an increasing function, the following inequalities hold (see Figure 3.15):

$$\text{avgRel}(G_n) \leq 1 - (1 - \text{Rel}_p(G_n, p_0))p_0 \leq 1.$$

Since $\text{avgRel}(G_n)$ approaches 1, then this implies that $1 - (1 - \text{Rel}_p(G_n, p_0))p_0$ approaches 1, so $(1 - \text{Rel}_p(G_n, p_0))p_0$ approaches 0 and since p_0 is fixed, then $\text{Rel}_p(G_n, p_0)$ must approach 1, as desired. $\square$

Similarly,

Theorem 3.4.17 Let $\mathcal{G} = \{G_n : n \geq 1\}$ be a family of graphs. Then,

$$\lim_{n \rightarrow \infty} \text{Rel}(G_n, p) = \begin{cases} 0 & \text{if } p \in [0, 1) \\ 1 & \text{if } p = 1 \end{cases}$$

if and only if $\text{avgRel}(G_n)$ tends to 0.

Proof. Suppose that for the family of graphs $\mathcal{G}$, the all terminal reliability of these graphs tends to 0 for $p \neq 1$, as n approaches infinity. Then clearly the average reliability tends to 0 as n approaches infinity, since the polynomial approaches the constant function 0 for values of $p \neq 1$. Now, let us suppose that the average reliability of the family of graphs $\mathcal{G}_n$ tend to 0 as n approaches infinity. Let $p_0 \in (0, 1)$.

Then, since $\text{Rel}(G)$ is an increasing function, the following inequalities hold (see Figure 3.15):

$$0 \leq (\text{Rel}_p(G_n, p_0))(1 - p_0) \leq \text{avgRel}(G_n).$$

Since $\text{avgRel}(G_n)$ approaches 0, then this implies that $(\text{Rel}_p(G_n, p_0))(1 - p_0)$ approaches 0, so $(\text{Rel}_p(G_n, p_0))(1 - p_0)$ approaches 0 and since p_0 is fixed, then $\text{Rel}_p(G_n, p_0)$ must approach 0, as desired. $\square$

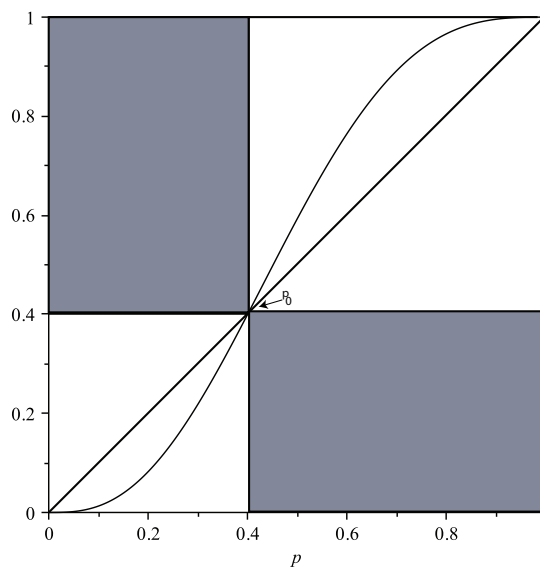


Figure 3.15: The reliability polynomial for a graph G . The upper shaded area is a rectangle with height $1 - p_0$ and width p_0 . The lower shaded rectangle has height p_0 and width $1 - p_0$.

3.4.4 Optimality and Average Reliability

We have seen in Chapter 2 that there are situations where a most optimal graph does not exist. If for a given n and m a most optimal graph does not exist, one can always choose the graph that is most optimal for values of p near 0 if the network is known to have low edge probabilities, or near 1 if the network is known to have high edge probabilities. Using the average reliability, we can have a new notion of optimality,

that depends on all $p \in [0, 1]$. For instance, we could use a graph with the largest average reliability as the optimal graph. This new notion of optimality will always provide an optimal graph. If an optimal graph should exist in the traditional sense, then it will also be the graph with the largest average reliability. So in search for an “optimal graph” if a most optimal graph does not exist, or is unknown to exist, the graph with the largest average reliability would be an appropriate choice.

When looking at simple graphs which have the largest average reliability, initial thoughts were that it would be the graph that is most optimal for values of p near 1, since the reliability polynomial is an increasing function and therefore the integral carries more weight near $p = 1$, but this is not the case. The graph G_1 in Figure 3.16 with $n = 7$, $m = 15$ has the largest average reliability, $\frac{28331}{51480} \approx 0.5503$, but it is in fact the graph that is most optimal for values of p near 0, whereas the graph G_2 is the most optimal for values of p near 1, and it has a smaller average reliability of $\frac{39623}{72072} \approx 0.5497$.

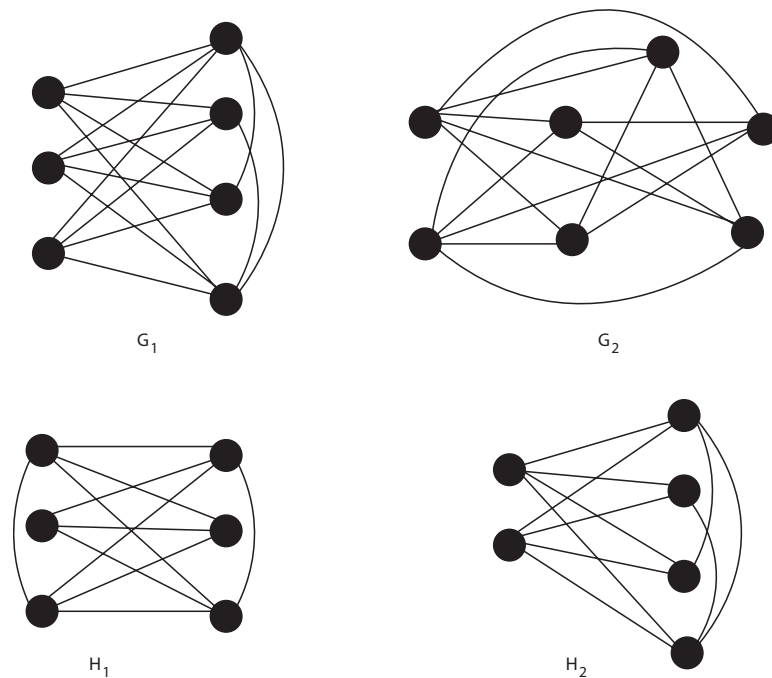


Figure 3.16: Graphs that are best near 0 and best near 1 for their families

So, perhaps it is the case that it is the graph which is most optimal for values of p near 0 which has the largest average reliability? Again, this is not the case. For $n = 6, m = 11$ we do not have a most optimal graph. The graph H_1 in Figure 3.16 is most optimal for values of p near 0, and it has an average reliability of $\frac{239}{462} = 0.5173$, but the graph H_2 in Figure 3.16 has the largest average reliability with a value of $\frac{598}{1155} = 0.5177$ and it is the graph that is most optimal for values of p near 1.

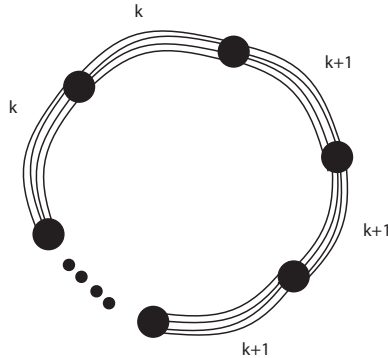


Figure 3.17: A cycle graph $C_{n,m}$, with bundles of size $k = \lfloor \frac{m}{n} \rfloor$ and $k + 1$

Since the graph with largest average reliability is not always the graph that is best for p near 0, or p near 1, we are back to the drawing board. Perhaps using the concept of average reliability we can come up with a ‘good’ family of graphs. We are unlikely to characterize what graphs have largest average reliability, since for those n and m where a most optimal graph exists, it will be this optimal graph which has the largest average reliability, and a characterization of all most optimal graphs is unknown. In light of this, coming up with a ‘good’ family of graphs is of interest.

A family of graphs, $\mathcal{G} = \{G_k : k \geq 1\}$, could be considered *good* if as k approaches infinity, the average reliability of this family tends to 1. Another definition of a good graph and perhaps a more appropriate one is the following,

Definition 3.4.18 *Let $\mathcal{G}$ be a family of graphs and let $G \in \mathcal{G}$ be the graph with the largest average reliability. The graph H is considered a C -good graph if $\frac{\text{avgRel}(H)}{\text{avgRel}(G)} \geq C$, where C is a given constant between $(0, 1)$.*

n	avgRel(K_n)	avgRel($C_{n,m}$)	Ratio
3	0.5	0.5	1
5	0.5866	0.5541	0.944
7	0.6564	0.6037	0.917
9	0.7085	0.6432	0.911
11	0.7418	0.6748	0.909
13	0.7691	0.7007	0.911
15	0.7905	0.77222	0.914
17	0.8079	0.7405	0.917
19	0.8222	0.7562	0.919
21	0.8343	0.7698	0.923
23	0.8448	0.7818	0.925
25	0.8538	0.7923	0.928
27	0.8617	0.8019	0.931
29	0.8687	0.8106	0.933

Table 3.8: The average reliability for K_n , n odd, the corresponding cycle bundle on n vertices with bundles of size $(n-1)/2$ and their ratio

For example, suppose we are to build a network on n vertices and m edges which has at least a particular edge connectivity. We do not know the most optimal graph for this family, or whether one even exists, so we will consider a graph good if its average reliability is at least 75% that of the largest possible average reliability for this family. Suppose we find a graph with the desired edge connectivity whose average reliability is 0.7661. Though we do not know what the largest average reliability is for this family we have found a graph whose average reliability is at least 75% the largest possible value, so by our definition, this is a good graph to use.

In this section, we will consider the graph, cycle bundles, of which we are quite familiar with now, and put it forward as a good graph to use if constructing a network on n vertices and m edges.

Suppose we fix the order, n , of our graph, then given m edges, $m = kn + r$, $0 \leq r < k$. Consider the graph with $n - r$ edges of bundles of size $k = \lfloor \frac{m}{n} \rfloor$ and r edges of bundle of size $k + 1$ (See Figure 3.17). Let this graph be denoted $C_{n,m}$. Now, this may not be the most optimal graph, since if $m = \binom{n}{2}$ the complete graph, K_n has

a higher average reliability, but $C_{n,m}$, by our definition, appears to be a (0.9)-good graph. (see Table 3.8).

To claim that given n and m it is the case that $C_{n,m}$ is a good graph, with a large value of C , we will find a lower bound on the average reliability of $C_{n,m}$. We will do so by assuming that all the bundles are the same size, k . This is a lower bound, since the addition of edges to a graph G increases the reliability. We will consider the reliability polynomial in the variable q . For each bundle of edges of size k , either all n bundles have at least one edge operational, which occurs with probability $(1 - q^k)^n$ or one bundle is completely non-operational and the other $n - 1$ bundles have at least one edge operational, which occurs with probability $nq^k(1 - q^k)^{n-1}$.

$$\begin{aligned}
 \text{Rel}_q(C_{n,m}, q) &\geq (1 - q^k)^n + nq^k(1 - q^k)^{n-1} \\
 &= \sum_{i=0}^n \binom{n}{i} (-1)^i q^{ki} + n \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i q^{ki+k} \\
 &= 1 + \sum_{i=1}^n \left((-1)^i \binom{n}{i} + (-1)^{i-1} n \binom{n-1}{i-1} \right) q^{ki}
 \end{aligned} \tag{3.4}$$

and since whether we look at the reliability polynomial in terms of p or q we obtain the same results, we will continue to consider the polynomial in the variable q .

It turns out that the average reliability of this graph involves the well known gamma function, $\Gamma(n)$, (the gamma function is an extension of the factorial function to real and complex numbers). For natural numbers, $\Gamma(n) = (n - 1)!$ and since we will be looking at applying the gamma function on the interval $[3, \infty)$ and another useful property is that it is a continuous, increasing function on this interval [1]. We find from Equation 3.4 that

$$\text{avgRel}(C_{n,m}) \geq \int_0^1 \left(1 + \sum_{i=1}^n \left((-1)^i \binom{n}{i} + (-1)^{i-1} n \binom{n-1}{i-1} \right) q^{ki} \right) dq$$

$$\begin{aligned}
&= q \Big|_0^1 + \left(\sum_{i=1}^n \frac{((-1)^i \binom{n}{i} + (-1)^{i-1} n \binom{n-1}{i-1})}{ki + 1} q^{ki} \right) \Big|_0^1 \\
&= 1 + \sum_{i=1}^n \frac{(-1)^i \binom{n}{i} + (-1)^{i-1} n \binom{n-1}{i-1}}{ki + 1} \\
&= \frac{n! \Gamma(2 + \frac{1}{k})}{\Gamma(n + 1 + \frac{1}{k})} - 1 + 1 \\
&= \frac{n! \Gamma(2 + \frac{1}{k})}{\Gamma(n + 1 + \frac{1}{k})}
\end{aligned}$$

From this we can see if we fix $n \geq 3$, we have

$$\lim_{k \rightarrow \infty} \frac{n! \Gamma(2 + \frac{1}{k})}{\Gamma(n + 1 + \frac{1}{k})} = 1$$

since the numerator approaches $n!$ and the denominator approaches $\Gamma(n + 1) = n!$, so for large values of k this is a good graph. Let us consider this graph for some particular values of n and m . First consider m such that the graph is forced to have multiedges, that is, $m > \binom{n}{2}$, so $k > \frac{n(n-1)}{2n} = \frac{n-1}{2}$.

We can see that

$$\frac{n! \Gamma(2 + \frac{1}{k})}{\Gamma(n + 1 + \frac{1}{k})} \geq \frac{n!}{\Gamma(n + 1 + \frac{1}{k})}$$

since $\Gamma(2 + \frac{1}{k}) > \Gamma(2) = 1$. We will look at finding values of $\frac{n!}{\Gamma(n+1+1/k)}$ for $k > \frac{n-1}{2}$ and show that $C_{n,m}$ is at least 0.75 times the largest average reliability for $n \geq 23$.

Looking at $\frac{n!}{\Gamma(n+1+\frac{1}{k})}$ as a function of k we see that it is increasing, since $\Gamma(n+1+\frac{1}{k})$ decreases, so for $k > (n-1)/2$,

$$\frac{n!}{\Gamma(n + 1 + \frac{1}{k})} > \frac{n!}{\Gamma(n + 1 + \frac{2}{n-1})},$$

so we will use $\frac{n!}{\Gamma(n+1+\frac{2}{n-1})}$ as a lower bound on the average reliability of $C_{n,m}$.

Another useful property of the gamma function is that $\Gamma(z + 1) = z\Gamma(z)$, for complex numbers z whose real parts are non-negative [1]. Using this we can get for

a positive integer n and $x \in (0, 1)$ that

$$\Gamma(n+x) = (n+x-1)(n+x-2)\dots(1+x)x\Gamma(x).$$

This can be used to show that $f(n) = \frac{n!}{\Gamma(n+1+2/(n-1))}$ is an increasing function. We will show that $f(n+1)/f(n) > 1$,

$$\begin{aligned} \frac{f(n+1)}{f(n)} &= \frac{(n+1)!\Gamma(n+1+\frac{2}{n-1})}{n!\Gamma(n+2+\frac{2}{n})} \\ &= \frac{(n+1)\Gamma(n+1+\frac{2}{n-1})}{\Gamma(n+2+\frac{2}{n})} \\ &= \frac{A}{B} \end{aligned}$$

where

$$A = (n+1)\left(n+\frac{2}{n-1}\right)\left(n-1+\frac{2}{n-1}\right)\dots\left(1+\frac{2}{n-1}\right)\left(\frac{2}{n-1}\right)\Gamma\left(\frac{2}{n-1}\right)$$

and

$$B = \left(n+1+\frac{2}{n}\right)\left(n+\frac{2}{n}\right)\left(n-1+\frac{2}{n}\right)\dots\left(1+\frac{2}{n}\right)\left(\frac{2}{n}\right)\Gamma\left(\frac{2}{n}\right).$$

We want $\frac{f(n+1)}{f(n)} > 1$ and this occurs when

$$\begin{aligned} \frac{(n+\frac{2}{n-1})(n-1+\frac{2}{n-1})\dots(1+\frac{2}{n-1})(\frac{2}{n-1})\Gamma(\frac{2}{n-1})}{(n+\frac{2}{n})(n-1+\frac{2}{n})\dots(1+\frac{2}{n})(\frac{2}{n})\Gamma(\frac{2}{n})} &> \frac{n+1+\frac{2}{n}}{n+1} \\ &= 1 + \frac{2}{n(n+1)}. \end{aligned}$$

We know that $\Gamma(\frac{2}{n-1}) > \Gamma(\frac{2}{n})$ and that $n-i+\frac{2}{n-1} > n-i+\frac{2}{n}$, $1 \leq i \leq n$. With this and the fact that $\frac{2/(n-1)}{2/n} = \frac{n}{n-1} = 1 + \frac{1}{n-1} > 1 + \frac{2}{n(n+1)}$, we have that $f(n)$ is an increasing function.

When $n \geq 7$ we have $\frac{n!}{\Gamma(n+1+\frac{2}{n-1})} \geq 0.5$, so for any graph on $n \geq 7$ the average reliability of $C_{n,m}$ is at least 50% of the largest average reliability for a graph with $n \geq 7$ and $m > \binom{n}{2}$. For $n \geq 11$ we have that $\frac{n!}{\Gamma(n+1+\frac{2}{n-1})} \geq 0.6$, and for $n \geq 23$ we have that $\frac{n!}{\Gamma(n+1+\frac{2}{n-1})} \geq 0.75$. This tells us that for values of $n \geq 23$ and a given $m > \binom{n}{2}$, the average reliability of $C_{n,m}$ is at least 0.75 times the largest average reliability. Thus when $m > \binom{n}{2}$, if a most optimal graph does not exist, or is unknown, then $C_{n,m}$ is a good graph to use, for n sufficiently large.

What can be said about simple graphs? When a matching is removed, as mentioned earlier, Kelmans proved that the resulting simple graph is most optimal. If a matching is removed, then at most $n/2$ edges have been removed, so $m \geq \binom{n}{2} - n/2 = \frac{n(n-2)}{2}$, which means that $k \geq \frac{n-2}{2}$. This gives us a lower bound of $\frac{n!}{\Gamma(n+1+\frac{2}{n-2})}$ for the average reliability of the corresponding cycle bundle $C_{n,m}$. When $n \geq 9$, $\frac{n!}{\Gamma(n+1+\frac{2}{n-2})} \geq 0.5$, so we know that for a given n and m , when looking at simple graphs, the average reliability of $C_{n,m}$ is at least 50% of the largest average reliability over all simple graphs on n vertices and m edges. For larger values of n , we can do even better, for $n \geq 12$, $\frac{n!}{\Gamma(n+1+\frac{2}{n-2})} \geq 0.6$ and for $n \geq 25$, $\frac{n!}{\Gamma(n+1+\frac{2}{n-2})} \geq 0.75$. So, for values of $n \geq 25$, the average reliability of $C_{n,m}$ is at least 75% of the largest average reliability over all simple graphs on n vertices and $m \leq \binom{n}{2}$ edges.

As a result of the above computations, we have the following result,

Theorem 3.4.19 • For $n \geq 23$, $m > \binom{n}{2}$ the cycle bundle graph $C_{n,m}$ is a (0.75)-good graph.

• For $n \geq 25$, $m > \binom{n}{2} - \frac{n}{2}$, the cycle bundle graph $C_{n,m}$ is a (0.75)-good graph.

The values of m considered thus far result in fairly dense graphs. What can be said about sparse graphs? We would not expect sparse graphs to be very reliable. When $m = n - 1$ and $m = n$, the cycle bundle $C_{n,n}$ is a tree and $C_{n,n-1}$ is a cycle. These are known to be most optimal in their classes. When $m = n + 1$, the most optimal

graph is a theta graph whose path lengths differ by at most 1. For a given n and $m = n + 1$, we have that

$$\begin{aligned}\text{Rel}(\Theta_{n_1, n_2, n_3}, p) &\leq p^{n+1} + (n+1)p^n(1-p) + 3\ell^2 p^{n-1}(1-p)^2 \\ &= (3\ell^2 - n)p^{n+1} + (n+1 - 3\ell^2)p^n + 3\ell^2 p^{n-1}\end{aligned}$$

where $\ell = \lceil \frac{n+1}{3} \rceil$. The average reliability of a graph on n vertices and $n+1$ edges is at most $\frac{2(n^2+n+3\ell^2)}{n(n^2+3n+2)}$.

The corresponding cycle bundle is $C_{n, n+1}$, which is a cycle with one edge replaced by a bundle of size 2, and

$$\begin{aligned}\text{Rel}(C_{n, n+1}, p) &= p^{n+1} + (n+1)p^2(1-p) + (2n-1)p^{n-1}(1-p)^2 \\ &= (n-1)p^{n+1} + (-3n+3)p^n + (2n-1)p^{n-1}\end{aligned}$$

so it has an average reliability of $\frac{2(n^2+3n-1)}{n(n^2+3n+2)}$. This means for a given n ,

$$\frac{\text{avgRel}(C_{n, n+1})}{\text{avgRel}(\Theta_{n_1, n_2, n_3})} \geq \frac{n^2 + 3n - 1}{n^2 + n + 3\ell^2}.$$

Setting $\ell = \frac{n+1}{3}$, for a given n , we have that

$$\begin{aligned}\frac{\text{avgRel}(C_{n, n+1})}{\text{avgRel}(\Theta_{n_1, n_2, n_3})} &\geq \frac{n^2 + 3n - 1}{n^2 + n + 3\left(\frac{(n+1)^2}{9}\right)} \\ &= \frac{3n^2 + 9n - 3}{4n^2 + 5n + 1} \\ &\geq 0.75\end{aligned}$$

for $n \geq 3$. This follows from the fact that with $f(n) = \frac{3n^2+9n-3}{4n^2+5n+1}$ we get $f'(n) = \frac{-3(7n^2-10n-8)}{4n^2+5n+1)^2} < 0$ for $n \geq 2$. Since $f(3) > 0.75$ and as n approaches infinity, $f(n)$

approaches 0.75, we have that the cycle bundle is at least 75% the largest average reliability for $m = n + 1$.

Taking into account the results obtain in this section, if given n and m and a most optimal does not exist or is unknown to exist, then we conjecture that the corresponding cycle bundle graph, $C_{n,m}$ is a C -good graph, with a large C .

3.5 Roots of All Terminal Reliability Polynomials

We have looked at several analytic properties of reliability polynomials. One natural question, which is algebraic and analytic in nature, concerns the location of the roots of the polynomial. Besides being an interesting question in its own right, it can also provide some information regarding the coefficients of the polynomial. For instance, a result of Newton's states that for a polynomial $f(x) = \sum a_i x^i$, where a_i are nonzero and real, if we find that the roots of $f(x)$ are always real, then we can say that the sequence $|a_0|, |a_1|, \dots, |a_n|$ is strictly log concave, and hence unimodal (see for example [43]). That is, the terms of the sequence are non-decreasing, then non-increasing and $|a_i|^2 > |a_{i+1}||a_{i-1}|$. If the coefficients are counting objects, like the number of independent sets in a graph or the number of faces in a simplicial complex or matroid, then whether the a_i are log-concave or unimodal is useful information to know, as it gives insight into the behaviour of the sequence and can assist in obtaining bounds on the terms of the sequence.

The roots of several graph polynomials, including flow polynomials, chromatic polynomials, and independence polynomials have been active areas of research. The location of the roots of reliability polynomials have also been studied. The roots of strongly connected reliability polynomials have been investigated by Brown and Dilcher [25] and have been proven by Brown and Cox [24] to be dense in the entire complex plane, answering an open problem.

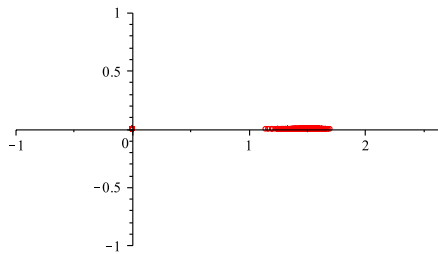


Figure 3.18: Real roots for all simple graphs on $n \leq 8$ vertices.

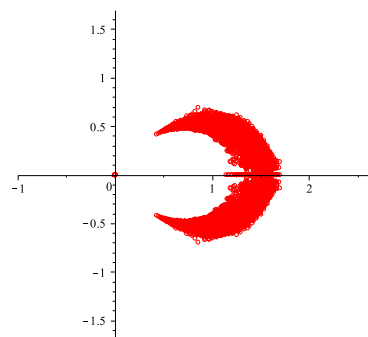


Figure 3.19: Complex roots for all simple graphs on $n \leq 8$ vertices.

Example 3.5.1 *Figure 3.18 and Figure 3.19 show the real and complex roots for all simple graphs on up to 8 vertices. The complex roots are in the disk $|z - 1| \leq 1$.*

It was proven in [21] that the real roots for all terminal reliability polynomials are in $\{0\} \cup (1, 2]$. In contrast to strongly connected reliability, whose roots are dense in the complex plane, it was conjectured by Brown and Colbourn in [21] that the roots lie in $|z - 1| \leq 1$ in the complex plane. There was evidence to support this conjecture. Wagner [74] showed it to be true for series-parallel graphs, but for an arbitrary graph G , it was proven false by Royle and Sokal [70]. The counterexample was a subdivision of K_4 and the root found outside the conjectured disk had modulus 1.04. Whether the roots are contained within a bounded disk is still an open problem. In this section,

we prove some results regarding the location of the roots of all terminal reliability polynomials for graphs with n vertices and m edges.

We will begin by looking at the real roots of all terminal reliability polynomials.

Theorem 3.5.2 *Let G be a graph on n vertices and consider the real roots R of $\text{Rel}_p(G, p)$. For any $r \in R - \{0\}$, $1 + \frac{1}{n-1} \leq r \leq 2$, with equality possible for all $n \geq 2$.*

Proof. Let G be a connected graph on n vertices and m edges and consider the H -form of the all terminal reliability polynomial. So we have that

$$\text{Rel}_p(G, p) = p^{n-1} \sum_{i=0}^{m-n+1} H_i (1-p)^i.$$

Consider

$$\sum_{i=0}^{m-n+1} H_i (1-p)^i$$

We will show that $(0, 1 + \frac{1}{n-1})$ is zero free. As we know, for any reliability polynomial there are no zeros in $(0, 1]$ since the reliability polynomial is an increasing function on this interval, so we need only show that $(1, 1 + \frac{1}{n-1})$ is zero free.

Let $q = 1 - p$, so we have $\sum H_i q^i = H_0 + H_1 q + \dots + H_{m-n+1} q^{m-n+1}$. Assume that $m - n + 1$ is odd and consider the grouping

$$(H_0 + H_1 q) + (H_2 q^2 + H_3 q^3) + \dots + (H_{m-n} q^{m-n} + H_{m-n+1} q^{m-n+1}).$$

If $m - n + 1$ is even, then consider the grouping

$$(H_0 + H_1 q) + (H_2 q^2 + H_3 q^3) + \dots + (H_{m-n-1} q^{m-n-1} + H_{m-n} q^{m-n}) + H_{m-n+1} q^{m-n+1}.$$

Consider the expressions $H_i q^i + H_{i+1} q^{i+1} = q^i (H_i + H_{i+1} q)$, i even and $q \in (\frac{-1}{n-1}, 0)$, which corresponds to $p \in (1, 1 + \frac{1}{n-1})$. Since i is even, $q^i > 0$.

As mentioned in the introduction of the thesis, in [23] it is stated that the H_i count the number of monomials of degree i in a pure multicomplex with $n - 1$ variables, so $(n - 1)H_i \geq H_{i+1}$, since given a monomial of degree i we can add in any of the $n - 1$ variables to obtain a monomial of degree $i + 1$. Thus as $q > -\frac{1}{n-1}$,

$$\begin{aligned} H_i + H_{i+1}q &\geq \frac{1}{n-1}H_{i+1} + H_{i+1}q \\ &= H_{i+1}\left(\frac{1}{n-1} + q\right) \\ &> 0. \end{aligned}$$

This means that $\sum H_i q^i > 0$ for $q \in (-\frac{1}{n-1}, 0)$. It then follows that $\text{Rel}_q(G, q) = (1 - q)^{n-1} \sum H_i q^i \neq 0$ for $q \in (-\frac{1}{n-1}, 0)$, that is, $\text{Rel}_p(G, p) \neq 0$ for $p \in (0, 1 + \frac{1}{n-1})$, so all roots lie in $[1 + \frac{1}{n-1}, 2]$.

The maximum value of 2 is achieved for any graph G whose edges are replaced by bundles of k edges, where k is an even number, since $(1 - (1 - p)^k)^{n-1}$ can be factored out of the reliability polynomial, giving us a root at 2.

The lower bound of $1 + \frac{1}{n-1}$ is tight as it is achieved for $G = C_n$, since

$$\text{Rel}(C_n, p) = p^n + np^{n-1}(1 - p) = p^{n-1}((1 - n)p + n)$$

and this has a root at $p = 1 + \frac{1}{n-1}$. □

The proof of Theorem 3.5.2 demonstrates a family of simple graphs, namely cycles, for which the lower bound of $1 + \frac{1}{n-1}$ for a real root is achieved. By adding a leaf attached to C_n by bundle of edges, we obtain a graph with multiple edges which also achieves this lower bound. The proof also mentions a family of graphs with multiple edges for which the upper bound of 2 is achieved. A complete graph on 75 vertices

has a root of approximately 1.9, so we can get close to 2, but it would be interesting to find a simple graph with a real root at 2.

We will now switch our focus to the complex roots of all terminal reliability polynomials. A useful result when investigating roots of polynomials, real and complex, is the Eneström-Kakeya Theorem, proven independently by Eneström and Kakeya.

Theorem 3.5.3 [36,51] *Let $f(x) = \sum b_i x^i$ be a polynomial with positive coefficients b_i . Then $f(x)$ has its roots in the annulus $\min\{b_i/b_{i+1}\} \leq |z| \leq \max\{b_i/b_{i+1}\}$.*

Investigating roots of reliability polynomials can also be accomplished by looking at the roots of other polynomials and mapping the roots of one polynomial to another, if possible. When comparing reliability polynomials to other polynomials linear fractional transformations of the form

$$g(z) = \frac{az + b}{cz + d}, ab - cd \neq 0,$$

will be useful.

Definition 3.5.4 *The transformation $g(z) = \frac{az+b}{cz+d}$, $ab - cd \neq 0$ where a, b, c, d are complex constants is called a linear fractional transformation.*

To have the extended z plane as the domain we require that $g(\infty) = \infty$ if $c = 0$ and if $c \neq 0$ then $g(\infty) = \frac{a}{c}$ and $g(-\frac{d}{c}) = \infty$. Under $g(z)$, circles and lines are transformed into circles and lines. Also, the interiors and exteriors of circles and half-planes are mapped to the interior or exterior of circles and half-planes.

Thus, if we have a graph G and polynomial $f(x)$ where the roots of $f(x)$ are found within a certain disk, and mapped to the roots of $\text{Rel}(G)$ by a linear fractional transformation, then we have information regarding the location of the roots of $\text{Rel}(G)$.

Using the H -form of the reliability polynomial, Theorem 3.5.3 and linear fractional transformations we will investigate the location of the roots and obtain some insight into their location in the complex plane. While it was conjectured that the roots were always found in $|z-1| \leq 1$, the counterexamples did not provide a disk that contained the roots. In this section we will provide, for a given n , a disk which contains the roots of the reliability polynomials for graphs of order n .

Recall that the all terminal reliability of a graph can be expressed in terms of its H -vector, giving us

$$\text{Rel}_p(G, p) = p^n \sum_{i=0}^{m-n+1} H_i (1-p)^i.$$

Consider the polynomial

$$H(G, z) = \sum_{i=0}^{m-n+1} H_i z^i.$$

The roots of this polynomial are mapped to the non-zero roots of the H -form of the all terminal reliability polynomial by the linear fractional transformation, $g(z) = 1-z$, so if the roots of $H(G, z)$ are found inside the disk $|z| \leq R$, then the roots of the all terminal reliability polynomial of G are found in $|z-1| \leq R$. The question now becomes, can we bound the roots of $H(G, z)$.

A theorem in Marden's book ([60], page 124) says

Theorem 3.5.5 *For any real numbers a and b , such that $a > 1, b > 1$ and $(1/a) + (1/b) = 1$, the polynomial $f(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n$, $c_n \neq 0$ has all its zeros in the circle*

$$|z| < \left(1 + \left(\sum_{j=0}^{n-1} \frac{|c_j|^a}{|c_n|^a} \right)^{b/a} \right)^{1/b}$$

We can use this to bound the roots of $H(G, z)$ and thus the roots of all terminal reliability polynomials. Let a and b satisfy the above conditions. It is known that

$$\sum_{i=0}^{m-n+1} H_i = F_{m-n+1}.$$

We get that

$$\begin{aligned} \sum_{j=0}^{m-n} \frac{|H_j|^a}{|H_{m-n+1}|^a} &= (1/H_{m-n+1})^a \left(\sum_{j=0}^{m-n} H_j^a \right) \\ &< (1/H_{m-n+1})^a \left(\sum_{j=0}^{m-n+1} H_j^a \right) \\ &< (1/H_{m-n+1})^a \left(\sum_{j=0}^{m-n+1} H_j \right)^a \\ &= \left(\frac{F_{m-n+1}}{H_{m-n+1}} \right)^a \\ &= (1/H_{m-n+1})^a F_{m-n+1}^a \end{aligned}$$

So we get that the roots of $H(G, z)$ lie in

$$\begin{aligned} |z| &< \left(1 + \left(\frac{F_{m-n+1}}{H_{m-n+1}} \right)^a \right)^{b/a}^{1/b} \\ &= \left(1 + (F_{m-n+1}/H_{m-n+1})^b \right)^{1/b} \\ &< 1 + F_{m-n+1}/H_{m-n+1} \end{aligned}$$

This gives the following result.

Theorem 3.5.6 *Let G be a graph of order n and size m . The moduli of the roots of $\text{Rel}(G, p)$ are bounded above by $1 + F_{m-n+1}/H_{m-n+1}$.*

It appears from Maple programing that for all simple graphs on $n \leq 8$ vertices, complete graphs are the extremal graphs for the ratio F_{m-n+1}/H_{m-n+1} and for complete graphs this ratio equals $\frac{n^{n-2}}{(n-1)!}$. So we do have a disk which contains the roots, which is an improvement, but we can do even better.

Theorem 3.5.7 *Let G be a graph of order n . The roots of $\text{Rel}(G, p)$ are found in the disk $\frac{1}{n-1} \leq |z - 1| \leq \frac{H_{m-n}}{H_{m-n+1}} \leq n - 1$.*

Proof. We can also use the fact that the cographic matroid for a graph G has an H -vector which is a pure O -sequence [62], that is, it is the degree sequence of some pure multicomplex. In [23] it was shown that a corresponding order ideal for the cographic matroid of a graph G can be represented as monomials in $n - 1$ variables $x_1, x_2, \dots, x_{n-1}$ of degrees at most $d_1 - 1, \dots, d_{n-1} - 1$.

Using the fact that the H -vector comes from a multicomplex, we can see that $(n - 1)H_i \geq H_{i+1}$, which gives us that $\frac{1}{n-1} \leq \frac{H_i}{H_{i+1}}$. This is since we could multiply the monomials by any of the $n - 1$ variables to obtain a monomial of size $i + 1$.

Now using the fact that the multicomplex is pure, we can see that $H_i \leq (n - 1)H_{i+1}$. This is since we could delete any one of the $n - 1$ variables, so we get that $\frac{1}{n-1} \leq \frac{H_i}{H_{i+1}} \leq n - 1$. By Theorem 3.5.3 and the fact that it was proven by Huh [48] that the H -vector of the cographic matroid is log concave, hence unimodal, and thus the modulus of the roots is bounded by H_{m-n}/H_{m-n+1} . This gives us that the roots of $H(G, z)$ are such that $\frac{1}{n-1} \leq |z| \leq \frac{H_{m-n}}{H_{m-n+1}} \leq n - 1$ and therefore the all terminal reliability roots are found in $\frac{1}{n-1} \leq |z - 1| \leq \frac{H_{m-n}}{H_{m-n+1}} \leq n - 1$. $\square$

Again, looking at simple graphs on up to 8 vertices, the ratio H_i/H_{i+1} appears to be largest for complete graphs and is equal to $H_{m-n}/H_{m-n+1} = \frac{n}{2} - 1$, which is better than the previous bound since

$$\frac{n^{n-2}}{(n-1)!} = \left(\frac{n}{n-1}\right) \left(\frac{n}{n-2}\right) \cdots \left(\frac{n}{2}\right) \left(\frac{1}{1}\right) > \frac{n}{2},$$

but still does not provide a fixed disk which contains all roots for all terminal reliability polynomials.

We will now investigate how various graph operations can affect the location of the roots of all terminal reliability polynomials. For a graph G , if we apply some graph operation to obtain a new graph H , the all terminal reliability polynomial of H may be an evaluation of the all terminal reliability of G . If that is the case, then we could study the roots of the reliability polynomial of G and map those roots to the roots of the reliability polynomial for H . For example, if we replace each edge of a graph with a bundle of k edges to obtain the graph G^k , then $\text{Rel}(G^k, p) = \text{Rel}(G, 1 - (1 - p)^k)$. The roots of $\text{Rel}(G^k, p)$ then occur when $1 - (1 - p)^k = r$, where r is a root of $\text{Rel}(G, p)$.

Recall that given a graph G and a gadget H , we can replace the edges of G with H to obtain the graph $G[H]$. We will study the roots of $\text{Rel}(G[H], p)$ by using linear fractional transformations. Before we examine the roots of $\text{Rel}(G[H], p)$, we will prove the following lemma, so that we can look at the roots of one polynomial and compare their location to the roots of a reliability polynomial for a graph G using a linear fractional transformation.

Lemma 3.5.8 *Let G be a graph with reliability polynomial*

$$\text{Rel}(G, p) = \sum_{i=0}^{m-n+1} F_i p^{m-i} (1-p)^i$$

and let

$$f(x) = \sum_{i=0}^{m-n+1} F_i x^i.$$

The non-zero roots of $\text{Rel}(G, p)$ map to the roots of $f(x)$ via the linear fractional transformation $g(z) = \frac{1-z}{z}$.

Proof. Note that

$$\begin{aligned}
 \text{Rel}(G, p) &= \sum_{i=0}^{m-n+1} F_i p^{m-i} (1-p)^i \\
 &= p^m \sum_{i=0}^{m-n+1} F_i p^{-i} (1-p)^i \\
 &= p^m \sum_{i=0}^{m-n+1} F_i \left(\frac{1-p}{p} \right)^i.
 \end{aligned}$$

Let

$$h(p) = \sum_{i=0}^{m-n+1} F_i \left(\frac{1-p}{p} \right)^i$$

and

$$f(x) = \sum_{i=0}^{m-n+1} F_i x^i$$

where F_i is the number of faces of size i in the cographic matroid of G . The roots of $h(p)$ are the non-zero roots of $\text{Rel}(G, p)$. If $p = r$ is a root of $h(p)$ then $\frac{1-r}{r}$ is a root of $f(x)$. This tells us that the roots of $h(p)$ map to the roots of $f(x)$ by the linear fractional transformation $g(z) = \frac{1-z}{z}$. $\square$

Recall from the thresholds section that for a graph G and gadget H we have

$$\text{Rel}(G[H], p) = \sum_{i=0}^{m-n+1} F_i(G) p_{\text{new}}(H)^{m-i} q_{\text{new}}(H)^i.$$

Lemma 3.5.9 *Let G be a graph, H a gadget and $G[H]$ as previously defined. If R is the set of non-zero roots of $\text{Rel}(G, p)$ then the roots of $\text{Rel}(G[H], p)$ occur when*

$$\frac{q_{\text{new}}(G)}{p_{\text{new}}(H)} = \frac{1-r}{r} \text{ for } r \in R.$$

Proof. Let G be a graph on n vertices and m edges and H a gadget. Consider $G[H]$.

We know that

$$\text{Rel}(G[H], p) = \sum_{i=0}^{m-n+1} F_i(G) p_{\text{new}}(H)^{m-i} q_{\text{new}}(H)^i.$$

We can use this fact to investigate the roots of $\text{Rel}(G[H])$ in terms of the roots of $\text{Rel}(G)$. We know that

$$\begin{aligned} \text{Rel}_p(G, p) &= \sum_{i=0}^{m-n+1} F_i(G) p^{m-i} (1-p)^i \\ &= p^m \sum_{i=0}^{m-n+1} F_i(G) p^{-i} (1-p)^i \\ &= p^m \sum_{i=0}^{m-n+1} F_i(G) \left(\frac{1-p}{p} \right)^i \end{aligned}$$

Let $q = 1 - p$ and consider the polynomial

$$\text{Rel}_{p,q}(G, p, q) = p^m \sum_{i=0}^{m-n+1} F_i(G) \left(\frac{q}{p} \right)^i. \quad (3.5)$$

Let R be the set of roots of the polynomial

$$f(x) = \sum_{i=0}^{m-n+1} F_i(G) x^i$$

In $\text{Rel}_{p,q}(G, p, q)$, replace p with $p_{\text{new}}(G)$ and q with $q_{\text{new}}(H)$ to obtain

$$\begin{aligned} \text{Rel}(G[H]) &= \text{Rel}_{p,q}(G, p_{\text{new}}(H), q_{\text{new}}(H)) \\ &= p_{\text{new}}(H)^m \sum_{i=0}^{m-n+1} F_i(G) \left(\frac{q_{\text{new}}(H)}{p_{\text{new}}(H)} \right)^i. \end{aligned}$$

The roots of $\text{Rel}(G[H])$ correspond to where $p_{\text{new}}(H) = 0$ or

$$\sum_{i=0}^{m-n+1} F_i(G) \left(\frac{q_{\text{new}}(H)}{p_{\text{new}}(H)} \right)^i = 0,$$

which means $f\left(\frac{q_{\text{new}}(H)}{p_{\text{new}}(H)}\right) = 0$. The latter occurs where $\frac{q_{\text{new}}(H)}{p_{\text{new}}(H)} = t$, for $t \in R$.

We know from Lemma 3.5.8 that the roots of $\sum F_i(G)x^i$ come from the roots of $\text{Rel}_p(G, p)$ under the mapping $h(z) = \frac{1-z}{z}$, so each $t \in R$ is such that $t = \frac{1-r}{r}$ for some non-zero root r of $\text{Rel}_p(G, p)$, so the roots of $\text{Rel}_{p,q}(G[H])$ occur where $\frac{q_{\text{new}}(H)}{p_{\text{new}}(H)} = \frac{1-r}{r}$ for r , a non-zero root of $\text{Rel}_p(G, p)$. $\square$

Example 3.5.10 Consider the graph C_n and the graph with multiedges, P_2^k , which is P_2 with $k \geq 1$ edges. We know that $\text{Rel}(C_n, p) = np^{n-1}(1-p) + p^n$, so we have that $\text{Rel}(C_n, p) = p^{n-1}(n(1-p) + p)$. This has roots at $p = 0$ and $p = 1 + \frac{1}{n-1}$. Under the linear fractional transformation $g(z) = \frac{1-z}{z}$ mentioned in the previous theorem, the non-zero root $1 + \frac{1}{n-1}$ maps to $-1/n$.

We have that $p_{\text{new}}(P_2^k) = 1 - (1-p)^k$ and $q_{\text{new}}(P_2^k) = (1-p)^k$ so $\text{Rel}(C_n[P_2^k])$ has roots where $p_{\text{new}}(P_2^k) = 1 - (1-p)^k = 0$ and $\frac{(1-p)^k}{1-(1-p)^k} = -1/n$.

Example 3.5.11 Suppose now we take C_n and replace each edge with a path, P_k , of length $k \geq 2$ to obtain $C_n[P_k]$. We know that we end up with a cycle of length nk , so using the method described, we should obtain roots at $p = 0$ and $p = \frac{nk}{nk-1}$. We know for cycles that $\text{Rel}(C_n, p) = np^{n-1}(1-p) + p^n$ which has roots at $p = 0$ and $p = 1 + \frac{1}{n-1}$. For P_k we have that $p_{\text{new}}(P_k) = p^k$ and $q_{\text{new}}(P_k) = k(1-p)p^{k-1}$.

We know that the linear fractional transformation $g(z) = \frac{1-z}{z}$ from the previous theorem sends the non-zero root $p = 1 + \frac{1}{n-1}$ to $-1/n$. The roots of $\text{Rel}(C_n[P_k], p)$ are where $p_{\text{new}}(P_k) = p^k = 0$, so $p = 0$ and where $\frac{k(1-p)p^{k-1}}{p^k} = -1/n$, which is when $p = \frac{nk}{nk-1}$ as expected.

Using the types of graph operations we considered in the examples above, we can show the following.

Theorem 3.5.12 *The roots of all terminal reliability polynomials are dense in the complex plane if and only if the roots are dense in $|z - 1| \leq 2$.*

Proof.

Clearly if the roots are dense in the entire complex plane, then they are dense in $|z - 1| \leq 2$.

We will be considering the all terminal reliability polynomial in the variable q , so we can assume that the roots are dense in $|z| \leq 2$ and then show they are dense in the entire complex plane.

Suppose that we know that there is a family of graphs $\mathcal{G}$ such that their all terminal reliability polynomial roots are dense in $|z| \leq 2$. Let $G \in \mathcal{G}$. We know that

$$\begin{aligned} \text{Rel}(G, p) &= \sum_{i=0}^{m-n+1} F_i p^{m-i} (1-p)^i \\ &= p^m \sum_{i=0}^{m-n+1} F_i \left(\frac{1-p}{p} \right)^i \\ \text{Rel}_{p,q}(G, p, q) &= p^m \sum_{i=0}^{m-n+1} F_i \left(\frac{q}{p} \right)^i \end{aligned}$$

The p^m only gives us a root at 0, so we are interested in the roots of

$$f(p, q) = \sum_{i=0}^{m-n+1} F_i \left(\frac{q}{p} \right)^i$$

Consider replacing each edge of G with the gadget P_k , $k \geq 2$, to obtain the graph $G[P_k]$. As seen in the example, $p_{\text{new}}(P_k) = p^k$ and $q_{\text{new}}(P_k) = k(1-p)p^{k-1}$. Looking at it in the variable q we have that $p_{\text{new}}(P_k) = (1-q)^k$ and $q_{\text{new}}(P_k) = kq(1-q)^{k-1}$.

If $r \neq 1$ is a root of $\text{Rel}_q(G, q) = (1 - q)^m \sum F_i(G)(q/1 - q)^i$ then $q = \frac{r}{1-r}$ is a root of $\sum_{i=0}^{m-n+1} F_i x^i$. To find a root of $\text{Rel}_q(G[P_k], q)$ we must solve $q_{\text{new}}(P_k) - (\frac{r}{1-r})p_{\text{new}}(P_k) = 0$. Let $R = \frac{r}{1-r}$, then we have the following sequence of equivalent statements.

$$\begin{aligned} kq(1 - q)^{k-1} - R(1 - q)^k &= 0, \\ (1 - q)^{k-1}(kq - R(1 - q)) &= 0, \\ kq &= R(1 - q), \\ q &= \frac{R}{k + R}. \end{aligned}$$

We know that the roots of $\text{Rel}_q(G, q)$ are dense in $|z| \leq 2$. Under the linear fractional transformation $g_1(z) = \frac{z}{1-z}$ the roots of $\text{Rel}_q(G, q)$ map to the roots of $f(1 - q, q)$. So under $g_1(z)$ the disk $|z| \leq 2$ maps to $|z + \frac{4}{3}| = \frac{2}{3}$, since the boundary points of $-2, 2, 2i$ map to $\frac{-2}{3}, -2$ and $\frac{2i}{5} - \frac{4}{5}$ respectively. Since the roots are dense in $|z| \leq 2$, we want to see where the interior of $|z| = 2$ maps to under $g_1(z)$. Since an interior point, like 0 goes to 0, the interior of $|z| = 2$ maps outside the disk $|z + \frac{4}{3}| = \frac{2}{3}$, thus the roots are dense for $|z + \frac{4}{3}| \geq \frac{2}{3}$.

We can now map the roots of $f(1 - q, q)$ to the roots of $\text{Rel}_q(G[P_2], q)$ using the linear fractional transformation $g_2(z) = \frac{z}{2+z}$. This gives us that the roots of $\text{Rel}_q(G[P_2], q)$ are dense in $\Re(z) \geq -\frac{1}{2}$, because this linear fractional transformation maps the circle $|z + \frac{4}{3}| = \frac{2}{3}$ to $\Re(z) = -\frac{1}{2}$, since the boundary points of $-2/3, -2$ and $\frac{2i}{5} - \frac{4}{5}$ map to $-\frac{1}{2}, \infty$ and $-\frac{1}{2} + \frac{i}{2}$ respectively. Testing a point in the region where the roots are dense, like $z = 0$, shows that the outside of $|z + \frac{4}{3}| = \frac{2}{3}$ lies to the right of $\Re(z) = -\frac{1}{2}$. (See Figure 3.20 to see the various linear fractional transformations discussed.)

As the roots of $\text{Rel}_q(G[P_2], q)$ are dense in $\Re(z) \geq -\frac{1}{2}$, we have that for roots z of $\text{Rel}_q(G[P_2], q)$, the closure of their modulus $|z|$ is $[0, \infty)$. So given any $r \in [0, \infty)$ and $\epsilon > 0$ there is an all terminal reliability polynomial which has a root whose modulus is within ϵ of r .

Let $z \in \mathcal{C}$ be any nonzero complex number and let θ be the argument of z . We will show that for any $\epsilon > 0$ there exists a root w of an all terminal reliability polynomial such that w is within ϵ of z .

First, pick L large enough such that for any complex number ρ there is an L -th root of unity of ρ with argument within ϵ of θ .

We will show that by replacing the edge of $G[P_2]$ with bundles of $\ell > L$ parallel edges, we will obtain that the closure of the roots is the entire complex plane. If G is a graph and G^ℓ is G with each edge replaced by a bundle of ℓ edges, then $\text{Rel}_q(G^\ell, q) = \text{Rel}_q(G, 1 - q^\ell)$. So if w_ℓ is a root of $\text{Rel}_q(G, q)$ then any q such that $w_\ell = 1 - q^\ell$ is a root of $\text{Rel}_q(G^\ell, q)$. This means that q runs over all the ℓ -th roots of $1 - w_\ell$ and since the closure of the modulus of the roots of all terminal reliability polynomials is $[0, \infty)$, we can pick w_ℓ such that $|1 - w_\ell|$ is within ϵ of $|z|$. Since we picked L such that any complex number has at least one root of unity with argument within ϵ of θ , given z there is a root of an all terminal reliability polynomial within ϵ of it. As z and ϵ were arbitrary then the closure is the entire complex plane. $\square$

A corollary of the theorem just proved is

Corollary 3.5.13 *If there is a graph G whose all terminal reliability polynomial has a root arbitrarily close to -1, then there exists a graph, H such that the all terminal reliability polynomial of H has a root of arbitrarily large modulus.*

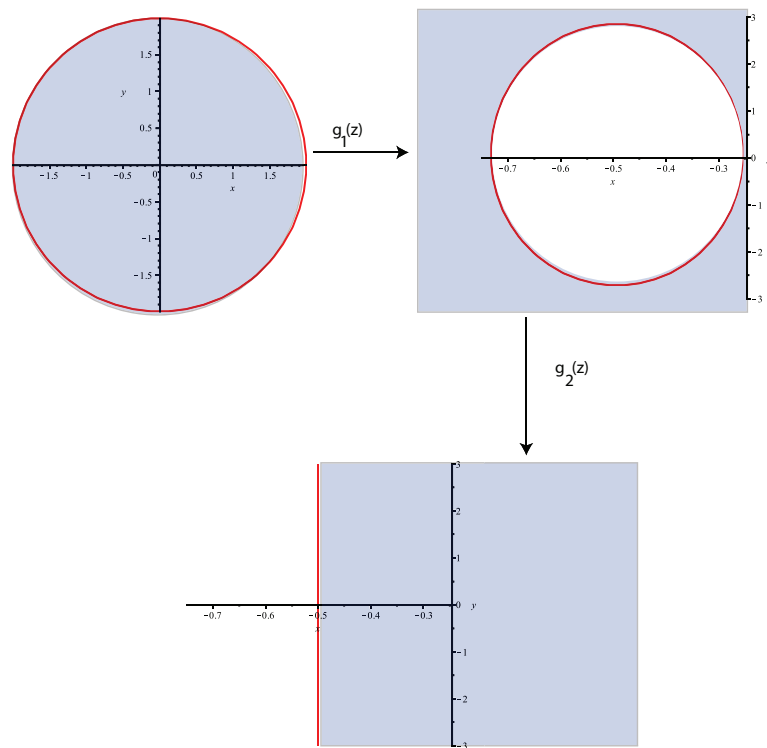


Figure 3.20: $|z| \leq 2$ under the $g_1(z) = z/(1 - z)$ and $|z + \frac{4}{3}| \geq 2/3$ under $g_2(z) = z/(z + 2)$

Chapter 4

k-clique Reliability

4.1 Introduction

In most of the reliability literature, it is assumed that the vertices are always operational and that it is the edges that can fail, but this is not the only model. The *vertex failure reliability* [37, 77] is a model where the vertices operate independently with probability $p \in [0, 1]$ and the desired property is that the subgraph induced by the operational vertices is connected. Areas of research regarding this model are the existence of optimal graphs and bounding techniques [37, 77].

When looking at reliability problems where the vertices operate independently with probability $p \in [0, 1]$, the requirement of having a connected subgraph operational is not the only model that has been studied. One particular reliability model that uses vertex failures is that of the *k-out-of-n reliability* [7, 18], where the components of the network are operational with probability $p \in [0, 1]$ and the system is considered functional if at least k of the n components are operational. In the representative graph, it is equal to the operational components inducing a connected subgraph. In this reliability problem, sometimes the components are edges, other times vertices. An example of *k-out-of-n* reliability is the following. A sewage treatment plant is operational if at least k of the n pumping stations are operational.

The all terminal reliability of a graph focuses on the global structure of the network, as it requires at least a spanning tree to be operational to be considered reliable. The *k-out-of-n* systems also have a global focus, as they require that the subgraph

induced by the operational components is connected. Here we introduce an extension of the k -out-of- n model, which focuses on local structure.

Let $k \geq 2$ be fixed. For a given simple graph G , suppose that each vertex is independently operational with probability $p \in [0, 1]$. What is the probability that the operational vertices contain a k -clique? We call this probability the k -clique reliability of a simple graph G , and denote it by $\text{crel}_k(G, p)$. That is, the system is considered reliable if there is at least a clique of size k operational.

The clique reliability is focused on a local structure, as it does not require the operational subgraph to be connected. This type of problem can arise when looking at small-world networks, such as Facebook, since a structural property of such networks is the existence of cliques, so one may want to know the probability that at least k of the vertices can fully communicate with each other. Another example where this type of problem could arise is the following. A processing plant may require that a workstation of at least k of the machines can send information directly to and from each other in order for the plant to be operational.

Example 4.1.1 *If considering the k -clique reliability of a complete graph, this problem becomes that of the well known k -out-of- n reliability. For a complete graph to have at least a k -clique operational, it is required that at least k of the n vertices be operational, which occurs with probability $\text{crel}_k(K_n, p) = \sum_{i=k}^n \binom{n}{i} p^i (1-p)^{n-i}$. This is the same as asking what the k -out-of- n reliability of K_n is, where the components are the vertices.*

For $k = 1$, it is rather trivial as we need at least one vertex operational, so $\text{crel}_1(G, p) = 1 - (1-p)^n$. For $k = 2$ this is a new and interesting reliability problem.

Example 4.1.2 *Consider C_4 . There are only two subsets of vertices of size at least 2 that do not contain an edge, namely the opposite pairs on the cycle, so it follows that $\text{crel}_2(C_4, p) = 4p^2(1-p)^2 + 4p^3(1-p) + p^4 = 4p^2 - 4p^3 + p^4$.*

For any graph, one may think that the k -clique reliability of a graph G is really just a coherent reliability problem in disguise, with minpaths of cardinality k , but it is in fact a proper subclass of them. For example, suppose C is any coherent reliability problem on a set S , and the minpaths of C are all of the same size, k . If the graph H is constructed with $V(G) = S$ and edge set $E(G) = \{xy \mid x \text{ and } y \text{ appear together in some element of } C\}$, then the k -clique reliability of H seems to be the same as the coherence reliability polynomial for C , but this is not the case. Consider the following counterexample.

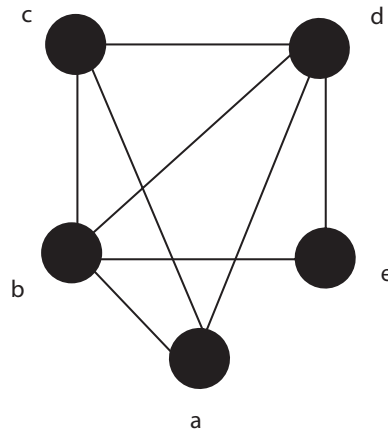


Figure 4.1: Graph G in Example 4.1.3

Example 4.1.3 Let C be a coherence reliability problem on the set $\{a, b, c, d, e\}$ with the following minpaths, $\{a, b, c\}$, $\{a, c, d\}$, $\{b, d, e\}$, thus

$$C = \{\{a, b, c\}, \{a, c, d\}, \{b, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, \{a, b, c, d, e\}\}.$$

This means that the coherence reliability polynomial for C is

$$3p^3(1-p)^2 + 5p^4(1-p)^2 + p^5 = -p^5 - p^4 + 3p^3.$$

Create the graph G whose vertex set $\{a, b, c, d, e\}$ and an edge appears between two vertices if they appear in the same minpaths of C , so

$$E(G) = \{xy \mid x \text{ and } y \text{ appear together in the same minpath of } C\}$$

(See Figure 4.1). The operational set for this graph is the same as C , with the addition of 2 new minpaths, as we've created two new triangles, namely $\{b, c, d\}$ and $\{a, b, d\}$. The 3-clique reliability of the G is $5p^3(1-p)^2 + 5p^4(1-p) + p^5 = p^5 - 5p^4 + 5p^3$, which clearly differs from the coherence reliability of G , since in forming G , we created those new triangles.

4.2 Computing k -clique Reliability

As with all terminal reliability, we have a few different forms for which we can express the k -clique reliability of a graph. One is the F -form,

$$\text{crel}_k(G, p) = \sum_{i=0}^{n-k} F_i p^{n-i} (1-p)^i,$$

where F_i is the number of ways to have i non-operational vertices such that the operational vertices induce a subgraph with at least a k -clique. The other is the N -form,

$$\text{crel}_k(G, p) = \sum_{i=k}^n N_i p^i (1-p)^{n-i},$$

where N_i is the number of ways to have i operational vertices which induce at least a k -clique. Note that $N_i = F_{n-i}$.

We can also calculate the k -clique reliability based on the inclusion and exclusion principle. Let K be the set of k -cliques of a graph G . Since we require that we have

at least a k -clique operational, we have that

$$\text{crel}_k(G, p) = \sum_{t \in K} p^k - \sum_{1 \leq i < j \leq |K|} p^{|t_i \cup t_j|} + \sum_{1 \leq i < j < \ell \leq |K|} p^{|t_i \cup t_j \cup t_\ell|} + \dots + (-1)^{|K|-1} p^{|t_1 \cup t_2 \cup \dots \cup t_{|K|}|}.$$

If we have a relatively small number of cliques, then this can be calculated quickly.

Example 4.2.1 Consider the graph, G in Figure 4.1 and $\text{crel}_3(G, p)$. As we saw, for this graph, $N_0 = N_1 = N_2 = 0$, $N_3 = 5$, $N_4 = 5$ and $N_5 = 1$. This gives us that $F_0 = 1$, $F_1 = 5$, $F_2 = 5$. As our cliques are $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$, $\{b, d, e\}$, we can use the inclusion and exclusion principle to calculate the reliability.

$$\text{crel}_3(G, p) = 5p^3 - (8p^4 + 2p^5) + (4p^4 + 6p^5) - (p^4 + 4p^5) + p^5 = p^5 - 5p^4 + 5p^3.$$

As we saw in the introduction of the thesis, for all terminal reliability we can use the deletion and contraction of an edge to recursively calculate the all terminal reliability polynomial. Though for a graph G with vertex v the number of cliques of size k can be calculated recursively by counting the k -cliques in $G - v$ and the number of $k - 1$ cliques in $G|_{N(v)}$, this deletion and contraction method does not extend to the probability of having at least a k -clique operational. When $k > 2$, we clearly need a k -clique operational in $G - v$, but when v is operational, we could have a k -clique in $G - v$ or a $(k - 1)$ -clique in $G|_{N(v)}$.

Example 4.2.2 Consider the wheel graph W_5 , $\text{crel}_3(W_5, p) = p^6 - 5p^4 + 5p^3$. Let v be a vertex, not the hub; then $\text{crel}_3(W_5 - v, p) = -2p^4 + 3p^3$ and $\text{crel}_3(W_5|_{N(v)}, p) = 2p^2 - p^3$, and so $(1 - p)(-2p^4 + 3p^3) + p(2p^2 - p^3) = 2p^5 - 6p^4 + 5p^3$.

So, we do not have a recursive formula for the k -clique reliability, unless $k = 2$. This is since when $k = 2$, either the edge containing a vertex $v \in V(G)$ is operational,

or not, so for a graph, G and $v \in V(G)$ we have,

$$\begin{aligned} \text{crel}_2(G, p) &= (1 - p)\text{crel}_2(G - v, p) + p(1 - (1 - p)^{\deg(v)}) \\ &\quad + p(1 - p)^{\deg(v)}\text{crel}_2(G - N[v], p). \end{aligned}$$

Example 4.2.3 Consider C_6 . Using the above formula for calculating the 2-clique reliability, we have that

$$\begin{aligned} \text{crel}_2(C_6, p) &= (1 - p)\text{crel}_2(P_5, p) + p(1 - (1 - p)^2) + p(1 - p)^2\text{crel}_2(P_3, p) \\ &= (1 - p)(p^5 + 5p^4(1 - p) + 9p^3(1 - p)^2 + 4p^2(1 - p)^3 \\ &\quad + p(2p - p^2) + p(1 - p)(p^3 + 2p^2(1 - p))) \\ &= -2p^6 + 6p^5 - 3p^4 - 6p^3 + 6p^3 \end{aligned}$$

Although in general for a given G and k , we don't have a recursive formula for the k -clique reliability, there are times when we can express the reliability of a graph G in terms of a subgraph of G .

Proposition 4.2.4 Let graph G . Join a vertex v to each vertex of G to obtain $G + v$. Then $\text{crel}_k(G + v, p) = (1 - p)\text{crel}_k(G, p) + p\text{crel}_{k-1}(G, p)$

Proof. Let H be a subgraph of $G + v$ that contains at least a k -clique. Clearly, either H contains v or it does not. If v is not operational then we need at least a k -clique operational in G , which occurs with probability $\text{crel}_k(G, p)$. If v is operational then we need at least a $(k - 1)$ -clique operational in G , which occurs with probability $\text{crel}_{k-1}(G, p)$, and as having at least a $(k - 1)$ -clique operational in G includes subgraphs contain at least a k -clique, this gives us that

$$\text{crel}_k(G + v, p) = (1 - p)\text{crel}_k(G, p) + p\text{crel}_{k-1}(G, p).$$

□

Example 4.2.5 Consider the wheel graph, which is $C_n + v$. By Proposition 4.2.4, we have that

$$\begin{aligned} \text{crel}_3(W_n, p) &= \text{crel}_3(C_n + v, p) \\ &= (1 - p)\text{crel}_3(C_n, p) + p\text{crel}_2(C_n, p) \\ &= p\text{crel}_2(C_n, p) \end{aligned}$$

Though there are many ways to express the k -clique reliability of a graph, in our study of k -clique reliability, we will also make use of two other polynomials, the k -clique generating polynomial and the k -clique-free polynomial. For a graph G of order n , the k -clique generating polynomial is

$$\text{cgen}_k(G, x) = \sum_{i=0}^n N_i x^i,$$

where N_i denotes the number of subsets of vertices of size i that induce at least a k -clique. The k -clique-free polynomial is

$$\text{cfree}_k(G, x) = \sum_{i=0}^n I_i x^i,$$

where I_i is the number of subsets of vertices of size i which do not induce a k -clique. When $k = 2$, the 2-clique-free polynomial is the well known and well studied independence polynomial, $I(G, p)$ [41, 57]. Clearly

$$\text{cgen}_k(G, x) + \text{cfree}_k(G, x) = (1 + x)^n$$

and moreover, we can show the following.

Lemma 4.2.6 *Let G be a graph on n vertices and $k \geq 2$. Then*

$$\text{crel}_k(G, p) = (1 - p)^n \text{cgen}_k(G, \frac{p}{1 - p})$$

and

$$\text{crel}_k(G, p) = 1 - (1 - p)^n \text{cfree}_k(G, \frac{p}{1 - p})$$

Proof. Let G be a graph on n vertices, then

$$\begin{aligned} \text{crel}_k(G, p) &= \sum_{i=k}^n N_i p^i (1 - p)^{n-i} \\ &= (1 - p)^n \left(\sum_{i=k}^n N_i p^i (1 - p)^{-i} \right) \\ &= (1 - p)^n \left(\sum_{i=k}^n N_i \left(\frac{p}{1 - p} \right)^i \right) \\ &= (1 - p)^n \text{cgen}_k(G, \frac{p}{1 - p}) \end{aligned}$$

and

$$\begin{aligned} \text{crel}_k(G, p) &= (1 - p)^n \text{cgen}_k(G, \frac{p}{1 - p}) \\ &= (1 - p)^n \left(\left(1 + \frac{1 - p}{p} \right)^n - \text{cfree}_k(G, \frac{p}{1 - p}) \right) \\ &= (1 - p)^n \left(\frac{1}{(1 - p)^n} - \text{cfree}_k(G, \frac{p}{1 - p}) \right) \\ &= 1 - (1 - p)^n \text{cfree}_k(G, \frac{p}{1 - p}), \end{aligned}$$

which completes the proof. □

Example 4.2.7 *We have seen that $\text{crel}_3(W_5, p) = p^6 - 5p^4 + 5p^3$. Using the formulas in Lemma 4.2.6, we can get the same results. Since $\text{cgen}_3(W_5, x) = 5x^3 + 10x^4 +$*

$5x^5 + x^6$ and $\text{cfree}_3(W_5, x) = 1 + 6x + 15x^2 + 15x^3 + 5x^4 + x^5$ we have that

$$\begin{aligned} \text{crel}_3(W_5, p) &= (1-p)^6 \text{cgen}_3(W_5, \frac{p}{1-p}) \\ &= (1-p)^6 \left(\frac{5p^3}{(1-p)^3} + \frac{10p^4}{(1-p)^4} + \frac{5p^5}{(1-p)^5} + \frac{p^6}{(1-p)^6} \right) \\ &= p^6 - 5p^4 + 5p^3 \end{aligned}$$

and

$$\begin{aligned} \text{crel}_3(W_5, p) &= 1 - (1-p)^6 \text{cfree}_3(W_5, \frac{p}{1-p}) \\ &= 1 - (1-p)^6 \left(1 + \frac{6p}{(1-p)} + \frac{15p^2}{(1-p)^2} \right. \\ &\quad \left. + \frac{15p^3}{(1-p)^3} + \frac{5p^4}{(1-p)^4} + \frac{p^5}{(1-p)^5} \right) \\ &= p^6 - 5p^4 + 5p^3 \end{aligned}$$

These relationships will be useful later in this chapter when we are investigating the location of the roots of these reliability polynomials.

We know that computing the all terminal reliability of a graph is $\#P$ -complete. What can we say about the complexity of k -clique reliability for a fixed $k \geq 2$? (It is trivial for $k = 1$ since we know that for a graph G of order n , $\text{crel}_1(G, p) = 1 - (1-p)^n$.) Let G be a graph and $k \geq 2$. As with any reliability polynomial, we have $\text{crel}_k(G, 0) = 0$ and if G has a k -clique, then $\text{crel}_k(G, 1) = 1$ (it is 0 otherwise). But how difficult is it to compute $\text{crel}_k(G, p)$ for any $p \in \mathbb{C} - \{0, 1\}$? We know that

$$\begin{aligned} \text{crel}_k(G, p) &= (1-p)^n \text{cgen}_k(G, \frac{p}{1-p}) \\ &= (1-p)^n \left(\left(1 + \frac{p}{1-p} \right)^n \text{cfree}_k(G, \frac{p}{1-p}) \right) \\ &= (1-p)^n \left(\frac{1}{(1-p)^n} - \text{cfree}_k(G, \frac{p}{1-p}) \right) \\ &= 1 - (1-p)^n \text{cfree}_k(G, \frac{p}{1-p}), \end{aligned}$$

therefore if it is $\#P$ -hard to compute $\text{cfree}_k(G, s)$ for any $s \in \mathbb{C} - \{0, 1\}$, then it is $\#P$ -hard to compute $\text{crel}_k(G, s)$. We will show that it is $\#P$ -hard to compute the k -clique reliability of a graph for any complex number s , $s \neq 0$ and $s \neq 1$, by looking at the complexity of computing $\text{cfree}_k(G, s)$.

Theorem 4.2.8 *For a fixed $k \geq 2$, computing $\text{crel}_k(G, s)$, $s \in \mathbb{C} - \{0, 1\}$ is $\#P$ -hard.*

Proof. If G is k -clique free then $\text{crel}_k(G, p) \equiv 0$. Suppose G has at least a clique of size k . We know that $\text{crel}_k(G, 0) = 0$ and $\text{crel}_k(G, 1) = 1$. Let $s \in \mathbb{C} - \{0, 1\}$. Consider the case when $k = 2$; then

$$\begin{aligned} \text{crel}_2(G, s) &= 1 - (1 - s)^n \text{cfree}_2(G, \frac{s}{1-s}) \\ &= 1 - (1 - s)^n I(G, \frac{s}{1-s}) \end{aligned}$$

In [47] it was shown that computing the independence polynomial for any non-zero complex number is $\#P$ -hard. Therefore calculating $\text{crel}_2(G, s)$ for any non-zero complex number is $\#P$ -hard.

Let $k \geq 3$. Consider the graph $G + v$ and $s \in \mathbb{C} - \{0, 1\}$. By Proposition 4.2.4, the k -clique reliability of this graph is

$$\text{crel}_k(G + v, p) = (1 - p)\text{crel}_k(G, p) + p\text{crel}_{k-1}(G, p).$$

This gives us that

$$\text{crel}_{k-1}(G, s) = \frac{(1 - s)\text{crel}_k(G, s) - \text{crel}_k(G + v, s)}{s}.$$

This means that if we can compute $\text{crel}_k(G, s)$ in polynomial time; then we can compute $\text{crel}_{k-1}(G, s)$ in polynomial time. As $\text{crel}_2(G, s)$ is $\#P$ -hard, so is $\text{crel}_3(G, s)$, and by induction, $\text{crel}_k(G, s)$ is $\#P$ -hard for all fixed $k \geq 2$. $\square$

4.2.1 Bounding and k -clique Reliability Complexes

We have seen that computing the k -clique reliability is $\#P$ -hard for values other than 0 and 1, so it would be of interest to find some bounds for the k -clique reliability of a graph. We can find some bounds by investigating combinatorial structures related to the reliability problem. One such example is using the inclusion and exclusion formula we saw earlier. Cutting the summation off after a positive term gives an upper bound, and cutting the summation off after a negative term gives a lower bound.

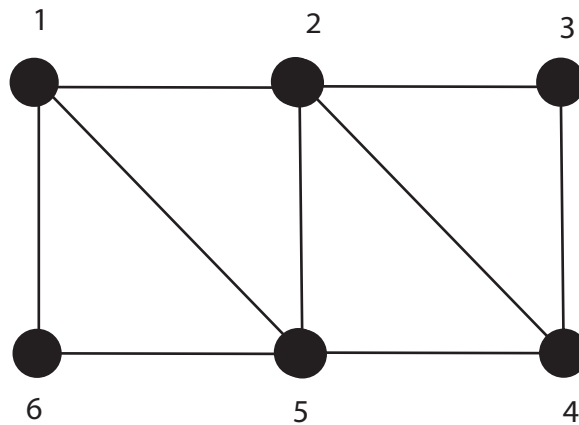


Figure 4.2: Graph for Example 4.2.9.

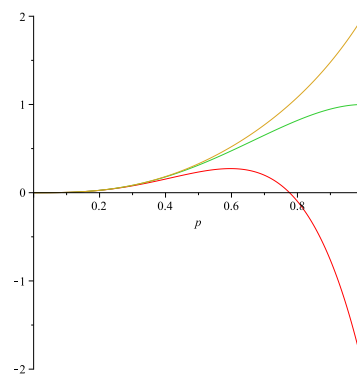


Figure 4.3: The green plot is the 3-clique reliability polynomial for the graph in Figure 4.2; the other plots are the upper and lower bounds achieved from the inclusion and exclusion formula having terms left off.

Example 4.2.9 Consider the graph G , in Figure 4.2. We will compute $\text{crel}_3(G, p)$. The minpaths (3-cliques) are $\{1, 2, 5\}$, $\{1, 5, 6\}$, $\{2, 4, 5\}$, $\{2, 3, 4\}$. Using the inclusion and exclusion formula to calculate the 3-clique reliability, we have

$$\text{crel}_3(G, p) = 4p^3 - (3p^4 + 2p^5 + p^6) + (2p^6 + 2p^5) - p^6 = 4p^3 - 3p^4 + 2p^6.$$

If we cut our formula off after a positive term, say the term $2p^6 + 2p^5$, we have $\text{crel}_3(G, p) \leq 4p^3 - (3p^4 + 2p^5 + p^6) + (2p^6 + 2p^5)$ (see Figure 4.3, the gold curve). If we cut it off after a negative term, say $-(3p^4 + 2p^5 + p^6)$, we have a lower bound, as $\text{crel}_3(G, p) \geq 4p^3 - (3p^4 + 2p^5 + p^6)$ (see Figure 4.3, the red curve).

Another way to bound the reliability polynomial is to bound the coefficients in the N - or F -form of the reliability polynomial. For example, with all terminal reliability we have a simplicial complex, the cographic matroid, whose F -vector is the coefficients of the F -form of the reliability polynomial, so bounds on the F -vector translate into bounds for the reliability polynomial (for example, see [22,23,31]). A natural question to ask is: are there any complexes associated with k -clique reliability for which we can use their combinatorial structure to bound the reliability polynomial?

Recall the general vertex failure problem, where the vertices operate independently with $p \in [0, 1]$ and the graph is reliable if the graph induced by the operational vertices is connected. This is not an example of a coherent system, as it is not closed under supersets. To add a vertex to a subset of vertices, which induce a connected subgraph, does not guarantee that this larger set of vertices will induce a connected subgraph. This means when expressing the vertex failure reliability of a graph G in the F -form, the F_i are not the components of the F -vector of a simplicial complex. For example, consider a 6-cycle with vertices 1, 2, 3, 4, 5, 6. To remove vertices 4, 5 and 6 leaves a connected subgraph, but to remove just 4 and 6 does not. To look at the subsets that do not induce a connected subgraph still does not provide a

complex, since a subset of vertices which do not induce a connected subgraph may have a subset of vertices which do induce a connected subgraph. Again, consider the example with the C_6 , the vertices 1, 3, 4 do not induce a connected subgraph, but the vertices 3, 4 do. The vertex failure model is not a coherent system.

In contrast, though the k -clique reliability problem is a vertex failure reliability problem, it is a coherent system and in fact there are two simplicial complexes associated with this problem that we can look at.

We have seen that we can express the k -clique reliability of a graph as

$$\text{crel}_k(G, p) = 1 - \sum I_i p^i (1 - p)^{n-i},$$

where I_i is the number of subsets of vertices of size i which do not contain a K_k . Since such a subset of vertices is closed under subsets, one complex we can look at has as faces the subsets of vertices which do not contain a K_k ,

$$I_k(G) = \{S \subseteq V(G) \mid S \text{ does not contain a } K_k\}.$$

This gives us $I_i = F_i$, where the F_i 's come from the F -vector of $I_k(G)$. When $k = 2$, $I_2(G)$ is the well known independence complex. Sometimes this complex is a pure complex and sometimes it is not.

Example 4.2.10 Consider $I_2(K_{n,n})$. This is a pure complex, as the maximal independent sets are the bi-partitions, both of which are size n . If we are looking at $I_3(W_n)$ then this is not a pure complex as we can have a facet consisting of all the vertices in the cycle or we can have a facet that contains the hub and $\lfloor \frac{n}{2} \rfloor$ vertices in the cycle.

We also know that the reliability polynomial can be expressed in the F -form as

$$\text{crel}_k(G, p) = \sum F_i p^{n-i} (1-p)^i,$$

where the F_i are the number of ways to remove i vertices and still have at least a k -clique. This gives us the complex

$$\Delta_k(G) = \{S \subseteq V(G) \mid G - S \text{ contains a } K_k\},$$

so the F -vector of this complex consists of the F_i in the F -form of the k -clique reliability polynomial. This is a pure complex, as the facets are complements of k -cliques. However, although it is a pure complex, it is not necessarily matroidal or shellable.

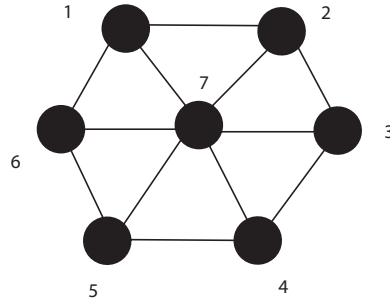


Figure 4.4: Graph for Example 4.2.11.

Example 4.2.11 Consider W_6 (Figure 4.4). We have

$$\begin{aligned} I_3(W_6) = & \{\{7, 1, 3, 5\}, \{7, 2, 4, 6\}, \{1, 2, 4\}, \{1, 2, 5\}, \{2, 3, 5\}, \{2, 3, 6\}, \\ & \{3, 4, 6\}, \{3, 4, 1\}, \{4, 5, 1\}, \{4, 5, 2\}, \{6, 5, 2\}, \{6, 5, 3\}\}, \end{aligned}$$

thus $\text{crel}_3(W_6, p) = 1 - (8p^3(1-p)^4 + 17p^4(1-p)^3 + 6p^5(1-p)^2 + p^6)$ and

$$\Delta_3(W_6) = \{\{3, 4, 5, 6\}, \{1, 4, 5, 6\}, \{1, 2, 5, 6\}, \{1, 2, 3, 6\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5\}\},$$

thus $\text{crel}_3(W_6, p) = 6p^3(1-p)^4 + 18p^4(1-p)^3 + 15p^5(1-p)^2 + 6p^6(1-p) + p^7$. We can see that $I_3(W_6)$ is not a pure complex and that $\Delta_3(W_6)$ is not a matroid (for example $\{1, 2, 3, 6\}$ and $\{1, 4, 5, 6\}$ do not have the exchange property), but it is shellable. A shelling order is $\{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 2, 3, 6\}, \{3, 4, 5, 6\}, \{1, 4, 5, 6\}, \{1, 2, 5, 6\}$.

There is a connection between these two complexes. The Alexander dual of a complex, $\mathcal{C}$ on a ground set V is

$$C^\vee = \{\sigma \subseteq V \mid V \setminus \sigma \notin C\}.$$

That is, the Alexander dual $C^\vee$ of a complex C has as facets the complements of the minimal non-faces of C . We know that the facets of $\Delta_k(G)$ are complements of k -cliques. The minimal non-faces of $I_k(G)$ are the k -cliques. This means that the Alexander dual of $I_k(G)$ is $\Delta_k(G)$. This relationship will be useful when investigating the combinatorial structures of these complexes. We should note that $(C^\vee)^\vee = C$.

Example 4.2.12 Consider the graph, W_6 in the previous example. The minimal non-faces of $I_3(W_6)$ are $\{1, 2, 7\}, \{2, 3, 7\}, \{3, 4, 7\}, \{4, 5, 7\}, \{5, 6, 7\}, \{1, 6, 7\}$, which are the complements of the facets of $\Delta_3(W_6)$, and thus $\Delta_3(W_6) = I_3^\vee(W_6)$.

The reason why one would be interested in the complexes associated with a reliability problem is that for complexes, there are bounds which hold for the F -vector and therefore these bounds can be used to bound the k -clique reliability polynomial. As mentioned in the background chapter, there are Sperner's bounds [71], which state that for a complex on a ground set of size m , $(m-i+1)F_{i-1} \geq iF_i$. There are also the Kruskal-Katona bounds (see [52, 55]) which are known to be better than Sperner's bounds.

If we know if a complex $\mathcal{C}$, associated with the k -clique reliability polynomial, is shellable, then we can express the reliability in the H -form,

$$\text{crel}_k(G, p) = p^k \sum_{i=0}^n H_i (1-p)^i,$$

where the H_i are from the H -vector of $\mathcal{C}$. We can then use the Ball-Provan bounds [4–6] to bound the reliability polynomial.

Example 4.2.13 *Consider the complete bipartite graph $K_{2,3}$. The 2-clique reliability for this graph is $(1 - (1-p)^2)(1 - (1-p)^3)$, since we need at least a vertex operational in each partition. Let vertices 1 and 2 be in the partition of size 2, and vertices 3, 4, 5 in the partition of size 3. We then obtain the complex*

$$\Delta_2(K_{2,3}) = \{\{2, 3, 5\}, \{2, 4, 5\}, \{2, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{1, 3, 4\}\}.$$

This gives us the F -vector $\langle 1, 5, 9, 6 \rangle$ and the H -vector $\langle 1, 2, 2, 1 \rangle$ since an interval partition of this complex is $[\{2, 3, 5\}, \emptyset], [\{2, 3, 4\}, \{4\}], [\{1, 3, 4\}, \{1\}], [\{1, 3, 5\}, \{1, 5\}], [\{1, 2, 4\}, \{1, 2\}], [\{1, 2, 5\}, \{1, 2, 5\}]$.

Overall, knowing some information regarding the combinatorial structure of complexes associated with reliability problems is useful, as the Ball Provan bounds can then be used.

The topological structure of $\Delta_1(G)$ is somewhat trivial, as it is a sphere. For $k \geq 2$, the structure seems much more delicate. We will investigate the topological structure of $\Delta_2(G)$ for a graph G . We will look to see when it is a matroid and when it is shellable (hence the reliability polynomial has an H -form). We focus on $\Delta_k(G)$ since it is a pure complex, and hence could be partitionable. It is known that Ball-Provan bounds are better than the Krustal-Katona bounds (for example, see [31]),

so it is desirable for an reliability polynomial to have an associated complex that is partitionable as there are good bounds to apply to the H -vector.

We will begin by investigating when $\Delta_2(G)$ is shellable by looking at what is called the *strong gcd condition*.

Definition 4.2.14 [50] *Given a simplicial complex Δ , a strong gcd order is a linear order $M_1, M_2, \dots, M_r$ of the minimal non-faces with the following property: if $1 \leq i < j \leq r$ and $M_i \cap M_j = \emptyset$ then there exists a $k > i$ such that $k \neq j$ and $M_k \subseteq M_i \cup M_j$. A simplicial complex Δ is said to satisfy the strong gcd condition if the minimal non-faces (circuits) of Δ admit a strong gcd order.*

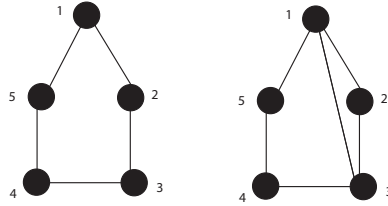


Figure 4.5: Graphs for Example 4.2.15

Example 4.2.15 *Consider the graph C_5 and its independence complex, $I(C_5)$, which has maximal faces*

$$\{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}.$$

The minimal non-faces are

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}$$

(the edges of the graph), and one can check that the independence complex for C_5 does not have a strong gcd order. This is because each pair of disjoint edges has one unique edge in their union and as each circuit (edge) must exceed another in the linear ordering, we can not have a strong gcd order since the first edge in our ordering will need to exceed another edge in the ordering.

Let G be the second graph in Figure 4.5, which is a C_5 with a chord. The independence complex for this graph has facets $\{\{1, 2\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}$. The circuits of this complex are $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{1, 3\}$ and they do have a strong gcd order, which is $\{1, 2\}, \{1, 5\}, \{2, 3\}, \{4, 5\}, \{3, 4\}, \{1, 3\}$.

A useful theorem regarding the strong gcd condition, a complex and its Alexander dual is the following.

Theorem 4.2.16 [9] *Let Δ be a complex whose circuits are size 2 (a flag complex), and $\Delta^\vee$ its Alexander dual. Then to say that Δ satisfies the strong gcd condition is equivalent to saying that $\Delta^\vee$ is shellable.*

A consequence of this is that when $k = 2$, $I_2(G)$ is the independence complex and we know the circuits are size 2, so if $I_2(G)$ satisfies the strong gcd condition, then $\Delta_2(G)$ is shellable.

For a graph G to be such that $\Delta_2(G)$ is shellable, it is necessary that G have no induced $2K_2$, else it would not have the strong gcd condition. It is also necessary that G not contain an induced C_5 since the edges of C_5 , as we saw in Example 4.2.15, do not have a strong gcd ordering. Hence if a graph has an induced C_5 , it cannot have a strong gcd ordering.

It is not clear whether not containing an induced $2K_2$ or C_5 is sufficient for $\Delta_2(G)$ to be shellable, but it is necessary. The next result will show that if we have graphs G_1 and G_2 whose independence complex also has the strong gcd condition, then G_1 and G_2 can be used to create another graph H whose independence complex has the strong gcd condition and hence $\Delta_2(H)$ is shellable.

First we need the following definition.

Definition 4.2.17 *Let $S = \{x_1, x_2, \dots, x_n\}$ be a set, and order the elements of S such that $x_i > x_j$ if $i > j$. Let $u = x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$ ($e_i \geq 0$) and $v = x_1^{f_1} x_2^{f_2} \dots x_n^{f_n}$*

($f_i \geq 0$). Then u comes lexicographically before v , $u >_{lex} v$, if for the minimal i such that $e_i \neq f_i$, $1 \leq i \leq n$, we have that $e_i > f_i$.

Example 4.2.18 Consider $K_{1,6}$ on vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, with central vertex v_1 . The edges of $K_{1,6}$ are v_1v_i , $i = 2 \dots 6$. Assume the ordering $v_1 > v_2 > v_3 > v_4 > v_5 > v_6$ on the vertices. Then the edges, in lexicographic order, are $v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_1v_6$. If we added the edge $\{v_2, v_3\}$ to our graph, then $\{v_2, v_3\} < e$ where e is any other edge in the graph, since e would contain v_1 and $v_1 > v_2$.

Lemma 4.2.19 Let G and H be graphs such that $\Delta_2(G)$ and $\Delta_2(H)$ are shellable. Then the $\Delta_2(G + H)$ is shellable.

Proof. We will use Theorem 4.2.16, so we will look at the independence complexes for these graphs.

Let G be a graph of order n_G and size m_G and let H be a graph of order n_H and size m_H . Let m be the product of n_G and n_H . Let $\{g_1, g_2, \dots, g_{m_G}\}$ be the strong gcd order for the edges of G . Let $\{h_1, h_2, \dots, h_{m_H}\}$ be the strong gcd order for the edges of H . Let $\{f_1, f_2, \dots, f_m\}$ be the edges that join G and H .

We want to show that the edges of $G + H$ exhibit a strong gcd order. We know that we can order the edges of G and H so that they exhibit a strong gcd order. It is also the case that the edges f_i , $i = 1 \dots m$ join G and H , so a pair of disjoint edges, one from G and one from H are connected by some edge f_i , $i \in \{1, \dots, m\}$. We also have that an edge f_i , $i \in \{1, \dots, m\}$ and any edge from G or H are connected by some other f_j , $j \neq i$, $j \in \{1, \dots, m\}$, so if we list all the edges of G , in their gcd order, followed by the edges of H in their gcd order, followed by some ordering of the edges f_i , $i = 1 \dots m$, then this has the potential to be a strong gcd order. It will be a strong gcd order if we can order the edges f_i , $i = 1 \dots m$ so that they exhibit a strong gcd order.

Let $\{a_1, a_2, \dots, a_{n_G}\}$ be the vertices of G and $\{b_1, b_2, \dots, b_{n_H}\}$ the vertices of H . An edge f_i has endpoints a_j and b_k for some $1 \leq j \leq n_G$ and $1 \leq k \leq n_H$.

Order the vertices of G such that $a_i > a_j$ if $i > j$ and order the vertices of H similarly, so that $b_i > b_j$ if $i > j$. If we order the edges $f_i, i = 1 \dots m$ lexicographically, then this is a strong gcd order on the edges between G and H . Suppose $f_i \cap f_\ell = \emptyset$, where $f_i = (a_k, b_j)$ and $f_\ell = (a_r, b_s)$. Without loss of generality, assume $k < r$. We know (a_r, b_j) is an edge between G and H and $(a_r, b_j) >_{lex} (a_k, b_j)$, so the gcd condition holds.

This means that the independence complex $I_2(G + H)$ has a strong gcd order, so $\Delta_2(G + H)$ is shellable. $\square$

We can characterize triangle-free graphs G for which the complex $\Delta_2(G)$ is shellable. Before we begin, we require the following definition.

Definition 4.2.20 [76] *A bipartite graph G with partitions A and B is called chain bipartite if and only if it is a bipartite graph and the vertices in each partition can be ordered linearly with respect to inclusion of their neighbourhoods.*

Theorem 4.2.21 *Let G be a triangle-free graph with no induced $2K_2$.*

- *If G is bipartite, then $\Delta_2(G)$ is shellable.*
- *If G is not bipartite, then $\Delta_2(G)$ is not shellable.*

Proof. Let G be a triangle-free graph with no induced $2K_2$ (hence G is connected, or is the disjoint union of a connected component of size at least 2 and isolated vertices). If G is not bipartite, then G can be obtained from a 5-cycle where each vertex, x , is replaced by an independent set (whose neighbours are the neighbours of x) [30], so it has an induced C_5 , and so $I_2(G)$ does not have the strong gcd condition, and hence $\Delta_2(G)$ is not shellable.

Now suppose that G is a bipartite graph with partitions A and B , with no induced $2K_2$. It is the case that $2K_2$ -free bipartite graphs are chain bipartite graphs [76]. Then we have that G is chain bipartite, so it has the property that for vertices x and y , $N(x) \subseteq N(y)$ or $N(y) \subset N(x)$ and the vertices in each partition (A and B) can be ordered under inclusion of their neighbourhoods.

Let $|A| = n_1$, $|B| = n_2$. Order the vertices in A so that $N(a_1) \supseteq N(a_2) \supseteq \dots \supseteq N(a_{n_1})$ and order the vertices $a_1, a_2, \dots, a_{n_1}$ by inclusion of their neighbours, so $a_1 > a_2 > \dots > a_{n_1}$. Now consider the ordering of the edges of G lexicographically. Suppose we have $\{a_i, b_j\} \cap \{a_r, b_s\} = \emptyset$ and assume without loss of generality that $i > r$. Since $i > r$, we know that $N(a_i) \subseteq N(a_r)$, so $\{a_r, b_j\}$ is an edge of G and $\{a_i, b_j\} < \{a_r, b_j\}$. Thus we have a strong gcd ordering of the edges of G and so $\Delta_2(G)$ is shellable. $\square$

So we know some situations where the complex is shellable. All matroids are shellable [31]. We will now classify the family of graphs for which $\Delta_2(G)$ is matroidal.

Theorem 4.2.22 *Let G be a graph of order $n \geq 4$. $\Delta_2(G)$ is a matroid if and only if G is the disjoint union of a complete multipartite graph and ℓK_1 , $\ell \geq 0$.*

Proof. Let $n \geq 4$. Let G be a graph and $V = V(G)$. Assume $\Delta_2(G)$ is a matroid. Let σ_1 and σ_2 be facets of $\Delta_2(G)$. Since $\Delta_2(G)$ is a matroid, the exchange axiom holds, so we know for each $x \in \sigma_1 - \sigma_2$ there exists a $y \in \sigma_2 - \sigma_1$ such that $(\sigma_1 - \{x\}) \cup \{y\}$ is a facet.

Suppose that G contains an induced graph that is a paw-graph (a triangle on $\{x, y, z\}$ with another vertex w , adjacent to z , but not to x or y). We know that $V - \{x, y\}$ and $V - \{z, w\}$ are facets of $\Delta_2(G)$. Let $\sigma_1 = V - \{x, y\}$ and let $\sigma_2 = V - \{z, w\}$. Since $\Delta_2(G)$ is a matroid, then we have that $(\sigma_1 - \{w\}) \cup \{x\}$ or $(\sigma_1 - \{w\}) \cup \{y\}$ is a facet, therefore either $\{w, x\}$ or $\{w, y\}$ is an edge of G , which

is a contradiction. So G is paw-free, thus by [68] G is triangle-free or a complete multipartite graph, possibly with isolated vertices.

First suppose that G is triangle-free. As $\Delta_2(G)$ is shellable, G contains no induced $2K_2$. By Theorem 4.2.21, if G is not bipartite, then $\Delta_2(G)$ is not shellable. If G is a bipartite graph, then $\Delta_2(G)$ is shellable, but this does not imply that $\Delta_2(G)$ is a matroid. If G is a connected bipartite graph, but not a complete bipartite graph, then there is at least one set of 4 vertices that induce a P_4 , say with edges $\{x, y\}, \{y, z\}, \{z, w\}$. Then $V - \{x, y\}$ and $V - \{z, w\}$ are facets of $\Delta_2(G)$ for which the exchange axiom does not hold (as $\{w, x\}$ and $\{w, y\}$ are not edges of G), and thus $\Delta_2(G)$ is not a matroid. So if $\Delta_2(G)$ is a matroid, then G is the disjoint union of a complete multipartite graph and ℓK_1 , $\ell \geq 0$.

Now suppose that G is the disjoint union of a complete multipartite graph and ℓK_1 , $\ell \geq 0$ and consider the complex $\Delta_2(G)$. Let σ_1 and σ_2 be facets of $\Delta_2(G)$. Let $x \in \sigma_1 - \sigma_2$. We show that there exists a $t \in \sigma_2 - \sigma_1$ such that $(\sigma_1 - \{x\}) \cup \{t\}$ is a facet.

Let $\sigma_1 - e_1 = V - \{z, w\}$ and $\sigma_2 = V - e_2 = V - \{x, y\}$. Assume first that $e_1 \cap e_2 = \emptyset$. We have that $\sigma_1 - \sigma_2 = \{x, y\}$ and $\sigma_2 - \sigma_1 = \{z, w\}$. It is the case that $(\sigma_1 - \{x\}) \cup \{z\}$ or $(\sigma_1 - \{x\}) \cup \{w\}$ is a facet if either $\{x, w\}$ or $\{x, z\}$ is an edge. Since the edges of G lie in a complete multipartite subgraph and $\{z, w\}$ is an edge and both z and w will not be in the same partition as x , the exchange axiom holds in this case.

Now assume that $e_1 \cap e_2 \neq \emptyset$. Let $\sigma_1 - e_1 = V - \{z, w\}$ and $\sigma_2 = V - e_2 = V - \{x, w\}$. We have that $\sigma_1 - \sigma_2 = \{x\}$ and $\sigma_2 - \sigma_1 = \{z\}$. We know that $(\sigma_1 - \{x\}) \cup \{z\}$ is a facet since $\{x, w\}$ is an edge.

Thus $\Delta_2(G)$ is a matroid when G is the disjoint union of a complete multipartite graph and ℓK_1 , $\ell \geq 0$. □

Since we know that $\Delta_2(G)$ is shellable for graphs G which are complete multipartite graphs with ℓK_1 , $\ell \geq 0$, we could use the Ball and Provan bounds to bound the k -clique reliability polynomial, but this really is not necessary, as we can explicitly write down the 2-clique reliability polynomial for such a graph. Let G be a graph that is a complete multipartite graph on n vertices with r partitions of sizes $n_1, n_2, \dots, n_r$ and ℓK_1 , $\ell \geq 0$.

Since the independence polynomials of disjoint graphs can be multiplied [57], the 2-clique free polynomial, or independence polynomial for G is

$$I(G, x) = (1 + x)^\ell \left(\sum_{i=0}^n \sum_{j=1}^r \binom{n_j}{i} x^i \right).$$

This comes from the fact that an independent set of the connected component consisting of the complete multipartite graph will consist of a subset of size i from exactly one of r partitions and the independence polynomial for the ℓK_1 is $(1 + x)^\ell$. From this, we can obtain the 2-clique reliability polynomial for G since

$$\begin{aligned} \text{crel}_2(G, p) &= 1 - (1 - p)^n \text{cfree}_2(G, \frac{p}{1-p}) \\ &= 1 - (1 - p)^n I(G, \frac{p}{1-p}) \end{aligned}$$

and we know $I(G, x)$.

4.3 Optimality of k -clique Reliability Polynomials

With any reliability problem, one is interested if an optimal graph exists. We have seen that with all terminal reliability it may be the case that a most optimal graph does not always exist. For k -clique reliability, we will look at both most and least optimal graphs. Before we begin, the following lemma will be useful, as it provides

some information as to the properties of the graphs that are most optimal for values of p near 0 and near 1.

Lemma 4.3.1 *Let G and H be graphs on n vertices and m edges and let $k \geq 2$.*

Then

$$\text{crel}_k(G, p) = \sum_{i=0}^n F_i(G) p^{n-i} (1-p)^i$$

and

$$\text{crel}_k(H, p) = \sum_{i=0}^n F_i(H) p^{n-i} (1-p)^i.$$

If we have that $F_i(G) = F_i(H)$ for $i > j$ but $F_j(G) > F_j(H)$, then $\text{crel}_k(G, p) > \text{crel}_k(H, p)$ for p close to 0.

If we have that $F_i(G) = F_i(H)$ for $i < \ell$ but $F_\ell(G) > F_\ell(H)$, then $\text{crel}_k(G, p) > \text{crel}_k(H, p)$ for p close to 1.

Proof. Let G and H be graphs on n vertices and m edges and assume $k \geq 2$ and consider

$$\text{crel}_k(G, p) - \text{crel}_k(H, p) = \sum_{i=0}^n (F_i(G) - F_i(H)) p^{n-i} (1-p)^i.$$

Let ℓ be such that $F_i(G) = F_i(H)$ for $1 \leq i \leq \ell - 1$ and let j be such that $F_i(G) = F_i(H)$ for $j + 1 \leq i \leq n$. Then we have that

$$\begin{aligned} \text{crel}_k(G, p) - \text{crel}_k(H, p) &= \sum_{i=\ell}^j (F_i(G) - F_i(H)) p^{n-i} (1-p)^i \\ &= p^{n-j} (1-p)^\ell \sum_{i=\ell}^j (F_i(G) - F_i(H)) p^{j-i} (1-p)^{i-\ell}. \end{aligned}$$

Let $F_i = F_i(G) - F_i(H)$, then

$$\begin{aligned} \text{crel}_k(G, p) - \text{crel}_k(H, p) &= p^{n-j}(1-p)^\ell \sum_{i=\ell}^j F_i p^{j-i} (1-p)^{i-\ell} \\ &= p^{n-j}(1-p)^\ell (F_\ell p^{j-\ell} + F_{\ell+1} p^{j-\ell-1} (1-p) + \\ &\quad \dots + F_{j-1} p (1-p)^{j-\ell-1} + F_j (1-p)^{j-\ell}) \end{aligned}$$

and from this we can see that for p near 0, $F_j(1-p)^{j-\ell}$ is dominant and so if $F_j(G) > F_j(H)$, then the k -clique reliability of G exceeds that of H for values of p near 0. For values of p near 1, the dominant term is $F_\ell p^{j-\ell}$, so if $F_\ell(G) > F_\ell(H)$ the k -clique reliability of G exceeds H for values of p near 1. $\square$

Therefore, if there is a graph $G \in \mathcal{S}_{n,m}$ such that for any other graph $H \in \mathcal{S}_{n,m}$, we have that $\text{crel}_k(G, p) > \text{crel}_k(H, p)$ for $p \in (0, 1)$, then G must have the maximum number of k cliques, that is, it must be most optimal for values of p near 0, and have the maximum connectivity. This means it is the graph which can have the most vertices non-operational and still have a K_k operational, so it is optimal for values of p near 1.

We will begin our study by looking at least optimal 2-clique polynomials. Though the main goal of a network is for it to be the most reliable, the study of least optimal 2-clique reliability polynomials tells us about graphs with a most optimal independence polynomial, which is an interesting problem in its own right.

Recall that for a graph G on n vertices and m edges, the independence polynomial for G is

$$I(G, x) = \sum_{j=0}^n I_j x^j,$$

where I_j is the number of independence sets of size j . The independence reliability polynomial for a graph G is defined as

$$\text{IRel}(G, p) = \sum_{j=0}^n I_j p^j (1-p)^{n-j},$$

it is the probability that an independent set is operational, given that the vertices independently operate with probability $p \in [0, 1]$.

Example 4.3.2 Consider C_6 . The independence polynomial is

$I(C_6, x) = 1 + 6x + 9x^2 + 2x^3$, and the independence reliability polynomial is

$$\begin{aligned} \text{IRel}(G, p) &= (1-p)^6 + 6p(1-p)^5 + 9p^2(1-p)^4 + 2p^3(1-p)^3 \\ &= 1 - 6p^2 + 6p^3 + 3p^4 - 6p^5 + 2p^6. \end{aligned}$$

We will show that to maximize $\text{crel}_2(G, p)$ we want to minimize the independence reliability polynomial, and to minimize $\text{crel}_2(G, p)$ we want to maximize the independence reliability polynomial. We know that

$$(1-p)^n I(G, \frac{p}{1-p}) = \sum_{j=0}^n I_j p^j (1-p)^{n-j} = \text{IRel}(G, p),$$

so this means that for $p \in [0, 1)$ we have

$$\text{IRel}(G, p) \leq \text{IRel}(H, p) \text{ if and only if } I(G, \frac{p}{1-p}) \leq I(H, \frac{p}{1-p}).$$

Since $p \in [0, 1]$, this means

$$I(G, \frac{p}{1-p}) \leq I(H, \frac{p}{1-p}) \text{ if and only if } I(G, x) \leq I(H, x)$$

for $x \in [0, \infty)$. Similarly,

$$\text{IRel}(G, p) \geq \text{IRel}(H, p) \text{ if and only if } I(G, x) \geq I(H, x)$$

where $x \in [0, \infty)$.

So the question of the existence of optimal graphs for 2-clique reliability polynomials becomes a question of the existence of optimal graphs for the independence polynomial on $[0, \infty)$.

If we can find at least one graph $G \in \mathcal{S}_{n,m}$ such that for all $H \in \mathcal{S}_{n,m}$ we have $I(G, x) \geq I(H, x)$ for $x \in [0, \infty)$, this graph has the least optimal 2-clique reliability polynomial. To show that such a graph exists, we will be using commutative algebra.

We know that the independence complex, $I_2(G)$, is a simplicial complex whose faces are the independent sets of G . The F -vector of $I_2(G)$ are the coefficients of the independence polynomial. For all graphs we have $F_0 = 1$, $F_1 = n$ and $F_2 = \binom{n}{2} - m$. One way to maximize the independence polynomial on $[0, \infty)$ is to find a family of graphs such that given F_1 and F_2 , we get the maximum possible values of F_i 's for $3 \leq i \leq n$.

To show that there exists a graph which, given n and m , has the maximal F_i 's, $3 \leq i \leq n$, we will look at another complex, the Stanley-Riesner complex and properties of its F -vector. As mentioned above, we will approach this problem via commutative algebra, so some background is necessary. A good reference for commutative algebra and its connections to simplicial complexes is [10].

Let $Q = \mathbf{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ be the *Kruskal-Katona ring*, where $\mathbf{k}$ is a field. For us, the x_i will be variables representing the vertices of a graph G . We will think of the ring Q as a vector space, where the monomials of Q form a basis for the vector space. Clearly the monomials in Q are square-free. Each set of degree d monomials

in our ring Q is also a vector space, called the d -th graded component of Q , denoted Q_d [45].

An ideal I of a ring R is a subring of R that is closed under left and right multiplication of elements of R . Given an ideal I of Q , the d -th graded component of I is the vector space $I_d = Q_d \cap I$ [45]. This vector space contains 0 and all the homogeneous polynomials of degree d which are in the ideal, I . The *Hilbert function* of a homogeneous ideal I is $H(I, d) = \dim_{\mathbf{k}} I_d$.

Example 4.3.3 Consider the graph C_5 with vertices x_1, x_2, x_3, x_4, x_5 and the Kruskal-Katona ring $Q = \mathbf{k}[x_1, x_2, x_3, x_4, x_5]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2)$. Also, consider the ideal, I of Q generated by the edges of G , $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5)$. $I_2 = Q_2 \cap I$ and the dimension of this vector space is 5. $I_3 = Q_3 \cap I$ and the dimension of this vector space is 10, $I_4 = Q_4 \cap I$ and the dimension of this vector space is 5, and $I_5 = Q_5 \cap I$ and the dimension of this vector space is 1.

As mentioned above, to find a family of graphs that have the largest possible F_i 's in the F -vectors of the independence complex, we will consider another simplicial complex, the Stanley-Reisner complex.

The *Stanley-Reisner complex* of an ideal, I of a ring R is the complex whose faces are the square-free monomials of R not in I [72].

Example 4.3.4 Consider the previous example. The Stanley-Reisner complex of $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5)$ of Q is $\Delta = \{x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5\}$.

In our study of optimality, we will look at ideals with a particular property.

Definition 4.3.5 [44] Let I be an ideal of R and (I_d) the ideal generated by the d -th graded component of I . We call (I_d) Gotzmann (that is, the d -th graded component of I is Gotzmann) if for all other ideals J of R with $H((I_d), d) = H(J, d)$ we have

that $H((I_d), d+1) \leq H(J, d+1)$. If each graded component of I is Gotzmann, then we call I a Gotzmann ideal.

Example 4.3.6 Consider yet again the examples above regarding C_5 .

The ideal $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5)$ of Q is not Gotzmann since the ideal $J = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3)$ of Q is such that $H((I_3), 3) = H(J, 3) = 5$, but $H(J, 4) = 8 < H((I_3), 4) = 10$.

We are considering the Kruskal-Katona ring, Q , which is square-free. Let I be an ideal of Q . If a monomial is in the d -th graded component of Q , but not in the d -th graded component of I , then it must be in the Stanley-Reisner complex. That is, given an ideal I of Q , any monomial in Q is either in I or in the Stanley-Reisner complex of I . Gotzmann ideals are ideals with the smallest Hilbert function growth, meaning if an ideal, (I_d) of a ring R is Gotzmann, their Hilbert function is smaller than the other homogeneous ideals of R with the same dimension in degree d . So, if (I_d) is Gotzmann, for any other homogeneous ideal $J \subseteq R$, if $H((I_d), d) = H(J, d)$ then $H((I_d), k) \leq H(J, k)$ for all $k \geq d$ [46].

So an ideal of Q which is Gotzmann has the smallest Hilbert function for each d -th graded component, and thus for each d , then the F_i of the Stanley-Reisner complex are the largest of the other homogeneous ideals of Q . We can use this to find a family of graphs for which the independence complex has the largest possible entries in the F -vector, and hence would have the least optimal 2-clique reliability polynomials.

For a simple graph G with vertex set $\{x_1, x_2, \dots, x_n\}$, the edge ideal of G is $I_G = (x_i x_j \mid \{x_i, x_j\} \in E(G))$. We will use the edge ideal in the Kruskal-Katona ring to show the existence of a graph that has the least optimal 2-clique reliability polynomial.

Example 4.3.7 Consider the graph G which is a 5 cycle with a chord, so the edge set of G is $\{x_1x_2, x_2x_3, x_3, x_4, x_4x_5, x_5x_1, x_1x_3\}$. The edge ideal of this graph is $I_G = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1, x_1x_3) \subseteq \mathbf{k}[x_1, x_2, x_3, x_4, x_5]$.

If a monomial of Q , that is a product of vertices of G , is not in I_G then that set of vertices cannot induce an edge in G , therefore it is an independent set. This means that the Stanley-Reisner complex of our edge ideal in the Kruskal-Katona ring is the independence complex of our graph G . If we can show that the edge ideal of a graph is Gotzmann in Q , then this means that each F_i , $i \geq 0$, for the independence complex has the largest entries in the F -vector when compared to any other graph of order n and size m and thus our graph has an independence polynomial that is most optimal for $\mathcal{S}_{n,m}$. We will use some facts about ideals which are known to be Gotzmann in the Kruskal-Katona ring.

Definition 4.3.8 [63] Let I be an ideal in a ring R with u and v being monomials of the same degree, with v in I . Then I is a lexicographic ideal if when $u >_{\text{lex}} v$ we have that u is also in I .

It is known that lexicographic ideals are Gotzmann in the polynomial ring and the Kruskal-Katona ring [59]. We assume the ordering $x_1 > x_2 > \dots > x_n$, where the x_i represent the vertices of G . We will show that the graph on n vertices and m edges, with edges added lexicographically, has an edge ideal that is lexicographic and hence Gotzmann in the Kruskal-Katona ring.

Theorem 4.3.9 Let G be a graph of size m with vertex set $\{x_1, x_2, \dots, x_n\}$, whose edges are added in lexicographic order. Then the edge ideal I_G of G in the Kruskal-Katona ring is a lexicographic ideal and hence Gotzmann.

Proof. Let G be a graph of size m and order n , with vertices $\{x_1, x_2, \dots, x_n\}$, whose edges are added in lexicographic order. Let Q be the Kruskal-Katona ring,

$Q = \mathbf{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ and I_G the edge ideal of G . Let u and v be monomials of Q of degree d , with $u \in I_G$. Suppose that $v >_{lex} u$.

First assume $d = 2$. Let $u = x_i x_j$, $i < j$. Since $v >_{lex} u$ if $v = x_i x_k$, $k < j$, this implies that v is in our edge ideal since v is an edge of G . The other option is that $v = x_l x_s$, where $l < i$ which again implies that $v \in I_G$, since v would be an edge of G , since edges are added in lexicographic order.

Now assume $d > 2$. Suppose that $v >_{lex} u$, however, a generator of I_G , that is, an edge of G , does not divide v . Then this means that the vertices of G which divide v , form an independent set of G . Since we have added our edges in lexicographic order, then this would imply that $u >_{lex} v$, which is a contradiction thus v is in our edge ideal and I_G is a lexicographic ideal, so I_G is a Gotzmann ideal in Q . $\square$

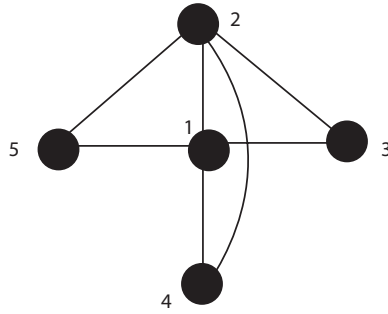


Figure 4.6: Example of a graph with edges added lexicographically using the vertex ordering $1 > 2 > 3 > 4$

For $m \leq n - 1$ the graphs that are formed by adding edges in lexicographic order are $K_{1,m}$ with possible isolated vertices. These were shown to be Gotzmann edge ideals in the polynomial ring (and in the Kruskal-Katona ring) in [46]. For $m > n - 1$ we start with a $K_{1,n-1}$ and the additional edges added lexicographically (see Figure 4.6). For these graphs, we know for given n vertices and m edges, so given F_1 and F_2 , that each F_i , $3 \leq i \leq n$ is maximal, since their edge ideals are Gotzmann in the Kruskal-Katona ring and thus have smallest Hilbert functions in each graded component of Q . This gives the following result.

Theorem 4.3.10 *A graph with the least optimal 2-clique reliability polynomial exists. For $m \leq n - 1$ the least optimal graph is a $K_{1,m}$, possibly with isolated vertices, and for $n - 1 < m \leq \binom{n}{2}$ the least optimal graph is a $K_{1,n-1}$, with additional edges added lexicographically.*

This also provides a result regarding most optimal independence polynomials.

Corollary 4.3.11 *A graph with the most optimal independence polynomial exists. For $m \leq n - 1$ the most optimal graph is a $K_{1,m}$, possibly with isolated vertices, and for $n - 1 < m \leq \binom{n}{2}$ the most optimal graph is a $K_{1,n-1}$, with additional edges added lexicographically.*

When looking for least optimal k -clique reliability polynomials for $k > 2$, the technique used for $k = 2$ will not work. This is because the nice thing about the case $k = 2$ is that we knew for a given n and m how many faces of size 1 and size 2 our independence (Stanly-Reisner) complex would have, but this is not the case for $k > 2$.

For values of $k > 2$, if m is such that a subgraph of a Turan graph $T(n, k - 1)$, (a complete $(k - 1)$ -partite graph) can be made, then this graph has the least optimal k -clique reliability polynomial, as $\text{crel}_k(T(n, k - 1), p) \equiv 0$.

We will now focus our attention to most optimal k -clique reliability polynomials. We start by looking at the existence of graphs with the most optimal 2-clique reliability polynomial. This corresponds to finding a graph with the least optimal independence polynomial. For any graph G on n vertices and m edges, we know that the faces of the independence complex is such that $F_0 = 1, F_1 = n, F_2 = \binom{n}{2} - m$. So, if we have a graph that has no independent sets of size 3 then the independence polynomial is $1 + nx + (\binom{n}{2} - m)x^2$. When a graph G with m edges has independence number 2, we know that adding edges to G decreases F_2 by 1 for each additional edge and that this graph has an independence complex with the smallest possible

F_i 's, given n and m . This means that we would like to know what is the least number, m_l , of edges so that there exists a graph on m_l edges that does not have an independence set of size 3 and what such a graph looks like.

Let G be a graph on n vertices. Suppose we have no independence sets of size 3. This means the vertex set of G can be partitioned into two cliques, one of size n_1 and one of size n_2 . Assume $n_1 \geq n_2$, so $n_1 = n_2 + r$ where $r \geq 0$.

$$m_l = \binom{n_1}{2} + \binom{n_2}{2} = n_2^2 + (r-1)n_2 + \frac{r(r-1)}{2}.$$

We want to minimize m_l , so we would like $r = 0$ or $r = 1$. If $r = 0$ then $m_l = n_2^2 - n_2$, and if $r = 1$ then $m_l = n_2^2$. We know that $n = 2n_2 + r$, so if n is divisible by 2 ($r = 0$) then G is $2K_{n_2}$, and if n is not divisible by 2 ($r = 1$) then G is $K_{n_2} \cup K_{n_2+1}$, with edges added as needed to achieve m edges.

Overall, what all this means is that for a given $n \geq 2$, $n = 2d + r$, $r \in \{0, 1\}$, we have that if $r = 0$ for $m \geq d^2 - d$, the graph with the least optimal independence polynomial is $2K_d$ with edges added as needed to achieve m edges. If $r = 1$ for $m \geq d^2$, the graph with the least optimal independence polynomial is $K_d \cup K_{d+1}$.

To find other families of least optimal independence polynomials (that is, most optimal 2-clique reliability polynomials) we will look at a graph operation which can be done to increase the value of the independence polynomial on $[0, \infty)$. Let G_1 be a graph which consists of a subgraph G (which may have more than one component) and a connected component, which is K_2 on vertices y, z . Let v be a vertex of G .

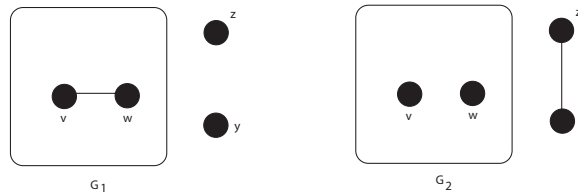


Figure 4.7: Graphs for Lemma 4.3.12. Let H_1 be the graph $G_1 \cup \{z, y\}$ and H_2 be the graph $G_2 \cup \{z, y\}$

Consider the graphs in Figure 4.7. We will show that shifting an edge from a K_2 to elsewhere in the graph will increase the independence polynomial.

Lemma 4.3.12 *For the graphs in Figure 4.7, we have $I(H_1, x) \geq I(H_2, x)$ on $[0, \infty)$.*

Proof. Note that

$$\begin{aligned} I(H_1, x) &= (1 + 2x + x^2)I(G_1, x) \\ &= (1 + 2x + x^2)(I(G_1 - v, x) + xI(G_1 - [v], x)) \\ &= (1 + 2x + x^2)(I(G_2 - v, x) + xI(G_2 - [v] - w, x)) \end{aligned}$$

and

$$I(H_2, x) = (1 + 2x)(I(G_2 - v, x) + xI(G_2 - [v], x))$$

We also have that

$$I(G_1 - [v], x) \leq (1 + x)I(G_2 - [v] - w, x) \quad (4.1)$$

since every independent set of $G_2 - [v] - w$ is an independent set of $G_1 - [v]$, but we could also put w in each set to obtain a new independent set.

Consider $F(x) = I(H_1, x) - I(H_2, x)$. Using the inequality (4.1) we can see that,

$$\begin{aligned} F(x) &= (1 + 2x)I(G_2 - v, x) + x^2I(G_2 - v, x) + \\ &\quad x(1 + 2x + x^2)I(G_2 - [v] - w, x) - (1 + 2x)I(G_2 - v, x) \\ &\quad - x(1 + 2x)I(G_2 - [v], x) \\ &= x^2I(G_2 - v, x) + x(1 + x)I(G_2 - [v] - w, x) \\ &\quad + x^2(1 + x)I(G_2 - [v] - w, x) - x(1 + 2x)I(G_2 - [v], x) \\ &\geq x^2I(G_2 - v, x) + xI(G_2 - [v], x) \end{aligned}$$

$$\begin{aligned}
& +x^2I(G_2 - [v], x) - xI(G_2 - [v], x) - 2x^2I(G_2 - [v], x) \\
& = x^2I(G_2 - v, x) - x^2I(G_2 - [v], x)
\end{aligned}$$

and since $I(G_2 - v, x) \geq I(G_2 - [v], x)$, we have $F(x) \geq 0$, so $I(H_1, x) \geq I(H_2, x)$. $\square$

Thus, if $m \leq n-1$, then the graph with the least optimal independence polynomial is the disjoint union of mK_2 and $(n-2m)K_1$. This gives us the following result.

Theorem 4.3.13 *Given n , if $m \leq \frac{n}{2}$, the graph with the most optimal 2-clique reliability polynomial consists of the disjoint union of mK_2 and $(n-2m)K_1$.*

This also provides a result regarding least optimal independence polynomial.

Corollary 4.3.14 *Given n if $m \leq \frac{n}{2}$, the graph with the least optimal independence polynomial consists of the disjoint union of mK_2 and $(n-2m)K_1$.*

We will now look at most optimal k -clique reliability polynomials for $k > 2$. We have seen in Lemma 4.3.1 that if a most optimal graph exists, then it must have the maximum number of k -cliques, since it will be most optimal for values of p near 0. In [19] it was shown that for a graph on n vertices and $m = \binom{d}{2} + r$ edges, the maximum number of cliques of size k , $k \geq 3$ is $\binom{d}{k} + \binom{r}{k-1}$. A graph which achieves such bounds is the graph which consists of a K_d and a vertex, x , with $N(x) \subseteq V(K_d)$ and $n-d-1$ isolated vertices (see Figure 4.8). This graph is not unique for values of $r < k-1$, since the addition of fewer than $k-1$ edges will not produce another k -clique, but for values of $r \geq k-1$, this graph is unique, since to obtain the maximum number of k -cliques, the edges will have to be added to the same vertex.

Theorem 4.3.15 *Given n and $m = \binom{d}{2} + r$, $0 \leq r \leq d-1$, if a graph with a most optimal k -clique reliability polynomial exists, then for $0 \leq r \leq k-2$ it is the graph which consists of a K_d , and the remaining edges added as needed. If $k-1 \leq r \leq$*

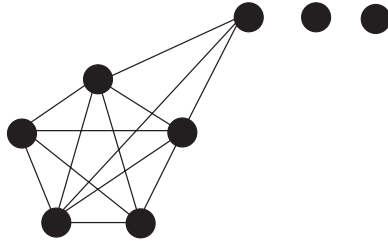


Figure 4.8: The graph with $n = 8$, $m = 13$ that has the most optimal 3 and 4-clique reliability polynomial.

$d - 1$ then it is the graph which is K_d with a vertex v of degree r , $N(v) \subseteq K_d$ and $(n - d - 1)K_1$ (see Figure 4.8).

4.4 Analytic Properties of k -Clique Reliability Polynomials

For all terminal reliability, we studied several analytic properties, namely thresholds, internal fixed points, inflection points, average reliability, and the location of the roots. We will now extend several of the results obtained in that section to k -clique reliability.

Recall, when looking at all terminal reliability we constructed graphs $G[H]$ which replaced each edge of G with a copy of H . We can do a similar type of operation on the vertices of G . That is we can replace each vertex of G by a copy of H to obtain the graph $G \times H$. This is the lexicographic product of G and H .

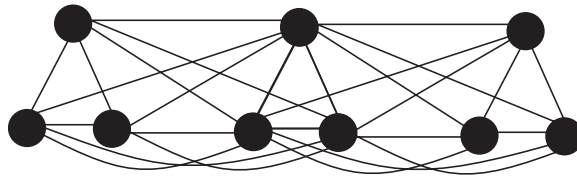


Figure 4.9: The lexicographic product of P_3 and C_3 , $P_3 \times C_3$.

Definition 4.4.1 Let G and H be graphs. The lexicographic product, $G \times H$ of G and H is the graph whose vertex set is the cartesian product of $V(G)$ and $V(H)$ and

is such that vertices (u, v) and (x, y) are adjacent if and only if u is adjacent to x in G or $u = x$ and v is adjacent to y in H .

We will be using complete graphs to extend some of the analytic property results from the all terminal reliability polynomial to the k -clique reliability polynomial. The k -clique reliability of the graph K_{k+1} is

$$\begin{aligned} \text{crel}_k(K_{k+1}, p) &= (k+1)p^k(1-p) + p^{k+1} \\ &= (-k)p^{k+1} + (k+1)p^k. \end{aligned}$$

The graph $K_{k+1} \times \overline{K_n}$ has the k -clique reliability polynomial of

$$\text{crel}_k(K_{k+1} \times \overline{K_n}, p) = (-k)(1 - (1-p)^n)^{k+1} + (k+1)(1 - (1-p)^n)^k,$$

since a subgraph of $K_{k+1} \times \overline{K_n}$, which contains at least a k -clique, comes from a subgraph of K_{k+1} (which contains a k -clique), where each operational vertex has at least one of the vertices in $\overline{K_n}$ operational.

From this, we can see that

$$\text{crel}_{m^k-1}(K_{m^k} \times \overline{K_{kj}}, p) = (1 - m^k)(1 - (1-p)^{kj})^{m^k} + m^k(1 - (1-p)^{kj})^{m^k-1}.$$

Letting $1 - p = q$ we obtain the same reliability polynomial as that of the cycle bundles, namely $(1 - m^k)(1 - q^{kj})^{m^k} + m^k(1 - q^{kj})^{m^k-1}$. This gives us the following results

Theorem 4.4.2 *Let $\mathcal{F}_{m,j} = \{K_{m^k} \times \overline{K_{kj}} \mid k \geq 1\}$ and let $t = 1 - \left(\frac{1}{m}\right)^{1/j}$. Then*

- $\mathcal{F}_{m,j}$ is a t -threshold family of graphs and the set of all such t is dense in $[0, 1]$.
- $\mathcal{F}_{m,j}$ has an internal fixed point tending to t and hence the internal fixed points of k -clique reliability polynomials over all k are dense in $[0, 1]$.

- $\mathcal{F}_{m,j}$ has an inflection point approaching t and hence their inflection points ranging over all m and k are dense in $[0, 1]$.
- The integrals, $\int_0^1 \text{crel}_{m^k-1}(K_{m^k} \times \overline{K_{kj}}, p) \, dp$ approaches t and hence the average reliabilities of this family is dense in $[0, 1]$

We will now look at another analytic property of k -clique reliability polynomials, the location of the roots. Even for a very simple family of graphs, the complete graphs, the location of the roots is nontrivial.

Recall that

$$\text{crel}_k(G, p) = (1 - p)^n \text{cgen}_k(G, \frac{p}{1-p}).$$

It follows that the roots of the k -clique reliability polynomial are the roots of the k -clique generating polynomial, under the linear fractional transformation $g(z) = z/(1 + z)$. Note that the circle $|z + 1| = R$ under $g(z)$ is transformed into the circle $|z - 1| = 1/R$, with the insides and outsides flipped.

Most often, the roots of graph polynomials turn out to be trivial for the simple family of complete graphs. For example, the chromatic polynomial of K_n is $\prod_{i=0}^{n-1} (x - i)$ and hence it has roots at the integers $0, 1, \dots, n - 1$. The independence polynomial of K_n is $1 + nx$, which has a single root at $x = -1/n$. The k -clique reliability of a complete graph of order n is

$$1 - \sum_{i=0}^{k-1} \binom{n}{i} p^i (1 - p)^{n-i}$$

and the k -clique generating polynomial of the complete graph K_n is given by

$$\text{cgen}_k(K_n, x) = (1 + x)^n - \sum_{i=0}^{k-1} \binom{n}{i} x^i,$$

which is just the binomial expansion with the lower terms removed. The location of the roots of the polynomial consisting of just the lower terms was studied in [49], but we need to investigate the binomial expansion where the lower terms are *truncated* from the polynomial.

Our first result is a tight annulus that contains the roots of the k -clique reliability polynomials of complete graphs.

Theorem 4.4.3 *For $k \geq 1$ the roots of $\text{crel}_k(K_n, z)$ are found in the annulus $1/k \leq |z - 1| \leq 1$, with roots occurring on the boundary.*

Proof. We show an equivalent result for $\text{cgen}_k(K_n, z)$, namely that the roots of $\text{cgen}_k(K_n, z)$ are in $1 \leq |z + 1| \leq k$, with roots on the boundary.

For K_n , we have that

$$\text{cgen}_k(K_n, z) = (1 + z)^n - \sum_{i=0}^{k-1} \binom{n}{i} z^i.$$

Setting $y = z + 1$, we now consider the roots of the polynomial

$$\begin{aligned} f(K_n, y) &= y^n - \sum_{i=0}^{k-1} \binom{n}{i} (y - 1)^i \\ &= \sum_{i=k}^n \binom{n}{i} (y - 1)^i \\ &= (y - 1)^k \sum_{i=k}^n \binom{n}{i} (y - 1)^{i-k} \\ &= (y - 1)^k \sum_{i=0}^{n-k} \binom{n}{k+i} \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} y^j. \end{aligned}$$

As $(y - 1)^k$ clearly gives us a root of 1, we are interested in the roots of

$$\sum_{i=0}^{n-k} \binom{n}{k+i} \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} y^j$$

Looking at this sum, we get that the coefficient of y^j is

$$\sum_{i=j}^{n-k} \binom{n}{k+i} \binom{i}{j} (-1)^{i-j}.$$

Simplifying this in Maple yields

$$\binom{n}{k+j} \frac{\Gamma(k+j+1)\Gamma(n-j)}{\Gamma(k)\Gamma(n+1)} = \frac{(n-j-1)!}{(n-k-j)!(k-1)!},$$

which gives us that

$$f(K_n, y) = \frac{(y-1)^k}{(k-1)!} \sum_{i=0}^{n-k} \frac{(n-i-1)!}{(n-k-i)!} y^i.$$

Let

$$g(y) = \sum_{i=0}^{n-k} a_i y^i = \frac{(k-1)! f(K_n, y)}{(y-1)^k},$$

where $a_i = \frac{(n-i-1)!}{(n-i-k)!}$.

To locate the roots of $g(y)$, recall Theorem 3.5.3, (the Eneström-Kakeya Theorem) which states that a polynomial $\sum b_i x^i$ with positive coefficients, b_i , has its roots in the annulus $\min\{b_i/b_{i+1}\} \leq |z| \leq \max\{b_i/b_{i+1}\}$. Since $\frac{a_i}{a_{i+1}} = \frac{n-i-1}{n-i-k} \geq 1$ we get that any root r of $g(y)$ satisfies $|r| \geq 1$. Also, a_i/a_{i+1} achieves a maximum value of k when $i = n - k - 1$, so the Eneström-Kakeya Theorem also gives us that $|r| \leq k$. Thus the zeros of $\text{cgen}_k(K_n, z)$ lie in the annulus $1 \leq |z+1| \leq k$, and hence (via the linear fractional transformation $z \mapsto z/(z+1)$), the roots of $\text{crel}_k(K_n, z)$ lie in the annulus $1/k \leq |z-1| \leq 1$. When $n = k+1$, $\text{cgen}_k(K_{k+1}, z) = z^k(k+1+z)$, which gives us zeros of 0 and $-k-1$, which occur on the boundary (and these yield zeros of 0 and $1+1/k$ which occur on the boundary of the annulus for $\text{crel}_k(K_n, z)$). $\square$

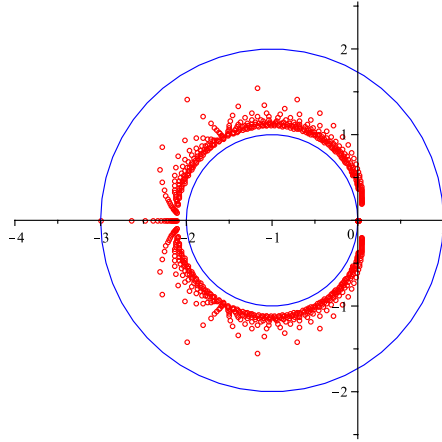


Figure 4.10: Roots of $\text{cgen}_2(K_n, z)$ for $2 \leq n \leq 40$.

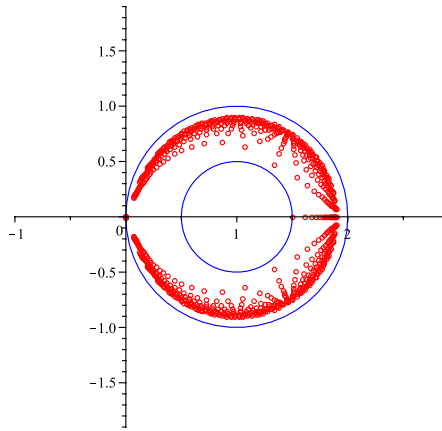


Figure 4.11: Roots of $\text{crel}_2(K_n, z)$ for $2 \leq n \leq 40$.

Clearly, the annulus that contains the roots of the k -clique reliability polynomials of complete graphs cannot be improved, but if we are interested in an asymptotic result, we find that the roots approach one boundary of the annulus.

Theorem 4.4.4 *For a fixed $k \geq 1$ and $\epsilon > 0$, the roots of $\text{crel}_k(K_n, z)$ lie in the annulus $1 - \epsilon \leq |z - 1| \leq 1$, provided that n is large enough.*

Proof. We show, equivalently, that for a fixed positive integer k and $\epsilon > 0$, the roots of $\text{cgen}_k(K_n, z)$ lie in the disk $|z + 1| = 1 + \epsilon$, provided that n is large enough.

We have that

$$\text{cgen}_k(K_n, z) = (1+z)^n - \sum_{i=0}^{k-1} \binom{n}{i} z^i.$$

Rouché's theorem (see, for example, [73]) states that if two complex-valued functions f and g are analytic inside and on some closed simple contour γ with $|g(z)| < |f(z)|$ for all z on γ , then f and $f+g$ have the same number of zeros, counting multiplicities, in γ . To make use of this theorem, let $f(z) = (1+z)^n$, $g(z) = -\sum_{i=0}^{k-1} \binom{n}{i} z^i$ and take γ to be the circle $|z+1| = 1+\epsilon$.

Clearly, $f(z)$ has all of its roots in γ so we would like to show that on γ , we have $|g(z)| < |f(z)|$. First, we can see that on γ , $|f(z)| = (1+\epsilon)^n$, and that

$$|g(z)| = \left| \sum_{i=0}^{k-1} \binom{n}{i} z^i \right| \leq \sum_{i=0}^{k-1} \binom{n}{i} |z|^i \leq \sum_{i=0}^{k-1} \binom{n}{i} (2+\epsilon)^i.$$

This means that if we want $|g(z)| < |f(z)|$ on γ then it is sufficient that

$$(1+\epsilon)^n > \sum_{i=0}^{k-1} \binom{n}{i} (2+\epsilon)^i,$$

which is certainly true for n sufficiently large, as k is fixed. □

One may wonder whether the bounding annuli in Theorem 4.4.3 extend to all graphs. In particular, for complete graphs the k -clique reliability polynomial has all its roots outside $|z-1| = 1/k$, so one may venture to conjecture that $\text{crel}_k(G, z)$ has all its roots in $|z-1| \geq 1/k$, or equivalently that $\text{cgen}_k(G, z)$ for any graph G has its roots in $|z+1| \leq k$. The next result will be used to demonstrate that this is not the case.

Theorem 4.4.5 *Let G and H be graphs on disjoint vertex sets. Then for disjoint graphs G and H ,*

$$\begin{aligned} \text{cgen}_k(G \cup H, z) &= \text{cgen}_k(G, z)(1+z)^{|V(H)|} + \text{cgen}_k(H, z)(1+z)^{|V(G)|} \\ &\quad - \text{cgen}_k(G, z)\text{cgen}_k(H, z). \end{aligned}$$

Proof. Since G and H are on disjoint vertex sets, in order for $G \cup H$ to have an induced subgraph, which contains a clique of size k , we need either G or H to have a clique of size k operational. Now $\text{cgen}_k(G, z)(1+z)^{|V(H)|}$ enumerates subsets of vertices for which the part in G contains a k -clique, and similarly, $\text{cgen}_k(H, z)(1+z)^{|V(G)|}$ enumerates subsets of vertices for which the part in H contains a k -clique. What we have over-counted are those subsets of vertices that contain a k -clique in both G and H , and $\text{cgen}_k(G, z)\text{cgen}_k(H, z)$ enumerates those. The result now follows. $\square$

As a consequence of this theorem we get that

$$\text{cgen}_k(2K_k, z) = 2(1+z)^k z^k - z^{2k} = z^k(2(1+z)^k - z^k)$$

and this has a root at $\frac{2^{1/k}}{1-2^{1/k}}$. Since

$$2^{1/k} + 2^{2/k} + \dots + 2^{k/k} = 2^{1/k} (2^{0/k} + 2^{1/k} + \dots + 2^{k-1/k}) = \frac{2^{1/k}}{2^{1/k} - 1}$$

and $2^{i/k} > 1$ for $1 \leq i \leq k-1$ and $2^{k/k} = 2$, we see that $\frac{2^{1/k}}{2^{1/k}-1} > k+1$. It follows that $\frac{2^{1/k}}{1-2^{1/k}}$ is less than $-k-1$ and is therefore outside the disk $|z+1| = k$. This means that $\text{crel}_k(2K_k, z)$ has a root z that lies in $|z-1| < 1/k$.

As Figure 4.12 shows, the roots of $\text{cgen}_2(G, z)$ can lie outside the disk $|z+1| = 2$, but they do not appear to be that far from -3 , so perhaps these roots are still

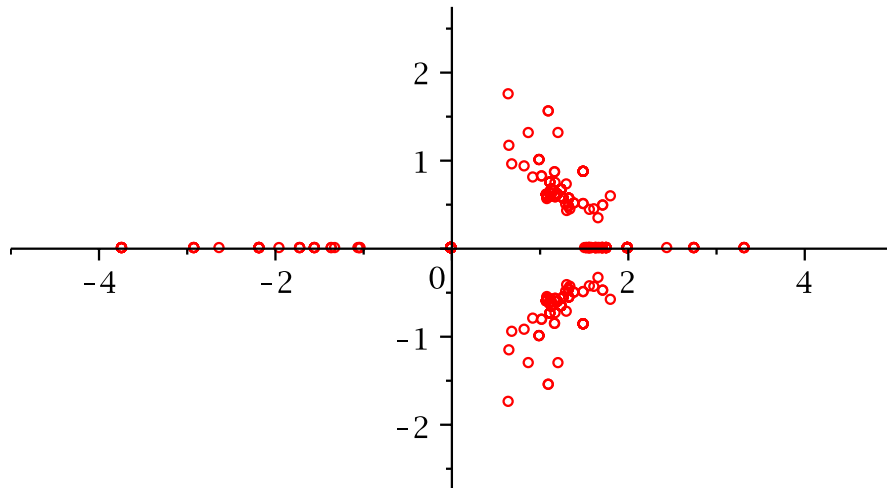


Figure 4.12: Roots of $\text{crel}_2(G, z)$ close to $z = 0$ for all connected graphs on 6 vertices.

bounded in modulus. However, as the next result shows, the roots of the k -clique reliability polynomial for any $k \geq 2$ are unbounded in moduli.

Theorem 4.4.6 *For $k = 1$, the roots of the k -clique reliability polynomials lie on the circle $|z - 1| = 1$ (and in fact, the closure of their roots is this circle). For $k \geq 2$, the moduli of the roots of k -clique reliability polynomials are unbounded.*

Proof. For $k = 1$, note that for a graph G of order n , $\text{crel}_1(G, p) = 1 - (1 - p)^n$, which has all of its roots on the circle $|z - 1| = 1$ (and, moreover, over all n the roots are dense on this circle).

For $k = 2$, consider $\text{cgen}_2(P_n, x)$ where P_n is the path of order n . Now

$$\text{cgen}_2(P_n, x) = (1 + x)^n - \text{cfree}_2(P_n, x),$$

where $\text{cfree}_2(P_n, x) = I(P_n, x)$ is the independence polynomial of G . In [26] it was shown that we can write the independence polynomial of paths as

$$I(P_n, x) = \beta_1 \lambda_1^n + \beta_2 \lambda_2^n,$$

where

$$\lambda_1(x) = \frac{1 + \sqrt{1 + 4x}}{2}, \quad \lambda_2(x) = \frac{1 - \sqrt{1 + 4x}}{2}$$

and

$$\beta_1(x) = \frac{\sqrt{1 + 4x} + (1 + 2x)}{2\sqrt{1 + 4x}}, \quad \beta_2(x) = \frac{\sqrt{1 + 4x} - (1 + 2x)}{2\sqrt{1 + 4x}}.$$

It follows that

$$\text{cgen}_2(P_n, x) = \alpha_0 \lambda_0^n + \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n, \tag{4.2}$$

where $\alpha_0 = 1$, $\alpha_1 = -\beta_1$, $\alpha_2 = -\beta_2$, and $\lambda_0 = 1 + x$.

Beraha, Kahane, and Weiss studied the *limit of zeros* of such functions (as arising in recurrences); a *limit of zeros* of a family of polynomials $\{P_n\}$ is a complex number z for which there are sequences of integers (n_k) and complex numbers (z_k) such that z_k is a zero of P_{n_k} , and $z_k \rightarrow z$ as $k \rightarrow \infty$. The theorem of Beraha, Kahane, and Weiss (BKW Theorem) [8] (see the original for the full statement) also requires some nondegeneracy conditions : as no α_i is identically 0, and as it is clearly not the case that $\lambda_i = \omega \lambda_k$ for any $i \neq k$ and any root of unity ω , these conditions hold. The BKW Theorem implies that the limit of zeros of (4.2) are precisely those complex numbers z such that either

- one of the $|\lambda_i(z)|$ exceeds the others, and $\alpha_i(z) = 0$, or
- $|\lambda_i(z)| = |\lambda_j(z)| > |\lambda_k|$ for $\{i, j, k\} = \{0, 1, 2\}$.

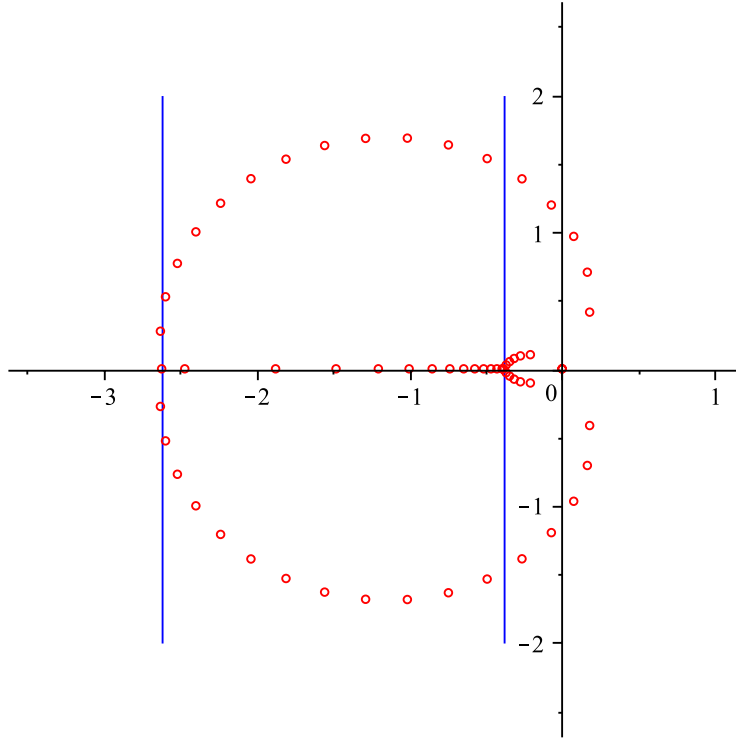


Figure 4.13: Roots of $\text{cgen}_2(P_{60}, x)$

Using the BKW Theorem to consider the limits of the roots of $\text{cgen}_2(P_n, z)$, we will look at the limit of the roots when $|\lambda_1| = |\lambda_2| > |\lambda_0|$. In [26] it was shown that $|\lambda_1| = |\lambda_2|$ precisely for $z \in (-\infty, -1/4]$. Now we get limit points for the roots of $\text{cgen}_2(P_n, z)$, by the BKW Theorem, whenever $|\lambda_1(z)| > |\lambda_0(z)|$ on this interval, that is, whenever

$$\left| \frac{1 + \sqrt{1 + 4z}}{2} \right| > |1 + z| \quad (4.3)$$

for $z \in (-\infty, -1/4]$.

Setting $z = -r$, where $r \geq 1/4$, we see that $1 + 4z = 1 - 4r \leq 0$. It follows that

$$\left| \frac{1 + \sqrt{1 + 4z}}{2} \right| = \sqrt{r}.$$

Now $|1 + z| = 1 - r$ if $r \leq 1$, and is $r - 1$ otherwise. Thus for $r \in [1/4, 1]$, (4.3) is equivalent to $1 - r < \sqrt{r}$, that is, $r^2 - 3r + 1 < 0$; for $r \geq 1$ (4.3) is equivalent to the same inequality. As the inequality is a parabola opening upwards, we need to be between the roots, and a small calculation shows that (4.3) on $(-\infty, -1/4]$ holds on the interval

$$\left(-\frac{3 + \sqrt{5}}{2}, -\frac{3 - \sqrt{5}}{2} \right)$$

(see Figure 4.13). As this interval contains -1 , under the transformation $z \mapsto z/(z + 1)$, there are roots of $\text{crel}_2(P_n, p)$ that have unbounded moduli. This proves the $k = 2$ case.

For $k \geq 3$, note that $K_{k-2} + P_n$, the graph formed from the disjoint union of K_{k-2} and P_n by adding in all edges between them, has

$$\text{crel}_k(K_{k-2} + P_n, p) = p^{k-2} \text{crel}_2(P_n, p)$$

as the k -cliques are precisely those subsets with the vertices of K_{k-2} and the endpoints of an edge of P_n (as P_n has no triangles). Thus the roots of k -clique reliability polynomials, for $k \geq 3$, contain the roots of the 2-clique reliability polynomials of paths, so we are done by the $k = 2$ case. $\square$

In addition to proving the existence of roots of arbitrarily large modulus, we can also show that the entire complex plane is the closure of the roots $\text{cgen}_k(G, x)$, over all k , and hence is also the closure of the roots of $\text{crel}_k(G, p)$ over all k .

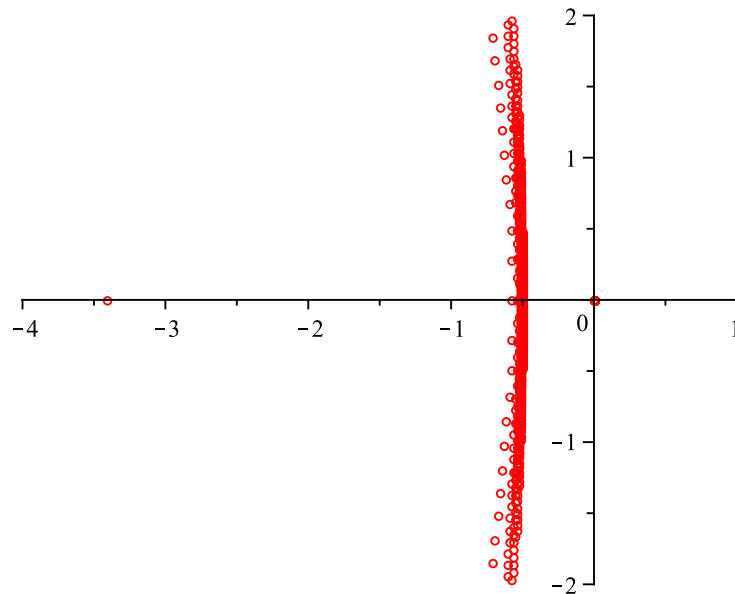


Figure 4.14: Roots of $\text{crel}_k(2K_k, z)$ for $2 \leq k \leq 40$ close to $\Re(z) = -1/2$

Theorem 4.4.7 *The closure of the roots of $\text{crel}_k(G, p)$, over all k is the entire complex plane.*

Proof. We first show that the closure of the roots of $\text{cgen}_k(G, x)$ over all k is the entire complex plane.

To begin, consider the graph $G_n \times \overline{K_m}$. Let $\{v_1, v_2, \dots, v_n\}$ be the vertices of G_n and let V_i be the set of the m vertices of the copy of $\overline{K_m}$ which replace v_i in $G_n \times \overline{K_m}$. The subgraphs of $G_n \times \overline{K_m}$ that contain a k -clique arise precisely from subgraphs H of G_n that contain a k -clique by replacing each vertex v_i of H by a nonempty subset of V_i . It follows that

$$\text{cgen}_k(G_n \times \overline{K_m}, x) = \text{cgen}_k(G_n, (1+x)^m - 1)$$

Using the lexicographic product of paths and $\overline{K_m}$ and of $2K_k$ and $\overline{K_m}$, we will show that the closure of

$$\{|z + 1| : z \text{ is a root of a } k\text{-clique generating polynomial}\}$$

is $[0, \infty)$ and then proceed to show that the closure of the roots is indeed the entire complex plane.

The graph $2K_k$ has the k -clique generating polynomial,

$$\text{cgen}_k(2K_k, x) = 2(x(1 + x))^k - x^{2k}$$

and as Figure 4.14 shows, the roots appear to be dense along $\Re(z) = -1/2$. Using the BKW Theorem, we can show this to be true for $\text{cgen}_k(2K_2, x)$, since $\alpha_0(z) = 2$, $\lambda_0(z) = z(1 + z)$, $\alpha_1(z) = 1$, $\lambda_1(z) = z^2$ and $|\lambda_0(z)| = |\lambda_1(z)|$ when $|z + 1| = |z|$ which is true when $\Re(z) = -1/2$. It follows that $|z + 1|$, ranging over all roots of k -clique generating polynomials, is dense in $[1/2, \infty)$.

From the proof of Theorem 4.4.6, we saw that the roots of $\text{cgen}_k(P_n, x)$ are dense in $\left(-\frac{3+\sqrt{5}}{2}, -\frac{3-\sqrt{5}}{2}\right)$, so the closure of $|z + 1|$ contains the interval $[0, 1/2]$, and hence the set of all $|z + 1|$ as z ranges over the roots of $\text{cgen}_k(G, x)$ is $[0, \infty)$. We now pick any complex number z and any $\epsilon > 0$; without loss of generality, $z \neq 1$, $\epsilon < r = |z + 1|$ and $\epsilon < 1$. We will show that there is a root w of a k -clique generating polynomial, such that w is within ϵ of z .

We observe that the region consisting of all z' whose moduli lie within $\rho = \epsilon/\sqrt{5r^2 + 4r + 4}$ of r and whose arguments are within a band of size 2ρ has diameter at most ϵ . To see this, rotate the region so that it lies in the first quadrant, with one side on the positive x axis. A pair of points furthest apart have cartesian coordinates

$((r - \rho) \cos \rho, (r - \rho) \sin \rho)$ and $(r + \rho, 0)$, with a squared distance of

$$\begin{aligned}
& ((r + \rho) - (r - \rho) \cos \rho)^2 + ((r + \rho) \sin \rho)^2 \\
\leq & \left((r + \rho) - (r - \rho) \left(1 - \frac{\rho^2}{2} \right) \right)^2 + ((r + \rho) \rho)^2 \\
\leq & \left(2\rho + \left((r - \rho) \frac{\rho^2}{2} \right) \right)^2 + ((r + \rho) \rho)^2 \\
< & (2\rho + r\rho^2)^2 + ((r + \rho) \rho)^2 \\
\leq & (2\rho + r\rho)^2 + (2r\rho)^2 \\
\leq & \rho^2 (5r^2 + 4r + 4) \\
= & \epsilon^2.
\end{aligned}$$

(We have used the fact that $\cos x > 1 - x^2/2$ for all x and $\sin x < x$ for all positive x .) The result now follows.

We know that $\text{cgen}_k(G_n \times \overline{K_m}, z) = \text{cgen}_k(G, (z + 1)^m - 1)$, so if z_m is a root of $\text{cgen}_k(G, x)$ then $(z_m + 1)^{(1/m)} - 1$ is a root of $\text{cgen}_k(G_n \times \overline{K_m}, x)$, where $(z_m + 1)^{(1/m)}$ ranges over all m -th roots of $z_m + 1$. Set $r = |z + 1| > 0$. We fix m large enough so that at least one of the m -th roots of unity has an argument within $\epsilon/(\sqrt{5r^2 + 4r + 4})$ of the argument of $z + 1$. From above, we can pick w a root of a k -clique generating polynomial, such that $|w + 1|^{1/m}$ is within $\epsilon/(\sqrt{5r^2 + 4r + 4})$ of $|z + 1|$. One of the m -th roots of $w + 1$ will also have argument within $\epsilon/(\sqrt{5r^2 + 4r + 4})$ of the argument of $z + 1$. The region described is precisely the difference between two sectors mentioned in the previous paragraph, translated, and hence $z + 1$ lies within ϵ of one of the m -th roots of $w + 1$. Thus z lies within ϵ of one of $(w + 1)^{(1/m)} - 1$, and is therefore within ϵ of a root of a k -clique generating polynomial. As z and ϵ were arbitrary, we conclude that the closure of the roots of the k -clique generating polynomials, over all k , is the entire plane.

As the image of a dense set under a continuous surjective map is dense, we conclude that the roots of k -clique reliability polynomials are also dense in the plane, and we are done. $\square$

So, as in the case of the all terminal reliability, the thresholds, inflection points, internal fixed points and average reliability of k -clique reliability are dense in $[0, 1]$. Unlike all terminal reliability, we have proved that the roots of k -clique reliability polynomials are dense in the complex plane.

Chapter 5

Conclusions

In this chapter, we will discuss some of the relationships between the various topics looked at in this thesis and give some suggestions for further research.

5.1 All Terminal Reliability

We know that if a family of graphs $\mathcal{G}$ is a t -threshold family of graphs then it has an internal fixed point approaching t . For $\epsilon > 0$, we have that for $t - \epsilon$ we are close to 0 and for $t + \epsilon$ we are near 1, so by the mean value theorem, the derivative near the threshold can be very large. One may also expect such a family to have an inflection point there as well. For example, we know that cycles are a 1-threshold family of graphs, and they have an inflection point at $(n - 2)/(n - 1)$, which approaches 1 as n approaches infinity.

From computations on simple graphs on $n \leq 8$ vertices, it appears that internal fixed points, and inflection points often occur relatively close together, such as is the case with graph G_1 in Figure 5.1. This graph has the reliability polynomial $14p^8 - 72p^7 + 142p^6 - 128p^5 + 45p^4$ and has an internal fixed point at approximately 0.47949 and an inflection point at approximately 0.47929. However, inflection points and internal fixed points can also be far apart, such as with the graph G_2 in Figure 5.1, which has reliability polynomial of $9p^8 - 24p^7 + 16p^6$; it has an inflection point at approximately 0.78178 and an internal fixed point at 0.90225. This then raises the question, can we have graphs with internal fixed points as far apart from all the inflection points as desired? For families of graphs, what is the asymptotic behaviour

of the internal fixed points and inflection points? In particular, for t -threshold families of graphs, will they have inflection points approaching t ?

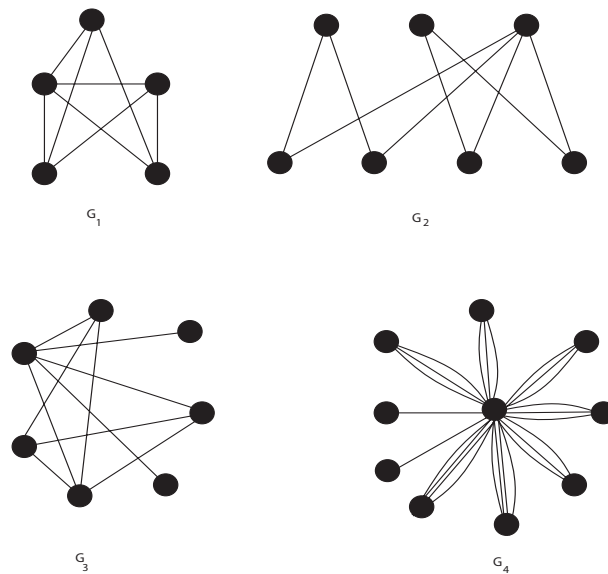


Figure 5.1: The graphs G_1 , G_2 , G_3 , G_4 .

Looking at inflection points, and specifically graphs with more than 1 inflection point, we see again that we can have 2 inflection points close together, like G_3 in Figure 5.1 with inflection points at approximately 0.81898 and 0.87866 or they can be farther apart, like G_4 in Figure 5.1, which has inflection points at approximately 0.79002 and 0.90369. Can we have a reliability polynomial whose inflection points are as close or as far apart as desired (on $(0, 1)$)?

These questions suggest that studying the analytic properties of the derivative of a nonzero all terminal reliability polynomial may be an interesting topic to pursue. The study of the derivative of the reliability polynomial is different from just studying the analytic properties of the all terminal reliability polynomial, since if a connected graph G has order at least 3, the derivative of the all terminal reliability polynomial

cannot be a reliability polynomial, by the following argument.

$$\text{Rel}'(G, p) = mp^{m-1} + F_1(-p^{m-1} + (m-1)p^{m-2}(1-p)) + \quad (5.1)$$

$$F_2(-2p^{m-2}(1-p) + (m-2)p^{m-3}(1-p)^2) + \dots \quad (5.2)$$

$$+ F_{m-n+1}((-)(m-n+1)p^{n-1}(1-p)^{m-n}$$

$$+ (n-1)p^{n-2}(1-p)^{m-n+1}$$

As we can see from Equation (5.1), $\text{Rel}'(G, 1) = m - F_1$ and if the derivative is a reliability polynomial, we'd expect $\text{Rel}'(G, 1) = 1$. This implies that we would need $F_1 = m - 1$, meaning G must have exactly one bridge. We saw in Theorem 3.3.3 that if a graph has edge connectivity 1, and has at least 3 vertices and exactly 1 bridge then it has an inflection point in $(0, 1)$, so the derivative is a decreasing function on some subinterval of $(0, 1)$. Thus it can not be a reliability polynomial.

As we saw in Chapter 2, for $n \geq 8$ there is more than one value of m such that a most optimal graph does not exist, but a characterization of the values of m for which a given n has no most optimal graph eludes us.

Without a complete characterization of when most optimal graphs exists, we turn to maximizing the average reliability as an alternative. Can we characterize what graphs will have the largest average reliability? Is it the graph that is most optimal for values of p near 0? near 1?

For the conditions studied in this thesis when $n \leq 8$ it was the case that either the graph most optimal near 1 or the graph most optimal near 0 had the largest average reliability. Of these two graphs, the one with the smallest internal fixed point was the graph in $\mathcal{S}_{n,m}$ with the largest average reliability. For instance, when $n = 6$, $m = 11$ the graph H_1 in Figure 5.2 has an internal fixed point of approximately 0.46112 and is most optimal for p near 0, while H_2 , which is most optimal for p near 1, has an internal fixed point of approximately 0.46059, which was the smallest internal fixed

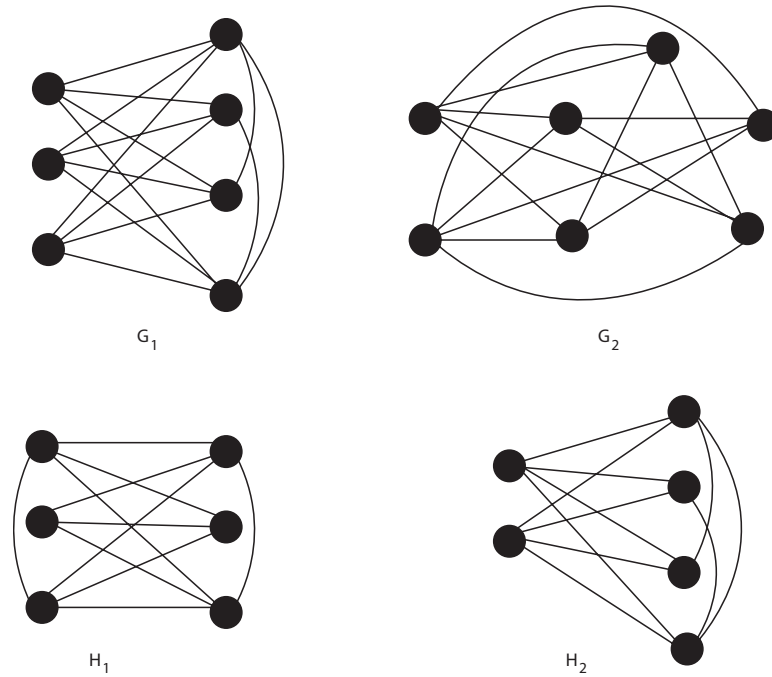


Figure 5.2: Optimal Graphs near $p = 0$ and $p = 1$ for their families

point in $\mathcal{S}_{6,11}$. It is H_2 which has the largest average reliability. Again, referring to Figure 5.2, for $n = 7$, $m = 15$ the graph G_1 , which is most optimal for p near 0, has an internal fixed point of approximately 0.40844 (smallest for $\mathcal{S}_{7,15}$), while G_2 , which is most optimal for p near 1, has one at approximately 0.40954. It is G_1 which has the largest average reliability.

This may lead one to conjecture that if a most optimal graph does not exist, the graph with the largest average reliability is either most optimal for p close to 0 or p close to 1, and of these graphs the one with the smallest internal fixed point has the largest average reliability. This is not the case. For $n = 8$ and $m = 14$, computations show that a most optimal graph does not exist, and even more interestingly, the graph with the largest average reliability is not the graph that is most optimal for values of p near 0 and it is not the graph that is most optimal for values of p near 1. Also, the pair $n = 8$ and $m = 14$ does not fall under the non-optimality conditions studied in this thesis. Refer to Figure 5.3. The graph G_1 is most optimal for values

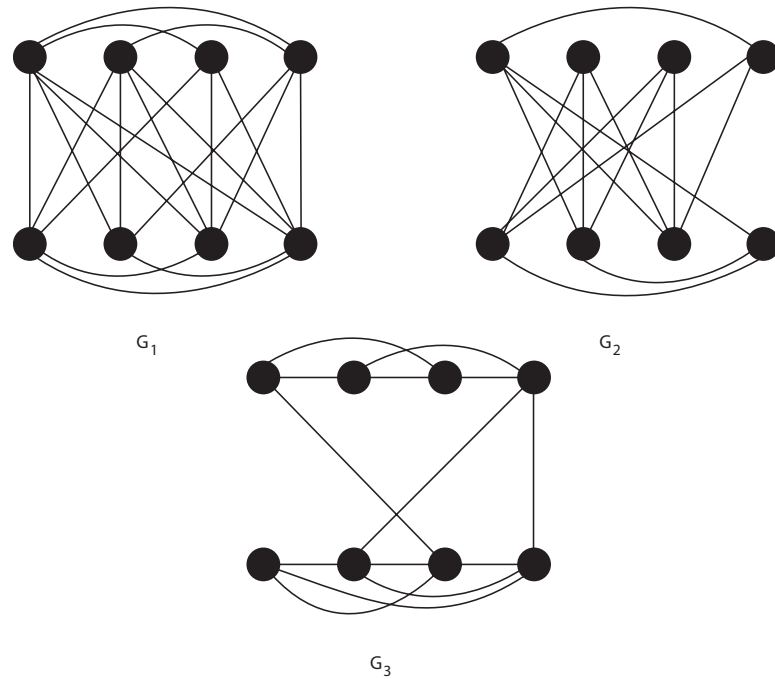


Figure 5.3: Potential graphs with large average reliability.

of p near 0 and has an internal fixed point of approximately 0.56815. The graph G_2 is most optimal for values of p near 1 and has an internal fixed point at approximately 0.56788, and lastly the graph G_3 has an internal fixed point of approximately 0.56782 (which is the smallest of all graphs $H \in \mathcal{S}_{8,14}$) and it has the largest average reliability. For the other n and m pairs with $n \leq 8$, where there is no most optimal simple graph, the graph with the largest average reliability is the graph $H \in \mathcal{S}_{n,m}$ with the smallest internal fixed point. This may be the case since if the internal fixed point is far to the left, then the reliability polynomial is above $y = p$ longer than the reliability polynomials of the other graphs and has more time to grow. In fact, for n and m where a most optimal simple graph does exist, since we know there is a graph G such that $\text{Rel}(G, p) \geq \text{Rel}(H, p)$ for all $H \in \mathcal{S}_{n,m}$, by Observation 3.2.7 the internal fixed point of G is the smallest for that given n and m . We also know that, given n and m , a least optimal graph always exists, and this graph has the smallest average reliability and an internal fixed point that is farthest to the right by Lemma 3.2.7.

Conjecture 5.1.1 *Let n and m be positive integers. The simple graph in $\mathcal{S}_{n,m}$ with the largest average reliability is the graph with the smallest internal fixed point over all graphs in $\mathcal{S}_{n,m}$.*

Another interesting observation is that for $n \leq 8$ all the simple graphs that are most optimal for $p = 1/2$ are also the graphs with the largest average reliability, and thus are also the graphs with the smallest internal fixed point. This means for n and m such that a most optimal simple graph does not exist and falls into one of the situations covered in this thesis, then the graph that is most optimal at $p = 1/2$ appears to be either the graph that is most optimal for p near 0 or p near 1, depending on which one has the smaller internal fixed point. For n and m where a most optimal simple graph does not exist and does not fall into one of the cases covered in the optimality section, then the graph that is most optimal at $p = 1/2$ does not appear to be the one that is most optimal for p near 0 or p near 1, but is still the graph with the smallest internal fixed point over all graphs in $\mathcal{S}_{n,m}$.

Conjecture 5.1.2 *Let $G \in \mathcal{S}_{n,m}$ be the graph that has the largest average reliability. Then for any other graph $H \in \mathcal{S}_{n,m}$, $Rel(G, 1/2) \geq Rel(H, 1/2)$ and $ifp(G) \leq ifp(H)$.*

(n, m)	Largest Average Reliability	Smallest Internal Fixed Point
(6, 11)	0.5177	0.4606
(7, 15)	0.5503	0.4084
(8, 14)	0.4567	0.5678
(8, 19)	0.5661	0.3866
(8, 22)	0.6125	0.3199
(8, 23)	0.6266	0.3015

Table 5.1: The Average Reliability and Internal Fixed Points for the optimal graphs in Figure 5.4

In Table 5.1 we can see the values for the average reliability and the internal fixed points for the graphs in Figure 5.4. It may also be of interest to find either a graph whose reliability polynomial has a root at -1 , so by Corollary 3.5.13 there is not a

fixed disk that contains the roots of the reliability polynomial, or to prove that there is a fixed disk, which contains the roots of the reliability polynomial.

5.2 k -clique Reliability

When looking at the k -clique reliability and the closure of the thresholds, internal fixed points, average reliability, and inflection points we took the union over all $k \geq 2$. It would be interesting to see if these results hold for a fixed k . Also, when investigating the roots, we showed that the closure was the entire complex plane, but again, we did not fix k . Therefore, finding a family of graphs for which, given a fixed k , the roots are dense in the complex plane would be a problem to pursue.

With regards to the optimality results surrounding k -clique reliability, we have a family of graphs which have the least optimal 2-clique reliability polynomials, but it would be nice to further investigate least optimal graphs for $k > 2$. When $k > 2$, if we know that a most optimal graph exists, then we know which graph it could be. However, for $k = 2$, whether a most optimal graph exists or not is an open problem which is interesting since it is related to least optimal independence polynomials on $[0, \infty)$. It may also be interesting to note that given n and m , the complement of the graph that has the least optimal 2-clique reliability polynomial for $\mathcal{S}_{n,m}$ is the conjectured most optimal graph for $k > 2$ in $\mathcal{S}_{n, \binom{n}{2} - m}$. This seems reasonable, as the least optimal graph for 2-clique reliability has the largest independence set, and the complement of that set is a clique.

We could also extend the concept of k -clique reliability and the idea of a local structure being the driving factor as to whether a network is reliable or not. Let G and H be graphs and assume that vertices operate independently with probability $p \in [0, 1]$. Let G be operational if at least a subgraph containing a copy of H is operational. Denote this by $\text{Rel}_H(G, p)$. When $H = K_k$, we have the k -clique

reliability and if the order of H is k , then the k -clique reliability is a lower bound for $\text{Rel}_H(G, p)$, since $H \subseteq K_k$.

5.3 Extending Results to Other Forms of Reliability

We saw that some of the analytic results regarding all terminal reliability could be extended to the k -clique reliability. These analytic properties can also be extended to other notions of network reliability, such as strongly connected reliability and two-terminal reliability to see what they say regarding the behaviour of those reliability polynomials on the interval $[0, 1]$.

Consider a tree having m^k vertices and replace each edge $\{x, y\}$ with two sets of bundles of size kj , one set directed from x to y , the other from y to x , to obtain a digraph. The strongly connected reliability for such a graph is $\text{Rel}(T_{m^k}^{kj}, q) = ((1 - q^{kj})^{m^k - 1})^2$. Using very similar proofs as those for the cycle bundles for all terminal reliability, we obtain the following results.

Corollary 5.3.1 *The internal fixed points for strongly connected reliability polynomials are dense in $[0, 1]$.*

Corollary 5.3.2 *The average reliabilities for strongly connected reliability polynomials are dense in $[0, 1]$.*

Corollary 5.3.3 *The inflection points for strongly connected reliability polynomials are dense in $[0, 1]$.*

Similarly, we can extend the density results to two-terminal reliability using the graph P_n with source v_0 and sink v_{n-1} . By having m^k vertices and by replacing each edge with bundles of size kj , we obtain the reliability polynomial $(1 - q^{kj})^{m^k - 1}$, and using a proof very similar to the cycle bundles for all terminal reliability, we get the following results.

Corollary 5.3.4 *The internal fixed points for two-terminal reliability polynomials are dense in $[0, 1]$.*

Corollary 5.3.5 *The average reliabilities for two-terminal reliability polynomials are dense in $[0, 1]$.*

and finally,

Corollary 5.3.6 *The inflection points for two-terminal reliability polynomials are dense in $[0, 1]$.*

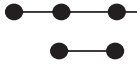
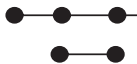
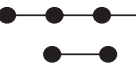
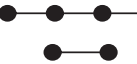
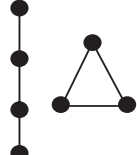
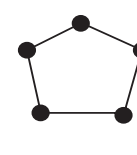
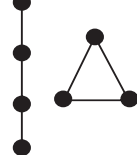
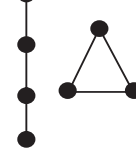
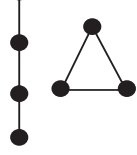
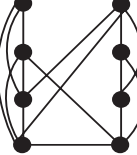
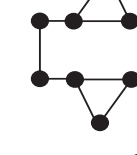
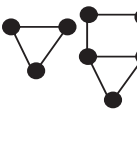
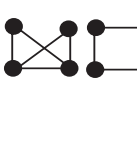
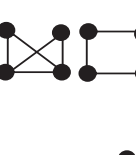
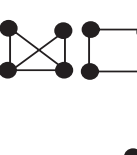
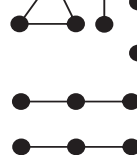
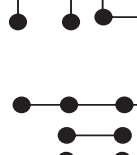
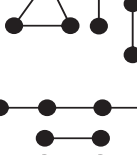
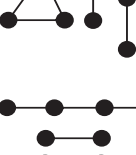
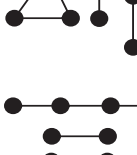
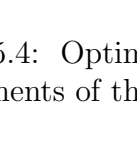
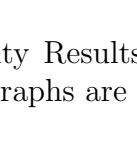
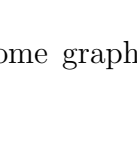
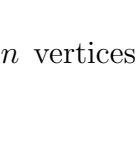
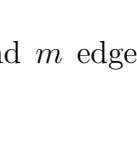
(n,m)	Most Optimal for p near 0	Most Optimal for p near 1	Most Optimal for p=1/2	Largest Average Reliability	Smallest Internal Fixed Point
(6,11)					
(7,15)					
(8,14)					
(8,19)					
(8,22)					
(8,23)					

Figure 5.4: Optimality Results for some graphs on n vertices and m edges. The complements of the graphs are shown.

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