Uniformization of curves

Archimedean vs non-archimedean

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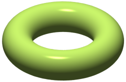
Outline

1. Uniformization of Riemann surfaces
2. Arithmetic analytic geometry
3. Universal Mumford curves over $\mathbb{Z}$

The uniformization theorem

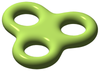
To a compact connected Riemann surface (= complex analytic curve) one associates a genus.

genus 0



genus 1

genus 2



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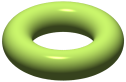

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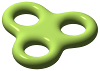

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Theorem (Fuchs uniformization)

A compact connected Riemann surface X^{an} of genus g is isomorphic to one of the following:

- *The Riemann sphere* $\mathbf{P}^{1,an}_{\mathbb{C}}$ *if* $g = 0$
- *A quotient* $\mathbb{C}/\Lambda$ *for some lattice* Λ *if* $g = 1$
- *A quotient* $\mathcal{H}/\Gamma$ *for* Γ *discrete subgroup of* $\mathrm{PSL}_2(\mathbb{R})$ *if* $g > 1$.

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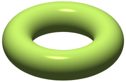

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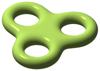

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What happens for X^{an} defined over other fields: $\mathbb{Q}_p$, $\mathbb{C}((t))$, $\mathbb{F}_p((t))$...?

Example: elliptic curves

Let $E(\mathbb{C}) = \{[x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : zy^2 = x^3 + az^2x + bz^3\}$ for some $a, b \in \mathbb{C}$.

Uniformization of E

$E(\mathbb{C})$ is a group, isomorphic to $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ is a lattice

This isomorphism is of an analytic nature:

$$\mathbb{C}/\Lambda \to E(\mathbb{C})$$
$$w \mapsto \begin{cases} [\wp(w) : \wp'(w) : 1] & \text{if } w \neq 0 \\ [0 : 1 : 0] & \text{if } w = 0 \end{cases}$$

where $\wp$ is the meromorphic Weierstrass $\wp$-function.

p-adic uniformization of elliptic curves

Use Schottky uniformization:

$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & & & \mathbb{C}/\Lambda \\ \downarrow & \searrow^{e^{2\pi i \cdot}} & & & \uparrow \wr \\ \mathbb{C}/\mathbb{Z} & \xrightarrow{\sim} & \mathbb{C}^* & \longrightarrow & \mathbb{C}^*/q^{\mathbb{Z}} \end{array}$$

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Over k non-archimedean, lattices are not discrete, but

$$k^*/q^{\mathbb{Z}}$$

still makes sense for $q \in k^*$ with $|q|_p < 1$ and it is (the set of k-points of) an elliptic curve.

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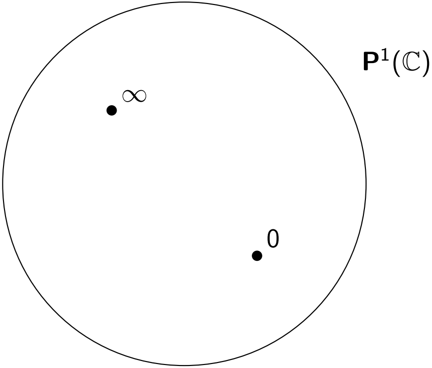
Over $k = \bar{k}$, not all elliptic curves arise this way: only those whose j invariant satisfies $|j(E)| > 1$ (Tate curves).

Schottky uniformization: geometric setting



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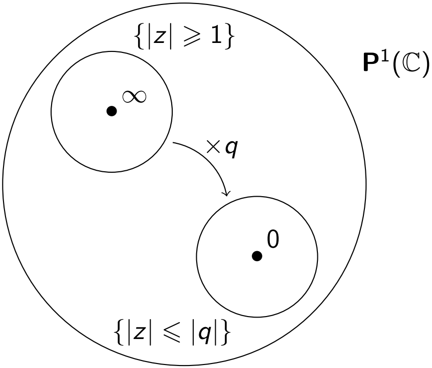

$$E \cong \mathbf{P}^1(\mathbb{C}) \setminus \{0, \infty\} \Big/ \left\langle \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

Schottky uniformization: geometric setting

Let $g \geqslant 1$. Let $D_{\pm 1}, \ldots, D_{\pm g}$ be disjoint open discs in $\mathbf{P}^1(\mathbb{C})$.
Let $\gamma_1, \ldots, \gamma_g \in \mathrm{PGL}_2(\mathbb{C})$ such that, setting $\gamma_{-i} := \gamma_i^{-1}$, we have

$$\forall i,\ \gamma_i(\mathbf{P}^1(\mathbb{C}) - D_{-i}) = \overline{D_i}.$$

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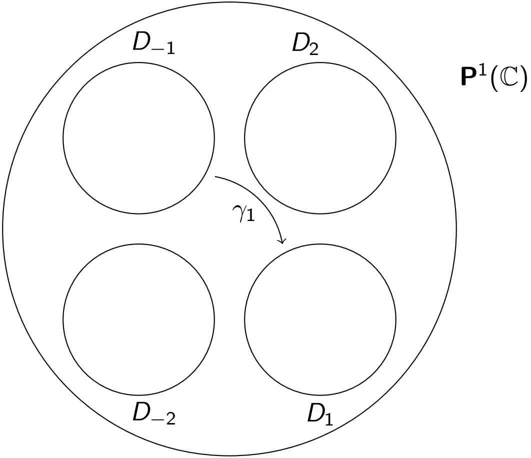

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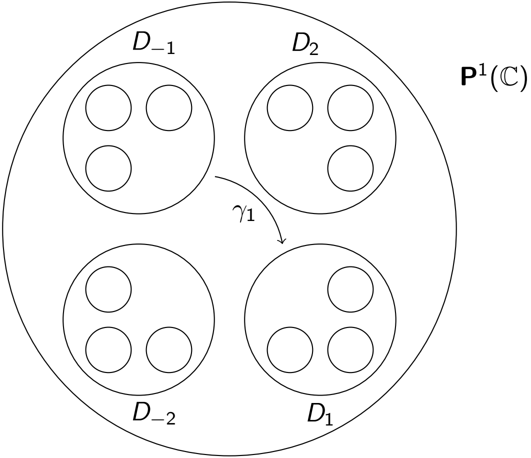

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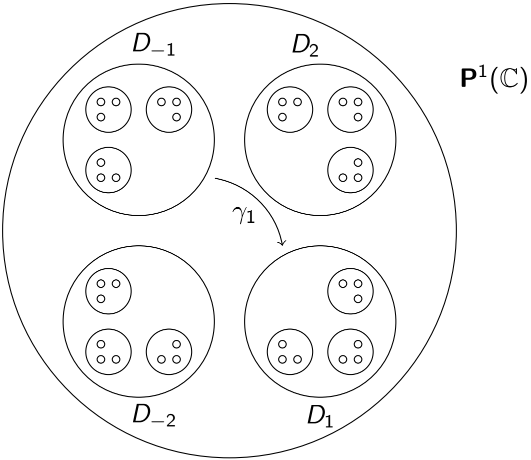

Schottky uniformization: properties

Set $\Gamma := \langle \gamma_1, \ldots, \gamma_g \rangle$. It is a free group of rank g.

Moreover, there exists a compact subset L of $\mathbf{P}^1(\mathbb{C})$ such that

- the action of Γ on $\mathbf{P}^1(\mathbb{C}) - L$ is properly discontinuous;
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. . . Where is the analysis here?

Outline

Some history

1960's John Tate introduces *rigid analytic geometry*

1970's Michel Raynaud links rigid spaces and Grothendieck's formal geometry

$\sim$1990 Vladimir Berkovich conceives a new theory using spaces of valuations and spectral theory

$\sim$2010 Jérôme Poineau develops the theory of Berkovich spaces over $\mathbb{Z}$

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What for?

- Arithmetic geometry: local Langlands program (étale cohomology on Berkovich spaces) and p-adic Hodge theory (Scholze's perfectoid spaces)
- Classical and combinatorial algebraic geometry (via connections to toric and tropical geometries)
- String theory (degeneration of Calabi-Yau, mirror symmetry, SYZ fibration)
- Dynamical systems and potential theory (dynamics on Berkovich spaces)
- p-adic differential equations (radii of convergence on Berkovich curves)
- ...

The Berkovich analytic space $\mathbf{A}_A^{n,\mathrm{an}}$

Let $(A, \|\cdot\|)$ be a commutative Banach ring with unit. Let $n \in \mathbb{N}$.

The analytic space $\mathbf{A}_A^{n,\mathrm{an}}$ is the set of multiplicative semi-norms on $A[T_1, \ldots, T_n]$ bounded on A, *i.e.* maps

$$|.| : A[T_1, \ldots, T_n] \to \mathbb{R}_+$$

such that

1. $|0| = 0$;
2. $\forall f, g \in A[T_1, \ldots, T_n]$, $|f + g| \leqslant |f| + |g|$;
3. $\forall f, g \in A[T_1, \ldots, T_n]$, $|fg| = |f|\,|g|$;
4. $\forall f \in A$, $|f| \leqslant \|f\|$.

The topology on $\mathbf{A}_A^{n,\mathrm{an}}$

The set $\mathbf{A}_A^{n,\mathrm{an}}$ is endowed with the coarsest topology such that, for any f in $A[T_1, \ldots, T_n]$, the evaluation function

$$\begin{array}{ccc} \mathbf{A}_A^{n,\mathrm{an}} & \to & \mathbb{R}_+ \\ |.|_x & \mapsto & |f|_x \end{array}$$

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Theorem (Berkovich)

The space $\mathbf{A}_A^{n,\mathrm{an}}$ is Hausdorff and locally compact.

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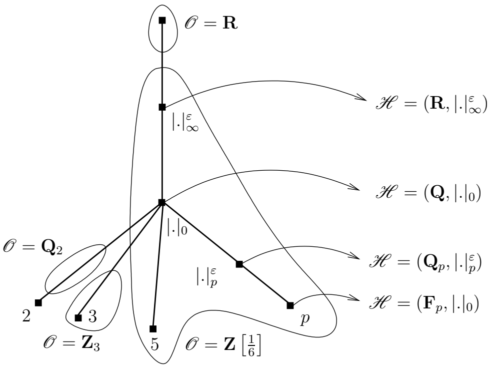
Theorem (Berkovich)

The space $\mathbf{A}_A^{n,\mathrm{an}}$ is Hausdorff and locally compact.

To each $x \in \mathbf{A}_A^{n,\mathrm{an}}$, we associate a residue field

$$\mathscr{H}(x) := \text{ completion of the fraction field of } A[T_1, \dots, T_n]/\mathrm{Ker}(|\cdot|_x)$$

For every open subset $U \subset \mathbf{A}_A^{n,\mathrm{an}}$, there is a ring $\mathscr{O}(U)$ of convergent functions on U.


$\mathscr{O} = \mathbf{R}$
$\mathscr{H} = (\mathbf{R}, |.|_\infty^\varepsilon)$
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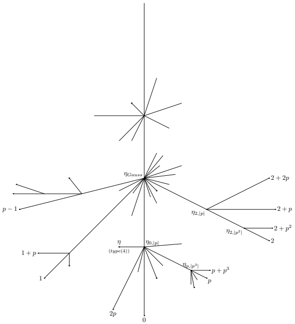
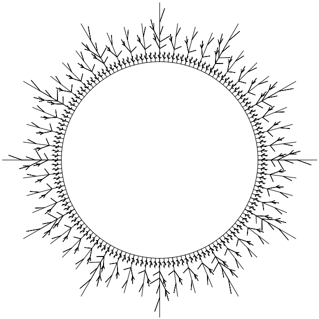
2
3
p
$\mathscr{O} = \mathbf{Z}_3$
5
$\mathscr{O} = \mathbf{Z}\left[\frac{1}{6}\right]$

Curves over $\mathbb{Q}_p$



$\mathbf{A}^{1,\mathrm{an}}_{\mathbb{Q}_p}$



$y^2 = x(x-1)(x-p)$

Properties of $\mathbf{A}_{\mathbb{Z}}^{n,\mathrm{an}}$

Theorem (Lemanissier)

The space $\mathbf{A}_{\mathbb{Z}}^{n,\mathrm{an}}$ is locally path-connected.

Theorem (Poineau)

- *For every x in $\mathbf{A}_{\mathbb{Z}}^{n,\mathrm{an}}$, the local ring $\mathscr{O}_x$ is henselian, noetherian, regular, excellent.*
- *The structure sheaf of $\mathbf{A}_{\mathbb{Z}}^{n,\mathrm{an}}$ is coherent.*

Theorem (Lemanissier - Poineau)

Relative closed and open discs over $\mathbb{Z}$ are Stein.

Outline

Mumford curves

Aim

Combine archimedean and non-archimedean approaches in a unique framework, using analytic geometry over $\mathbb{Z}$.

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If $g = 1$, the subset $\mathscr{S}_1 \subset \mathbf{A}_{\mathbb{Z}}^{1,\mathrm{an}}$ defined by

$$\mathscr{S}_1 := \{x \in \mathbf{A}_{\mathbb{Z}}^{1,\mathrm{an}} : 0 < x(T_1) < 1\}$$

is a universal parameter space for uniformizable elliptic curves.

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Combine archimedean and non-archimedean approaches in a unique framework, using analytic geometry over $\mathbb{Z}$.

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Theorem (Poineau - T.)

Let $g \geq 2$. There exists a connected open subset $\mathscr{S}_g \subset \mathbf{A}_{\mathbb{Z}}^{3g-3,\mathrm{an}}$ parametrizing Schottky groups of rank g with a choice of an ordered basis.

Universal Mumford curve

For $g = 1$, we have a *universal uniformization*

$$(\mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_1} \setminus \{0, \infty\}) \to \mathscr{X}_1 := (\mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_1} \setminus \{0, \infty\})/\langle z \mapsto T_1 z \rangle.$$

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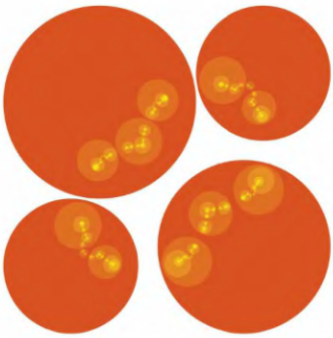
Theorem (Poineau - T.)

There exist $\Gamma \subset \mathrm{PGL}_2(\mathscr{O}(\mathscr{S}_g))$ *and a closed subset* $\mathscr{L}$ *of* $\mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_g} := \mathscr{S}_g \times_{\mathbb{Z}} \mathbf{P}^{1,\mathrm{an}}_{\mathbb{Z}}$ *such that*

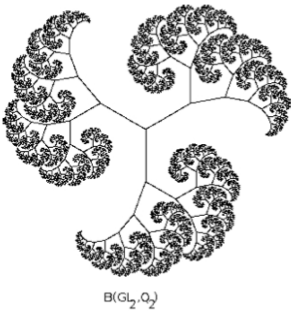
1. *for each* $z \in \mathscr{S}_g$, $\mathscr{L} \cap \mathrm{pr}_1^{-1}(z)$ *is the limit set of* Γ_z;
2. *we have a commutative diagram of analytic spaces*

$$\begin{array}{ccc} \mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_g} - \mathscr{L} & \searrow & \\ \downarrow & & (\mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_g} - \mathscr{L})/\Gamma =: \mathscr{X}_g \;. \\ & \swarrow & \\ \mathscr{S}_g & & \end{array}$$

Fractal nature of limit sets



Archimedean world



Non-archimedean world

What's next?

- Homotopy type of $\mathscr{S}_g$, connections with tropical geometry and geometric group theory (Culler-Vogtmann "Outer space")
- Compute Hausdorff dimension and capacity of limit sets
- Periods $(q_{i,j})_{1\leqslant i,j\leqslant g}$ and Jacobians (Manin-Drinfeld, Myers)
- q-expansions of modular forms (Ichikawa)
  Schottky problem (= characterize Jacobians inside $\mathscr{A}_g$)
- Gauß-Manin connections
  Picard-Fuchs equations (Gerritzen):

$$\text{for } 1 \leqslant i \leqslant g, \begin{cases} \nabla\left(\dfrac{du_i}{u_i}\right) = \displaystyle\sum_{j=1}^{g} \beta_j \otimes \dfrac{dq_{i,j}}{q_{i,j}}; \\ \nabla(\beta_i) = 0. \end{cases}$$

- Notions of hyperbolicity and Teichmüller space over $\mathbb{Z}$