Arithmetic geometry via group actions and combinatorics

Statement of past, current, and planned research

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Summary

My research lies at the intersection of algebraic geometry, number theory, representation theory, and analytic geometry. In this document, I present a combination of research projects aimed at studying an arithmetic variety X using group actions and their combinatorics. The fundamental idea is to realize X as the quotient Y/G of a better understood variety Y by the action of a suitable group G. This approach is certainly not recent – it can be traced back to the work of Klein and Poincaré – but maintains its surprising potential when applied to contemporary problems. In my previous and current work, I combine arithmetic versions of this idea with the recently established theories of non-archimedean analytic geometry to study arithmetic curves, their moduli spaces, and their Galois covers. To do this, it is often sufficient to know how G acts on certain combinatorial structures embedded in the Berkovich analytification of Y. My current and future research aims to strengthen these results and apply this method to other arithmetic-geometric objects. This will allow me to bring significant advances in the resolution of fundamental problems in arithmetic geometry and to expand my views towards possible applications to other branches of mathematics.

Research Statement

The common thread that underlies my research is the study of arithmetic-geometric objects[1] arising as quotients by the action of a group G. More specifically, denoting by X such an object, I have studied the following situations, each discussed in the text in an independent section:

1. Let X be the symplectic group over a local field, and $G = R^\times$ be the group of invertible elements in an integral domain R. Then, a G-cover of X is provided by the *metaplectic group with coefficients in* R. In [CT15], this notion is introduced in order to study certain representations attached to it in the ℓ-adic ($R = \mathbb{Z}_\ell$) and ℓ-modular ($R = \mathbb{F}_\ell$) cases;

2. Let X be a smooth projective curve and G be the Galois group of a finite ramified cover $Y \to X$. A much studied arithmetic question is the problem of lifting such covers from characteristic p to characteristic zero. In [Tur19], I study the geometry of the ramification locus of liftings, introducing the notion of Berkovich-Hurwitz graphs. In [Tur20], I investigate in more detail the case where $G = (\mathbb{Z}/p\mathbb{Z})^n$ and the ramification locus of the cover is sufficiently simple, establishing new obstructions for lifting $(\mathbb{Z}/3\mathbb{Z})^2$-covers;

[1] In this text the term *arithmetic-geometric object* is used to mean an analytic or algebraic space defined over a non-archimedean field, a field of characteristic $p > 0$, or a global field.

3. Let X be an analytic space defined over a discretely valued field K, and $G = \mathrm{Gal}(L|K)$ be the Galois group of a finite Galois extension L of K. In [FT18] we address the problem of classifying tame K-forms of $X_L := X \times_K L$ in dimension one, providing a classification when X_L is a disc or an annulus. In [FT20], we apply our previous work to the global case: we let X be the analytification of a K-algebraic curve and prove results on the minimal extension $L|K$ required for X_L to have a semi-stable model. This is achieved by looking at the action of G on finite combinatorial structures (called *triangulations*) appearing in the Berkovich analytification of X_L;

4. Let X be a moduli space (e.g. the space $\mathcal{M}_g$ parametrizing smooth curves of genus g). Then, a cover of X can often be obtained by adding extra structures on the objects that are parametrized. A classical example is the universal covering $\mathcal{T}_{g,\mathbb{C}} \longrightarrow \mathcal{M}_{g,\mathbb{C}}$ from the Teichmüller space $\mathcal{T}_{g,\mathbb{C}}$, parametrizing smooth curves with complex structures. Over a non-archimedean valued field k, a partial analogue of this cover is provided by the Schottky space $\mathcal{S}_{g,k}$ studied by Gerritzen and Herrlich. This is a covering of the form $\mathcal{S}_{g,k} \to \mathrm{Mumf}_{g,k}$, where $\mathrm{Mumf}_{g,k}$ is the moduli space of Mumford curves over k and the automorphism group of the cover is $\mathrm{Out}(F_g)$, whose elements are the outer automorphisms of the free group with g generators. In [PT21], we survey the theory of Mumford curves in the setting of Berkovich spaces, clarifying certain arithmetic aspects. In [PT20], we provide a unified framework for Schottky spaces, that allows to study the covering spaces of $\mathcal{M}_g$ in the archimedean and non-archimedean settings at once. More precisely, we define Schottky spaces as Berkovich analytic spaces over $\mathbb{Z}$, encoding in a single object both archimedean and non-archimedean uniformizations.

Each of these lines of research has a natural continuation that I plan to pursue in the near future, as well as enhance with methods from computer algebra, tropical geometry, and combinatorics. Moreover, I have developed specific tools that are common to more than one project (e.g. the linearization of actions of finite groups on non-archimedean discs and annuli, and results on group actions on Mumford curves) and whose potential I wish to explore fully. I have encountered several open problems that I believe my methods can contribute to solve: the problem of determining the wild monodromy of a curve, rationality results for motivic Zeta functions of wildly ramified abelian varieties, lifting problems for covers of singular curves, and conjectures about the arithmetic geometry of moduli spaces of curves and abelian varieties. A more detailed list of short- and long-term goals for my research is presented in the last section of this document.

1 The integral Weil representation

In the paper [CT15], joint with Gianmarco Chinello, we define and describe explicitly the Weil representation over an integral domain. Over the complex numbers, this representation has been introduced together with the metaplectic group by André Weil in his *Acta* paper [Wei64] in order to shed light on the work of Siegel on theta functions. This led to various developments in number theory, for example the work of Shimura on modular forms of half-integral weight and the one of Jacquet and Langlands on automorphic representations of adele groups.

The construction of Weil is as follows: he considers a local field F, a finite dimensional F-vector space X and the symplectic group $\mathrm{Sp}(W)$ over $W = X \times X^*$. He denotes by $\mathbb{T}$ the complex unit circle and by $\chi : F \to \mathbb{T}$ a non-trivial continuous character. Using χ, he builds an action φ of $\mathrm{Sp}(W)$ over the Heisenberg group, defined up to multiplication by an element of $\mathbb{T}$. Then, he constructs a cover $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$ such that φ lifts to a complex infinite representation of $\mathrm{Mp}(W)$, the now-so-called *Weil representation*. Finally he shows that $\mathrm{Mp}(W)$ contains a double cover of $\mathrm{Sp}(W)$ over which the Weil representation can be restricted.

We suppose that F is non-archimedean of characteristic $\neq 2$ and residue field of cardinality $q = p^e$. We replace $\mathbb{T}$ by an integral domain S such that $p \in S^\times$, S contains a square root of q and p^n-th roots of unity for every n, to ensure the existence of a nontrivial smooth character $\chi : F \to S^\times$. In this generality, we are able to define a *reduced metaplectic group* in the following way. Firstly we construct the metaplectic group $\mathrm{Mp}(W)$ together with a non-split short exact sequence

$$1 \longrightarrow S^\times \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1. \qquad (\star)$$

Then, we prove our main theorem:

Theorem 1. *Let* $\mathrm{char}(S) \neq 2$. *There exists a subgroup* $\mathrm{Mp}_2(W)$ *of* $\mathrm{Mp}(W)$ *such that the short exact sequence* $(\star)$ *restricts to a short exact sequence*

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_2(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1 \qquad (\star\star)$$

that does not split.

The main challenge in achieving this generalization is to overcome the lack of complex conjugation and complex absolute value, that are fundamental to Weil's approach. This is addressed by considering Haar measures with values in S and operators acting over the space of S-valued Schwartz functions over an F-vector space instead of L^2-functions, using Vignéras' approach. A nice aspect of our proof of Theorem 1 is that it provides an explicit description of the Weil representation over any integral domain. This can be applied to provide examples and shed new light on local Langlands correspondences, especially in the modular case ($S = \overline{\mathbb{F}}_\ell$) and in the integral case ($S = \mathbb{Z}_\ell^{ur}$).

2 Lifting local actions and Berkovich-Hurwitz graphs

Lifting problems aim to uncover the relationship between positive characteristic and characteristic zero phenomena. The following lifting problem is a core question in my research.

Problem 2 (**Local lifting problem for curves**). *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group and $\lambda : G \hookrightarrow \mathrm{Aut}_k(k[[t]])$ an action of G on the k-algebra of formal power series $k[[t]]$. Find a complete discrete valuation ring R with residue field k and a R-linear action $\Lambda : G \hookrightarrow \mathrm{Aut}_R(R[[T]])$ such that the reduction of Λ modulo the maximal ideal of R coincides with λ.*

The local lifting problem for curves has been widely studied since the '80s and is essentially equivalent to the problem of liftings G-Galois covers of smooth projective k-curves to R. Oort conjectured in [Oor87] that the answer to Problem 2 is always positive when G is cyclic, a problem recently proved by Obus-Wewers [OW14] and Pop [Pop14], building on decades of work by several researchers.

In the paper [Tur20], I study liftings with a special property called *equidistance*. A lifting is *equidistant* if all the branch points of the associated G-Galois cover of rigid open disks are at the same mutual distance. Equidistant liftings are remarkable for several reasons. First of all, one can show that if such a lifting exists, then $G \cong C \ltimes (\mathbb{Z}/p\mathbb{Z})^n$ for C cycluc of order coprime with p and $n > 0$. Moreover, the theory of deformation from Artin-Schreier to Kummer [OSS89] can be applied to find the following interpretation of equidistant liftings.

Fact 1. *A local action $\lambda : (\mathbb{Z}/p\mathbb{Z})^n \hookrightarrow \mathrm{Aut}_k k[[t]]$ with conductor m has an equidistant lifting if and only if there exists a space $L_{m+1,n}$, that is, a n-dimensional $\mathbb{F}_p$-vector space of logarithmic differential forms on $\mathbb{P}^1_k$ with $m+1$ distinct poles and a unique zero at ∞.*

The work of Raynaud [Ray90] and Matignon [Mat99] leads naturally to search a criterion for existence of equidistant liftings of $(\mathbb{Z}/p\mathbb{Z})^n$ that depends only on the ramification number m. Results in this sense were obtained by Pagot [Pag02] who shows that $m+1$ needs to be a multiple of p so that one can write $m+1 = \mu p$. Moreover, he studies the cases $\mu \in \{1, 2, 3\}$ and shows that a space $L_{m+1,2}$ exists if and only if $\mu | (p-1)$. In [Tur20], I study the cases where $p = 3$ and $\mu = 4, 5$ using combinatorial number theory in characteristic p, showing that Pagot's condition does not generalize further. More precisely, I introduce a new invariant of a space $L_{m+1,n}$ the *residual dispersion index* ρ, that allows me to quantify the variation of residues at the poles, and to establish when $p = 3$ that equidistant liftings can exist only when ρ is sufficiently high:

Theorem 3. *Let $p = 3$ and $m+1 = \mu p$. If $1 \le \mu \le 5$ every space $L_{m+1,2}$ verifies $\rho \ge \mu$.*

The same strategy used in the proof of this result allows me to produce examples of spaces $L_{15,2}$ with high residual dispersions, proving that Pagot's condition needs to be modified in order to treat higher values of μ. I am currently working to extend Theorem 3 in the case of higher values of p and μ, tackling the higher computational difficulties with the assistance of computer algebra systems that allow me to compute Gröbner bases for polynomal systems related to the existence of spaces $L_{m+1,n}$.

A more abstract point of view on the local lifting problem is provided by the theory of Hurwitz trees: combinatorial objects parametrizing local actions in characteristic zero. The idea at the core of Hurwitz trees, introduced by Henrio [Hen] and Brewis-Wewers [BW09] is to describe liftings of local actions by gluing together equidistant liftings. The Hurwitz tree associated to a local action in characteristic zero Λ is a metric rooted tree $\mathcal{H}_\Lambda$ with additional data (called *Hurwitz data*), attached to vertices and edges, encoding invariants measuring wild ramification and the reduction properties of Λ. Hurwitz trees turn out to be central in many strategies for solving Problem 2: they are used in [BW09] to show that some actions of the generalized quaternionic group Q_{2^m} do not lift; in [BWZ09] to show positive answers to the lifting problem for metacyclic groups; in [Hen] to describe all liftings of actions of order p.

In my PhD thesis [Tur14], I give a geometric interpretation of Hurwitz trees in the framework of Berkovich analytic spaces over the field of fractions of R. The main ingredient that I use is the *skeleton* of a Berkovich curve, a graph onto which the curve retracts by deformation in many good cases. Skeletons can be associated also to higher dimensional Berkovich spaces, they are central in understanding topological and arithmetic properties, and they bring out connections with tropical, toric, and logarithmic geometries. In [Tur14], I use the theory of skeletons to prove the following result:

Theorem 4. *For a local action $\Lambda : G \hookrightarrow Aut_R(R[[T]])$, there is a metric embedding*

$$\mathcal{H}_\Lambda \hookrightarrow \mathbb{D}^\circ(0,1)$$

of the Hurwitz tree in the Berkovich open unit disc such that the image is contained in the set of points with non-trivial stabilizer for the action induced by Λ on $\mathbb{D}^\circ(0,1)$.

Thanks to theorem 4 one can assign a Berkovich closed disc $\mathbb{D}^\bullet(v)$ (resp. a Berkovich open disc $\mathbb{D}^\circ(e)$) to every vertex v (resp. to every edge e) of $\mathcal{H}_\Lambda$. With these identifications, the Artin character a_e and depth character δ_v of [BW09] are reconstructed in terms of piecewise linear functions on the Hurwitz tree.

Theorem 5. *Let e be an edge of $\mathcal{H}_\Lambda$, with starting vertex $s(e) = v$. Let δ_v and a_e be the depth and Artin characters, associated to v and e respectively. Then there is a rank two valuation $x : R[[T]] \otimes_R K \to \mathbb{Q} \times \mathbb{Z}$ corresponding to the pair $(\mathbb{D}^\bullet(v), \mathbb{D}^\circ(e))$ such that*

$$x(\sigma(T) - T) = (\delta_v(\sigma), a_e(\sigma)) \;\; \forall\, \sigma \in G.$$

As the general philosophy of skeletons suggests, one shall be able to generalize these results from the case of the open disc to more general curves. In the paper [Tur19], I define the *Berkovich-Hurwitz graph* associated to any finite Galois cover of Berkovich curves $\varphi : X \to Y$ over a discretely valued field. In particular, when X and Y are open discs, one finds again the Hurwitz tree of Theorem 4. In this definition, the representation theoretical characters are replaced by an adaptation of the different function of Cohen-Temkin-Trushin [CTT16] and the differential data in the case of a p-cyclic Galois cover by a metrized line bundle over X, called the *Swan bundle*.
This line of research fits in between the theory of Berkovich skeletons (in the spirit of work of Baker-Payne-Rabinoff [BPR16] and Amini-Baker-Brugallé-Rabinoff [ABBR15]) and the ramification theory for Berkovich curves (as investigated by Faber [Fab13]; Temkin [Tem17], Brezner-Temkin [BT20] and Bojkovic-Poineau [BP17]). These techniques are proving to be fruitful for the study of a wide range of problems in arithmetic geometry, and are undergoing rapid developments. In my future research, I hope both to give contributions to and to benefit from the current building of these theories.

3 Minimal triangulations and semi-stable reduction of curves

Let K be a complete discretely valued field (not necessarily of mixed characteristic) and X a K-Berkovich curve. Then the skeleton of X is well understood if X has a semi-stable model. Otherwise, the situation is more involved and should be investigated using objects called *triangulations*. A triangulation on X is a subset $S \subset X$, such that the connected components of $X \setminus S$ are either forms of discs or forms of annuli (i.e. K-analytic spaces that become discs or annuli after base-change with a finite extension of K). It was shown by Ducros that every quasi-smooth Berkovich curve admits a triangulation, and often

there is one that is minimal by set inclusion, called *minimal triangulation* of X. Knowing a minimal triangulation is enough to determine the skeleton of X.
Jointly with Lorenzo Fantini, we started a program to investigate how the minimal triangulation of a curve determines its behaviour under base-change. As a first step, we turned our attention on classifying forms of discs and annuli, proving the following theorem (see [FT18]).

Theorem 6. *Let $L|K$ be a tamely ramified extension. Let A be a K-analytic subset of X such that $A \times_K L$ is isomorphic to an annulus. Then A is an annulus if the group $Gal(L|K)$ fixes the skeleton of the annulus $A \otimes_K L$, and a twist of an annulus (whose equation we explicitly describe) otherwise.*

The triviality of tame forms of discs was previously known by results of Ducros [Duc13] and Schmidt [Sch15], but it is a new result for annuli. Our approach provides a new proof of their results, and is substantially different from theirs: it does not rely on graded reduction, but on the notion of *canonical (formal) model of a semi-affinoid space*. We are able to retrieve the canonical model of A as a suitable formal completion of the Galois-fixed locus of the Weil restriction of the canonical model of $A \otimes_K L$. This technique, which is inspired by a construction of Edixhoven [Edi92] is the key ingredient to classify non-trivial tame forms of annuli by means of explicit equations.
In [FT20] we give a bound on the degree of the minimal extension $L|K$ such that $X \times_K L$ is semi-stable in terms of a local invariant of a minimal triangulation S of X, the collection of multiplicities $m(x)$ for all $x \in S$.

Theorem 7. *Let X be the analytification of a geometrically connected, smooth, and projective K-curve. Let $V_{\mathrm{min-tr}}$ be the minimal triangulation of X and let L be the minimal Galois extension of K such that $X \times_K L$ has a semi-stable model. Then*

$$\mathrm{lcm}\{m(x) \,|\, x \in V_{\mathrm{min-tr}}\} \,\big|\, [L : K].$$

In the case where $L|K$ is tamely ramified, we prove that there is equality of the two terms above. Moreover, combining Theorem 7 and Theorem 6 we can show that the information provided by the minimal triangulation is essentially equivalent to the one given by invariants of a minimal regular model with normal crossing, reproving a result of Saito [Sai87]. When $L|K$ is wildly ramified, minimal regular models are not enough to compute L, and our goal for the continuation of this project is to prove that minimal triangulations can be enhanced with wild ramification invariants to provide a satisfactory description of the behaviour of analytic curves under base-change.

4 Universal Mumford curves over $\mathbb{Z}$

Mumford constructed in [Mum72] a beautiful theory of Schottky uniformization for certain non-archimedean curves, called Mumford curves. The theory of skeletons is tightly connected to this uniformization property, as an algebraic curve of genus g is a Mumford curve if and only if the skeleton of its analytification is a graph of Betti number g. Over the complex numbers the picture is quite different, for example every Riemann surface admits Schottky uniformization, a fact that has been exploited to study the action of the mapping class group on the Teichmüller space. Such an action gives rise to the universal covering of $\mathcal{M}_{g,\mathbb{C}}$, the moduli space of Riemann surfaces of genus g, that factors through the space of marked Schottky groups (as surveyed in [Her15]). In ongoing work with Jérôme Poineau, we study the p-adic and the complex uniformizations in a single framework, by defining for every $g \geq 1$ a Schottky space $\mathcal{S}_g$ in the context of Berkovich geometry over $\mathbb{Z}$, as introduced by Poineau in [Poi10]. In analogy with the complex case, we prove the following results.

Theorem 8. *1. If $g \geq 2$, the Schottky space $\mathcal{S}_g$ is an open path-connected subset of $\mathbb{A}_{\mathbb{Z}}^{3g-3,\mathrm{an}}$, while $\mathcal{S}_1$ is the relative open unit disc inside $\mathbb{A}_{\mathbb{Z}}^{1,\mathrm{an}}$.*

2. There is a universal limit set $\mathfrak{L}_g \subset \mathbb{P}^{1,\mathrm{an}}_{\mathcal{S}_g}$ *such that for every $x \in \mathcal{S}_g$, the fiber $\mathfrak{L}_{g,x}$ is the limit set of the group associated with the point x.*

3. There is a universal Schottky group $\mathcal{G}_g \subset PGL(\mathcal{O}(\mathcal{S}_g))$ *whose action on $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{S}_g} - \mathfrak{L}_g$ is free and proper.*

As a result of the theorem, the quotient of the action of $\mathcal{G}_g$ on $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{S}_g} - \mathfrak{L}_g$ is a Berkovich space over $\mathbb{Z}$ that we call the *universal Mumford curve of genus g over $\mathbb{Z}$*. It is a parameter space for all the curves that admit uniformization by a Schottky group, both archimedean and non-archimedean. Among other things, we believe that this is the natural framework for the theory of Teichmüller modular forms, introduced by Ichikawa in [Ich94] and [Ich00]. In this setting, the q-expansion of such a modular form is given to the evaluation on the universal Mumford curve over $\mathbb{Z}$. In future work we would like to use this framework to produce arithmetic differential equations, in order to understand phenomena of uniformity between the complex and p-adic worlds.

Research Plan

The work presented above has several ramifications, that I intend to investigate in the future following four main lines of research, called Projects A–D in this section. All these projects are devoted to study arithmetic aspects of geometric objects, but each of them requires working with tools of different nature, and is therefore treated separately from the others. Project A is about using non-archimedean analytic geometry to compute fundamental arithmetic invariants of algebraic varieties; Project B proposes to investigate the global structure of analytic moduli over $\mathbb{Z}$ (as those presented in Section 4) in order to solve two long-standing problems in arithmetic geometry: the Schottky problem over number fields and the Bombieri-Dwork conjecture; Project C concerns the development of non-archimedean degeneration techniques for complex varieties, with applications to hyperbolic and tropical geometry; Project D builds on the construction of Berkovich-Hurwitz trees, framing ramification theory for group actions in geometric terms, and giving applications to arithmetic questions such as lifting theorems and convergence of p-adic differential equations.
The techniques I intend to use to implement my research plan cover a vast range of topics in algebraic geometry, representation theory, and number theory. I believe that the best way to acquire the relevant knowledge needed to achieve my goals is by working with experts in these fields. As a result, I am highly committed to pursuing new collaborations

and active interaction with other researchers.

A. Arithmetic invariants of varieties

As discussed in Section 2 of this document, there is a very useful object relating non-archimedean analytic geometry with combinatorics and tropical geometry, the *skeleton* of a Berkovich space, which is a finite polyhedral structure embedded in this space. This construction is well understood for curves and abelian varieties over algebraically closed fields, but remains highly mysterious in general, especially over a field of residue characteristic $p > 0$. In this project, I propose to apply a generalization of skeletons, provided by the theory of *minimal triangulations*, to study fundamental arithmetic invariants of curves and abelian varieties. The *minimal triangulation* of a smooth projective curve C over a non-archimedean field k is a finite set of points inside the Berkovich analytification of C, which correspond to the vertices of the skeleton of C when k is algebraically closed. Minimal triangulations were introduced very recently by Ducros in [Duc] to generalize skeletons of analytic curves that have a semi-stable model. Minimal triangulations are fundamental objects to describe arithmetic curves and several deep questions about them remain unanswered. In this project I propose to shed new light on these questions and obtain a satisfactory classification of non-archimedean analytic curves. Most importantly, this will allow me to introduce new tools for the computation of fundamental arithmetic invariants of algebraic curves over a discretely valued field, such as the reduction type, the Artin conductor, the discriminant, Tamagawa numbers, and the set of Weierstrass points. These invariants are central in many problems in number theory, so that any advance in this project will have far reaching consequences in a variety of relevant subjects such as Diophantine geometry, Galois representations, and function field arithmetic.

In the paper [FT2], we study the relationship between minimal triangulations and the minimal extension necessary for C to have a semi-stable model. In a work in progress with Andrew Obus, we study minimal triangulations of curves with potentially multiplicative reduction in order to give an answer to a question asked by Halle–Nicaise [HN16, Question 10.1.1]. To successfully apply this approach to arithmetic invariants, I need to solve three successive problems: **(A1)** study and classify the residue fields at the points of the minimal triangulation V of C. This is related to the classification of certain valuations of arithmetic surfaces, and I intend to solve it by generalizing existing results on divisorial valuations; **(A2)** investigate the singularities of the connected components of $C \setminus V$. This will allow to canonically associate a regular model to the minimal triangulation, and it is equivalent to study quotient singularities on arithmetic surfaces, a problem that is quite well understood for tame quotients but very difficult for wild quotients. To overcome this difficulty, I propose to decompose the generic fiber of these singularities in simpler pieces, obtained by refining the triangulation V, that will allow me to better control the ramification. Then, I plan to retrieve the arithmetic invariants of C from this decomposition by **(A3)** adapting to the arithmetic (i.e. not algebraically closed) setting the theory of wild ramification for Berkovich curves introduced by Temkin and his collaborators [CTT16], [Tem17]. A first concrete result produced with these techniques will be the answer of two questions posed by Lorenzini [Lor10, Questions 1.1 and 1.4]

Once established these results for curves, I plan to study similar phenomena in higher dimensions. The first step in this long journey will consist in **(A4)** establishing the consequences of the computations of arithmetic invariants for Jacobians, such as those associated with the Néron model and, in characteristic $p > 0$, the p-rank. In the long term, I wish to apply this work to prove mixed characteristic analogues of known results established by Nicaise for skeletons of K3 surfaces and Calabi–Yau varieties in equicharacteristic 0, surveyed in [Nic16].

B. Arithmetic aspects of analytic moduli over $\mathbb{Z}$

In this project, that continues the work presented in Section 4 and includes planned collaborations with Jérôme Poineau, I propose to define analogues of some classical moduli spaces as analytic spaces over $\mathbb{Z}$ and study their properties. Due to the global nature of these objects, the new tools that I will use to achieve the goals of this project will mainly stem from the theory of Berkovich spaces over $\mathbb{Z}$. The first milestone in this project is **(B1)** the introduction of the moduli space $\mathcal{A}_g^{\mathbb{Z}}$ of principally polarized abelian varieties and the moduli space $\mathcal{J}_g^{\mathbb{Z}}$ of Jacobians of Mumford curves of genus g as analytic objects over $\mathbb{Z}$. As an application of (B1), I want to realize the moduli space of tropical abelian varieties as a polyhedral complex inside a suitable compactification $\overline{\mathcal{A}_g^{\mathbb{Z}}}$.
In order to define $\mathcal{A}_g^{\mathbb{Z}}$ I will use the same strategy as the one achieved in [PT20]. The main idea is to compare the archimedean and non-archimedean analytic uniformizations of abelian varieties, and realize the construction of a universal abelian variety of given genus and reduction type. In order to build the space $\mathcal{J}_g^{\mathbb{Z}}$, I plan to extend the theory of Schottky automorphic functions and period matrices in such a way to apply it to the universal Mumford curve defined in Section 4. This theory is already available in the non-archimedean setting thanks to work of Manin-Drinfeld [MD73], Myers, and Gerritzen-Van der Put [GvdP80, §6] and has an analogous description over the complex numbers. My strategy will consist in a careful patching of these known structures that respects the analytic structure of the universal Mumford curve over $\mathbb{Z}$. As a byproduct of this construction, I will be able to define a natural Torelli map $\mathrm{Mumf}_g \to \mathcal{A}_g^{\mathbb{Z}}$. An interesting application of this construction would be the investigation of possible enhancements of the tropical Torelli map, that is not injective. More precisely, I hope to use the analytic approach to clarify this lack of injectivity and to define additional structure on tropical Jacobians that allows to upgrade the Torelli map to an injection. Attaining (B1) will open the possibility to globalize the construction of periods of Mumford curves in the framework of the universal Mumford curve over $\mathbb{Z}$. This has several important arithmetic consequences that I intend to explore in this project. Milestone **(B2)** consists in defining Gauss–Manin connections associated with the universal Mumford curve over $\mathbb{Z}$ and give applications of this theory to the study of arithmetic Picard–Fuchs equations arising from families of curves. Gauss–Manin connections are objects that encode variations of differential equations in families of varieties. In the case of families of p-adic Schottky groups, they have been studied by Gerritzen in [Ger86]. I intend to extend Gerritzen's construction to the universal Mumford curve. The technical part of this process, that I wish to attack using techniques from Poineau's theory, consists in finding a basis for the first De Rham cohomology of the universal Mumford curve. As a concrete application, my plan is to study solutions of Picard–Fuchs equations arising from Gauss–Manin connections over specific one-dimensional families of Mumford curves. In particular, for solutions that can be written as power series, I hope to determine the ring generated by the coefficients of

a solution of the studied Picard-Fuchs equations. This problem is directly connected to important long-standing problems in number theory such as Grothendieck's p-curvature conjecture, the Bombieri–Dwork conjecture and would answer to questions formulated by Beukers in [Beu02] and Zagier in [Zag09].

Milestone **(B3)** consists in the solution of an arithmetic version of the Schottky problem, asking for a description of the image of the Torelli map in $\mathcal{A}_g(k)$ for a number field k. The Schottky problem over the complex numbers is a famous question in algebraic geometry, that has been solved in some of its versions. However, for a given number field k, to understand which k-points of $\mathcal{A}_g$ represent Jacobians of curves is a much harder problem, that remains widely open at present. My approach to this question consists in combining the study of periods over $\mathbb{Z}$ defined in (B1), ideas from tropical geometry (introduced by Mikhalkin, Sturmfels, and many others), and a description of the role of the Schottky space $\mathcal{S}_g$ in the understanding of Teichmüller modular forms: global sections of a certain automorphic line bundle on $\mathcal{M}_g$ studied by Ichikawa [Ich94]. In this approach, the point of view of Berkovich spaces is crucial to compute the q-expansions of these modular forms as formal power series that converge in a certain domain of the affine line over the ring of integers of k.

Finally, a longer term milestone in this project is **(B4)** to replicate the construction of global moduli spaces in other situations of arithmetic significance, such as moduli of polarized K3 surfaces, Prym varieties, Hurwitz spaces, and Teichmüller spaces. This would allow, among other things, to study Brill-Noether theory for K3 surfaces in a tropical fashion and improve our understanding of degenerations of geometric objects from the complex world to the non-archimedean world, as outlined in Project C below.

C. Hybrid analytic moduli and degenerations

The compatibility between archimedean and non-archimedean phenomena arising in the global study of moduli (see Project B) are often related to phenomena of *degeneration*. My aim in this project is to clarify this correspondence by using the theory of *hybrid Berkovich spaces*, introduced by Berkovich [Ber09] and further developed by Jonsson [Jon15], who established links between this theory and the tropicalization of toric varieties. Hybrid spaces allow to study degenerations of families of complex analytic varieties as non-archimedean analytic spaces defined over a field of Laurent series $\mathbb{C}((t_1, \dots, t_n))$. Several results showing continuity of invariants with respect to these degenerations are now available, including continuity of Hodge numbers [IKMZ18], canonical measures [AN20], and volume forms [BJ17].

Hybrid Berkovich spaces naturally appear as subspaces of Berkovich spaces over $\mathbb{Z}$, and therefore the theory of [PT20] and its generalizations proposed in project B can be explored in this light and lead to improvements in tropical geometry and complex analytic geometry. A first result in this direction will be to **(C1)** study the restriction of the Schottky space $\mathcal{S}_g$ in the hybrid setting, providing a partial compactification of the complex Schottky space. By establishing continuity results for invariants of divisors on curves, I will give an interpretation of certain phenomena appearing in tropical Brill-Noether theory as degenerations from the complex locus of the moduli space of smooth curves $\overline{\mathcal{M}}_g$ to the non-archimedean locus of Mumford curves. On the same lines, I intend to **(C2)** establish continuity results for hybrid analytic objects, such as the Hausdorff di-

mension and the capacity of limit sets of Schottky groups. This will allow to transfer the results linking Hausdorff dimension to the geometry of a Riemann surface from the complex setting to the non-archimedean one. Moreover, it will enable to stratify the space $\mathcal{S}_g$ according to these new invariants, enhancing fundamental results in the theory of Berkovich curves that establish links between the skeleton of a Berkovich curve and the arithmetic properties of the curve.
In the long term, I aim to go beyond the case of curves and apply the hybrid analytic approach to moduli spaces of tropical and complex analytic relevance such as character varieties, Grassmanians, and the Teichmüller space. The main difficulty in doing so consists in the fact that these spaces do not admit obvious toroidal compactifications, a technique classically used to define tropicalizations. However, in very recent times, techniques to tropicalize cluster varieties have been made available (see for example [BFMMNC20]), and I intend to exploit these to **(C3)** give an interpretation of degenerations of cluster structures in the context of hybrid spaces and offer a comparison with the relevant tropical theory. This will provide an enhancement of the tropical picture that can hopefully contribute to advances in related topics, such as the Gross–Siebert program. In a similar fashion, I plan to **(C4)** build non-archimedean compactifications of the Teichmüller space and explore possible applications to hyperbolic geometry and geometric group theory. It has been suggested by several authors that the Culler–Vogtmann outer space should be seen as a tropical analogue of the Teichmüller space, and we have shown in [PT20, Theorem 5.2.1] that a non-archimedean fiber of the Schottky space over $\mathbb{Z}$ is tightly related to the outer space. The main effort involved in the achievement of milestone (C4) will then consist in showing that one can modify $\mathcal{S}_g^{\mathbb{Z}}$ by replacing the complex Schottky space with the Teichmüller space on the archimedean fibers, and get a nice topological space after suitably enhancing the non-archimedean fibers. The successive step will be to show continuity results for metric structures on the hybrid Teichmüller space constructed in this way.

D. Ramification theory of group actions on curves

The Hurwitz data attached to Hurwitz trees for studying wild ramification (see Section 2) are derived from Kato's theory of refined Swan conductors [Kat87] introduced in the 80's but still intensively studied nowadays (see for instance [KLS19]). In this line of research I propose to translate these connections in the language of Berkovich curves, use this translation to enhance the current picture, and apply the results obtained to arithmetic and representation theoretical questions. First, I want to focus on establishing new results on the 'differential' part of these ramification invariants: more specifically, milestone **(D1)** will consist in the completion of a joint project with Michel Matignon where we study obstructions to the existence of the vector spaces of differential forms $L_{m+1,2}$ described in Fact 1. Our work improves the strategy of Matignon [Mat99], Pagot [Pag02], and myself [Tur20] by introducing new invariants for these spaces and exploiting tools of algebraic combinatorics, stemming from the theories of Schur polynomials and permutation polynomials, to show that these invariants can occur only for special values of m and p. The second part of this project **(D2)** deals with the variation of the refined Swan conductor for covers of curves when moving on a Berkovich curve. This phenomenon is already apparent in Obus–Wewers theory of wild ramification kinks [OW14], as well as

Abbes–Saito's approach to wild ramification [AS02], and in both cases formulated in the language of rigid geometry. I plan to reformulate it in terms of Berkovich spaces as this will help to clarify important wild ramification phenomena. For instance, piecewise linear variation of ramification invariants and the computation of their slopes, which is a very important tool in the proof of Oort conjecture, directly descends from the description of these functions as sections of vector bundles on a Berkovich curve. More importantly, I plan to use this reformulation to generalize Theorem 6 in the wildly ramified case. While a complete characterization of isomorphism classes of wild forms of discs and annuli seems out of reach, this method will allow to classify them according to the ramification properties of the Galois action induced by the base-change. When combined with the arithmetic reformulation of Temkin's theory proposed in ($A3$), this classification can be used to extend the picture provided by Brezner-Temkin of wild ramification invariants attached to certain finite morphisms of order p of Berkovich curves. My aim is to **(D3)** generalize the main result of [BT20], to the case of general $\mathbb{Z}/p^n\mathbb{Z}$-covers, first by proving that the ramification invariants introduced are fine enough to be essentially uniquely determined by a given cover, and then by showing how a given Berkovich-Hurwitz tree endowed with the relevant ramification invariants can be lifted to a cover, provided that these invariants satisfy extra compatibility conditions. I plan to derive these conditions from the theory of deformations of torsors under group schemes of roots of unity, as introduced by Sekiguchi-Suwa [SS01] and revisited by Mezard-Romagny-Tossici [MRT14]. Other possible applications of (D3) are a description of liftings of covers of curves with 'mild' singularities and results on fundamental groups and fundamental group schemes of curves in positive characteristic, such as the 'Abhyankar type' conjectures exposed in [HOPS17].

The interplay between Berkovich-Hurwitz graphs and refined Swan conductors is also tightly related to the theory of p-adic differential equations. In the case of Hurwitz trees, this has been first observed by Kedlaya in [Ked16, §11], where he translates the local lifting problem into a question of existence of suitable connections on $\mathbb{P}^1_K$. As a long term milestone, I aim to **(D4)** use the generalization provided by Berkovich-Hurwitz graphs and previous milestones achieved in this project to establish the existence of connections on non-archimedean curves and study geometrically the convergence of p-adic differential equations arising from these.

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