EQUIDISTANT LIFTINGS OF ELEMENTARY ABELIAN GALOIS COVERS OF CURVES

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Abstract. In this paper, we discuss the problem of lifting actions of elementary abelian p-groups from positive characteristic p to characteristic 0. Studying logarithmic differential forms linked to deformations of $(\mu_p)^n$-torsors, we show necessary conditions on the set of ramification points in order to get liftings with a geometric property called *equidistance*. Such conditions of combinatorial nature lead us to show new obstructions to lifting actions of the group $(\mathbb{Z}/3\mathbb{Z})^2$.

Introduction

The problem of relating the Galois theory of curves in characteristic $p > 0$ to that in characteristic 0 is both interesting in itself and carries several important applications. Fundamental groups of curves in positive characteristic, still widely unknown objects, constitute the main examples of the relevance of this technique. For smooth projective curves, addressing this problem locally is enough to get an understanding of the global behavior. Concretely, this results in an increasing importance of studying Galois actions over rings of formal power series.

Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, and $\lambda : G \hookrightarrow \mathrm{Aut}_k(k[[t]])$ an injective homomorphism into the group of k-automorphisms of formal power series. Such a λ is called a local action in characteristic p. A necessary and sufficient condition for the existence of such λ is that G is a *cyclic-by p* group, i.e. the semi-direct product of a cyclic group of order prime to p by a normal p-subgroup. The local lifting problem asks to determine which local actions in characteristic p can be related to analogous actions in characteristic 0. One says that the action λ lifts to characteristic zero, if there exist a complete discrete valuation ring of characteristic zero R having k as residue field and an injection $\Lambda : G \hookrightarrow \mathrm{Aut}_R(R[[T]])$ whose special fiber is λ. When dealing with this lifting problem, the fundamental question is the following:

Question 1. Is there a criterion to decide whether λ lifts to characteristic zero?

A first attempt to answer Question 1 has been made by fixing the group G. We know that every local G-action lifts in the following cases:

- when G is cyclic, by recent results of Pop [Pop14] and Obus-Wewers [OW14], building on the work of Green-Matignon [GM98];
- when $G = D_{2p}$ by Pagot [Pag02b] for $p = 2$ and Bouw-Wewers [BW06] for general p;
- when $G = D_{18}$ by Obus [Obu15];
- when $G = A_4$ by Obus [Obu16].

Whenever G is not cyclic, D_{2p^n} or A_4, we know that there exists at least one local action that does not lift. This is a result by Chinburg-Guralnick-Harbater (Corollaries 3.5 and 4.6 in [CGH08]. See also [CGH15] for an up-to-date discussion on their results). Finally, there are groups for which there is no action of G admitting a lifting to characteristic zero. For example, for $(m, p) = 1$ one can show that $G = (\mathbb{Z}/p\mathbb{Z})^2 \times \mathbb{Z}/m\mathbb{Z}$ is such a group. As a consequence, every local action in characteristic p of a finite group containing an abelian subgroup that is

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neither cyclic nor a p-group does not lift to characteristic 0 (see Lemma 3.1, and Proposition 3.3 in [Gre03], for a full proof of these facts).

The ramification properties of liftings play an important role in many proofs of the aforementioned results. Then, it seems natural to investigate the existence of liftings from the point of view of ramification theory, starting with the simplest configuration possible, the equidistant one. A local action in characteristic zero, is called *equidistant* if the distance between two distinct geometric ramification points does not depend on the choice of such points. We are interested in the following question:

Question 2. When does a local action in characteristic p lift to an equidistant action in characteristic zero?

Equidistant liftings are useful for two reasons. First of all, they have been the first examples of liftings to characteristic zero to be computed explicitly. Moreover, one hopes to use them to construct more involved examples. In order to understand how effective this strategy is, there is another question of major importance for us. We can loosely formulate it as follows.

Question 3. Let λ be a local G-action in characteristic p. What is the relationship between a lifting of λ and its sub-actions with the equidistant property?

In this paper, we investigate the three questions when $G = (\mathbb{Z}/p\mathbb{Z})^n$ by studying logarithmic differential forms related to deformation of $(\mu_p)^n$-torsors. The technique of relating equidistant liftings and such differential forms appears for the first time in Pagot's Ph.D. thesis [Pag02b], inspired by results of Matignon, who gives in [Mat99] the first examples of liftings of p-elementary abelian local actions. In this work, we provide obstructions to liftings of $(\mathbb{Z}/3\mathbb{Z})^2$-actions that were not previously known. This gives further evidence for a criterion for lifting that depends only on the upper ramification filtration associated to λ.

The paper is structured as follows. In a first part, we introduce the local lifting problem and related techniques: ramification theory, the rigid ramification locus, and good deformation data. Then, we apply such tools to show obstructions to lifting of local actions of $(\mathbb{Z}/p\mathbb{Z})^n$ satisfying an "equidistant property". In this case, the existence of liftings is equivalent to the existence of some n dimensional $\mathbb{F}_p$-vector spaces of logarithmic differential forms with $m+1$ poles, denoted by $L_{m+1,n}$. In Pagot's thesis, this condition is formulated in terms of explicit equations in characteristic p, with the poles of these differential forms as unknowns, and the residues as parameters. These equations have been studied by Pagot in the case of small conductors ($m + 1 \leq 3p$) and we generalize his approach. In the main part of the paper, we will work with the following assumption

Assumption 0.1 (Partition Condition)**.** *Let $\omega \in L_{m+1,n}$, and let $\mathbf{h}$ be the set of residues at the poles of ω. Then there is a maximal $\mathbf{h}$-adapted partition $\mathcal{P}$ of $\{0, \dots, m\}$ such that $|\mathcal{P}| \leq \left[\frac{m}{p}\right] + 1$.*

The first contribution of the present paper is the following result:

Proposition 0.2. *Let $m + 1 = \lambda p$, and assume that the partition condition is satisfied. Then, after possibly renumbering of the poles,*

$$\begin{cases} h_i = h_0 & \textit{if } i \leq p-1 \\ h_i = h_p & \textit{if } p \leq i \leq 2p-1 \\ \dots & \\ h_i = h_{(\lambda-1)p} & \textit{if } p(\lambda-1) \leq i \leq \lambda p - 1. \end{cases}$$

In the case where $p = 3$, the partition condition is always satisfied. Therefore, we can simplify the equations determining the existence of logarithmic differential forms. In this way, the equations get simple enough to use the combinatorics of the poles to reprove the results of Pagot, and to go further by studying more ramification points. The main theorem is the following

Theorem 0.3. *Let $p = 3$ and $m+1 \leq 15$. Then, a two dimensional space of logarithmic differential forms exists if and only if $m+1 = 6, 12$.*

Due to the previous work of Pagot, Green and Matignon, the novelty in the proof of Theorem 0.3 consist in finding a contradiction when $m+1 = 15$. This is done thanks to a careful study of the combinatorics of some symmetric functions on the poles of a space $L_{15,2}$. The last section contains an outline of work in progress about geometric techniques that can be used to improve the results of the present paper.

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1. Local actions

In this section we review well known facts about properties of local actions (both in characteristic 0 and in positive characteristic), in order to build up the setup for our main result. We distinguish between properties of geometric and of arithmetic nature, with a particular emphasis on the interactions between these two aspects. Throughout the rest of the paper, k will denote an algebraically closed field of positive characteristic p, and R a complete discrete valuation ring of characteristic zero with residue field k, uniformizer π, and fraction field K.

Some geometry of local actions. Let $\Lambda : G \hookrightarrow \mathrm{Aut}(R[[T]])$ be a local action in characteristic zero. Then Λ induces an action of G over $\mathrm{Spf}(R[[T]])$ and its geometric and special fibers. In particular, there is an action of G over the rigid generic fiber $\mathsf{X} = \mathrm{Spf}(R[[T]])_\eta$. Notice that, by the Weierstrass preparation theorem, X can be identified with the open disc $\{x \in \overline{K} : |x| < 1\}/Gal(\overline{K}|K)$, where $\overline{K}$ is a fixed algebraic closure of K.

Definition 1.1. We call **rigid ramification locus** of Λ, the set

$$\mathfrak{R}_\Lambda = \{x \in \mathsf{X} : \sigma.x = x, \exists\, \sigma \in G \setminus \{1\}\}$$

of rigid points that have non trivial stabilizer for the action induced by Λ.

Since Λ acts with trivial inertia, one can show that the rigid ramification locus of Λ is a finite set. In fact, for every $\sigma \in G \setminus \{1\}$, there is a finite number of rigid points fixed by σ (Claim 3.3 in [GM98]), and G is finite. Then, up to finite extensions of complete discretely valued fields, we can and do suppose that the points of $\mathfrak{R}_\Lambda$ are K-rational.

Definition 1.2. A local action in characteristic zero Λ is said to be equidistant if $|\mathfrak{R}_\Lambda| > 1$ and all the elements of $\mathfrak{R}_\Lambda$ are at the same mutual distance, after identification of X with the unit disc.

For Λ an equidistant local action, it is not restrictive to suppose that $0 \in \mathfrak{R}_\Lambda$, after possibly reparametrizing the disc. Then, there is a unique closed disc $\mathsf{D}(0, \rho) \subset \mathsf{X}$ such that all the points of $\mathfrak{R}_\Lambda \setminus \{0\}$ lie on the boundary of D. By restricting the action on such disc and studying its reduction, one can compare the ramification theory in characteristic zero and in characteristic p. In this sense, we can say that equidistant actions are more elementary.

Ramification theory of local actions. Let $\lambda : G \hookrightarrow \mathrm{Aut}(k[[t]])$ be a local action in characteristic p. By Cohen structure theorem ([Ser67], II, Theorem 2), the ring $k[[t]]^G$ of elements that are invariant under λ is abstractly isomorphic to $k[[z]]$. The corresponding extension of fraction fields $k((t))/k((z))$ is Galois, with $\mathrm{Gal}(k((t))|k((z))) = G$. One can show that such G is a semi-direct product $P \rtimes \mathbb{Z}/n\mathbb{Z}$ of a p-group and a cyclic group of order prime to p.
To study such an extension of local fields, one can consider the ramification filtrations attached to it (see Chapter IV of [Ser68]). There is one filtration $G = G_0 \geq P = G_1 \geq \cdots \geq \{1\}$ given by

$$G_i = \{\sigma \in G : v_t(\sigma(t) - t) \geq i\},$$

where v_t is the t-adic valuation. It is classically called the *filtration for the lower numbering.* We have a corresponding filtration $G = G^0 \geq P = G^{\frac{1}{n}} \geq \cdots \geq \{1\}$ *for the upper numbering*, related to the former by means of the Herbrandt formula. The jumps of such filtrations, i.e. those rational numbers $i \in \mathbb{Q}$ such that $G_i \neq G_{i+\epsilon}$ for every $\epsilon > 0$, are such that $G_i/G_{i+\epsilon}$ is elementary abelian, for ϵ small enough. They lie at the heart of the arithmetic of local actions.

In the same spirit, one can study local actions in characteristic 0. In this case, the Galois theory of extensions of $R[[Z]]$ is quite well understood, and the study of good reduction of such extensions can be checked by computing their different (see for example [GM98, Chapter 3]). A ramification theory for such extension exists as well, and it is studied by Kato in [Kat87]. Let us briefly recall the basic definitions, for the convenience of the reader.

Definition 1.3. Let $\Lambda : G \hookrightarrow \mathrm{Aut}(R[[T]])$ be a local action in characteristic zero, and let $v_\eta : R[[T]] \to \mathbb{Q}$ be the Gauss valuation, given by $v_\eta(\sum a_i T^i) = \min\{v_R(a_i)\}$. Then, the *Kato ramification filtration* associated to the Galois extension $R[[T]] \otimes_R K$ of $R[[T]]^G \otimes_R K$ is given for the lower numbering by

$$G_{\eta,i} = \{\sigma \in G : v_\eta(\sigma(T) - T) \geq i\} \subseteq G,$$

and for the upper numbering by means of the Herbrandt formula.

Consider the map

$$\begin{aligned} R[[T]] &\longrightarrow k[[t]] \\ f &\longmapsto \bar{f} \end{aligned}$$

given by reducing the coefficients of f modulo π, and set $v_t(f) = \mathrm{ord}_t\big(\overline{\frac{f}{\pi^{v_\eta(f)}}}\big)$, where ord_t is the usual t-adic valuation on $k[[t]]$.

Definition 1.4. With notations as in 1.3, the *residual ramification filtration* associated to a Galois extension $R[[T]]$ of $R[[T]]^G$ is given for the lower numbering by

$$G_{res,i} = \{\sigma \in G : v_t(\sigma(T) - T) \geq i\} \subseteq G,$$

and for the upper numbering by means of the Herbrandt formula.

The equidistant case. The geometry and arithmetic of local actions are often intertwined. Here we are mainly interested in the geometric property of equidistance, and we show how this influences the arithmetic of the action.

Proposition 1.5. *Let λ be a local action of a p-group P in characteristic p, and suppose that it admits an equidistant lifting to characteristic zero. Then, both the Kato and the residual ramification filtrations attached to λ have a unique jump.*

Proof. Let $\Lambda : P \hookrightarrow \mathrm{Aut}(R[[T]])$ be a lift to characteristic zero of λ. Let $\sigma \in P \setminus \{1\}$ be a non-trivial automorphism. By equidistance, there is at least a ramification point for σ, and we can suppose it to be 0. Then, by Weierstrass preparation, $\frac{\sigma(T)-T}{T} = u(T)P(T)$, with $u \in R[[T]]^\times$ and P a distinguished polynomial of degree m.
Now, notice that the zeroes of $P(T)$ give the ramification points other than 0 and look at the Newton polygon of P. The equidistant property tells us that there is a unique side, of negative slope $-s$. The same argument works for any other non-trivial automorphism σ', yielding that m is again the degree of the distinguished polynomial attached to σ'. We can also see that the polynomial $\overline{\frac{P(\pi^s T)}{\pi^{ms}}} \in k[t]$ must have m distinct roots in k. By adding the previous remark, we get that

$$\forall \sigma \in P,\ v_\eta(\sigma(T) - T) = v_K(ms) \text{ and } v_t(\bar{\sigma}(t) - t) = m + 1.$$

Applying the characterization of ramification filtration given in Definition 1.3, we get the claim. □

One easy consequence is that equidistant liftings happen only in very specific cases.

Corollary 1.6. *Let $\lambda : G \hookrightarrow \mathrm{Aut}(k[[t]])$ be a local action in characteristic p having an equidistant lifting to characteristic 0. Then G is either cyclic of order prime-to-p or an elementary abelian p-group.*

Proof. Write G as a cyclic-by-p group $P \rtimes \mathbb{Z}/n\mathbb{Z}$, and suppose there exists a lifting of λ. If p divides $|G|$ and $n \geq 1$, the ramification filtration has at least two jumps. The same happens when G is a p-group that is not elementary abelian. By Proposition 1.5, the only cases left are those of the statement. $\square$

Remark 1.7. It is not true in general that all the actions of elementary abelian group are equidistant. For example, in [GM99, Chapter III, Remark 5.2.1], Green and Matignon give explicit equations of local actions of $\mathbb{Z}/p\mathbb{Z}$ that are not equidistant.

2. Lifting intermediate $\mathbb{Z}/p\mathbb{Z}$-extensions

The classical approach to lift actions of elementary abelian groups is to consider liftings of their cyclic subextensions and then discuss compatibility conditions. We recall in this section the conditions for lifting $\mathbb{Z}/p\mathbb{Z}$ actions given by the existence of some logarithmic differential forms in positive characteristic, the so-called *good deformation data*. Then, we discuss a condition for the existence of such data, due to Henrio, that we analyze in the special case where the number of poles is a multiple of p. In this situation we prove an elementary lemma, that gives a first combinatorial condition on the poles of a good deformation datum and sets the stage for the study of liftings of actions of elementary abelian groups.

2.1. **Good deformation data.** The theory of deformation data (see section 5.1 in [BW12] for an account on this) shows that an equidistant lifting to characteristic zero of a local actions of $\mathbb{Z}/p\mathbb{Z}$ with conductor m is equivalent to the existence of a **multiplicative good deformation datum**, namely a logarithmic differential form $\omega \in \Omega_{\mathbb{P}^1_k}$, having a unique zero of order $m-1$ in ∞.
Let $x_i \in k$ be the poles of ω and let $h_i \in \mathbb{F}_p^\times$ be the residue of ω in x_i. Finally, let x be a parameter for $\mathbb{P}^1_k$. Then we can write

$$\omega = \sum_{i=0}^{m} \frac{h_i}{x - x_i} dx = \sum_{i=0}^{m} \frac{h_i x_i}{1 - x_i z} dz$$

as well as

$$\omega = \frac{u}{\prod_{i=0}^{m}(x - x_i)} dx = \frac{u z^{m-1}}{\prod_{i=0}^{m}(1 - x_i z)} dz$$

after change of parameter $z = \frac{1}{x}$.
A comparison of the two formulae leads to the following set of conditions that have to be satisfied by poles and residues:

$$\begin{cases} \sum_{i=0}^{m} h_i x_i^k = 0 \qquad \text{for } 1 \leq k \leq m-1 & (2.1) \\ \prod_{i<j} (x_i - x_j) \neq 0 & (2.2) \end{cases}$$

Conversely, any $m+1$-uple of couples $\{(x_i, h_i) \in k \times \mathbb{F}_p^\times\}_{i=0,\dots,m}$ satisfying these equations gives rise to a unique multiplicative good deformation datum ω. We call any set of pairs $\{(x_i, h_i)\}$ satisfying conditions (2.1) and (2.2) the **characterizing datum** of ω. Henrio states a criterion for the existence of a multiplicative good deformation datum ([Hen, Proposition 3.16]), formulated in terms of partitions of the set of residues of ω.

Definition 2.1. Let $\mathbf{h} = \{h_0, \dots, h_m\}$ be a $m+1$-uple of elements of $\mathbb{F}_p^\times$ such that $\sum h_i = 0$. A partition $\mathcal{P}$ of $\{0, \dots, m\}$ is called $\mathbf{h}$-*adapted*, if $\sum_{j \in J} h_j = 0$ for every $J \in \mathcal{P}$. A $\mathbf{h}$-adapted partition $\mathcal{P}$ is called *maximal* if there is no $\mathbf{h}$-adapted partition refining $\mathcal{P}$.

Using the definition, we can state the criterion in the following way.

Proposition 2.2 (Partition condition)**.** *If there is a maximal* $\mathbf{h}$*-adapted partition* $\mathcal{P}$ *of* $\{0,\dots,m\}$ *such that* $|\mathcal{P}| \le \lfloor \frac{m}{p} \rfloor + 1$*, then there is a* $m+1$*-uple* $\{x_0,\dots,x_m\}$ *of elements of* k *and a multiplicative good deformation datum* ω*, such that* $\{(x_i,h_i)\}$ *is the characterizing datum of* ω*.*

Remark 2.3. Note that the partition condition is always satisfied when $p=3$, but when $p>3$, this is no-more the case (e.g. by considering the set $\mathbf{h} = \{1,1,-1,-1\}$). For a thorough discussion with more examples, see [Pag02b, 2.2.1].

When the number of poles of ω is a multiple of p this condition becomes quite restrictive:

Proposition 2.4. *If* $m+1=\lambda p$*, then the partition condition is equivalent to ask that, after possibly renumbering of the poles,*

$$\begin{cases} h_i = h_0 & \text{if } 0 \le i \le p-1 \\ h_i = h_p & \text{if } p \le i \le 2p-1 \\ \dots & \\ h_i = h_{(\lambda-1)p} & \text{if } p(\lambda-1) \le i \le \lambda p - 1. \end{cases}$$

Proof. Let $\mathcal{P}$ be a maximal $\mathbf{h}$-adapted partition of $\{h_i\}$ with $|\mathcal{P}| \le \lambda$. First notice that every set $J \in \mathcal{P}$ is of cardinality exactly p. In fact, suppose that J strictly contains a subset of indices $\{i_1,\dots,i_p\}$ of cardinality p. Then J can-not be in a maximal partition, since the set $\{h_{i_1}, h_{i_1}+h_{i_2}, h_{i_1}+h_{i_2}+h_{i_3},\dots,h_{i_1}+\cdots+h_{i_p}\}$ contains necessarily the element 0. Hence $|J| \le p$. Moreover $|J| \ge p$ because $|\mathcal{P}| \le \lambda$. Hence $|J| = p$ for every $J \in \mathcal{P}$.
Let us fix $J = \{i_1,\dots,i_p\}$. We want to show that $h_{i_j} = h_{i_k}$ for every j and k. To do this, consider the set $\{h_{i_1}, h_{i_1}+h_{i_2},\dots,h_{i_1}+h_{i_2}+\cdots+h_{i_p}\}$. As above, it contains all the elements of $\mathbb{F}_p$ because any repetition would result in a contradiction of the maximality of $\mathcal{P}$. It is the same for the set $\{h_{i_2}, h_{i_1}+h_{i_2},\dots,h_{i_1}+h_{i_2}+\cdots+h_{i_p}\}$, so that $h_{i_1} = h_{i_2}$. With analogous arguments, one shows that $h_{i_1} = h_{i_j}$ for every j. □

Remark 2.5. By writing the equality $\sum_{i=0}^m \frac{h_i x_i}{1-x_i z} dz = \frac{u z^{m-1}}{\prod_{i=0}^m (1-x_i z)} dz$ in formal power series in the variable z, one finds other conditions on a characterizing datum of a good deformation datum, namely

$$\begin{cases} \sum_{i=0}^m h_i x_i^m = u \\ \sum_{i=0}^m h_i x_i^{m+k} = u \cdot c_k(x_0,\dots,x_m) \quad \forall\, k \ge 1, \end{cases}$$

where c_k is the degree k complete homogeneous symmetric polynomial in $m+1$ variables.
While the conditions in the second line are dependent on equations (2.1), one can deduce nontrivial relations from writing them in this form. For instance, as remarked by Pagot (see the proof of Theorem 2.2.5 of [Pag02b]), when $m+1=\lambda p$, we get $0 = \left(\sum_{i=0}^m h_i x_i^\lambda\right)^p = \sum_{i=0}^m h_i x_i^{m+1} = u \cdot \sum_{i=0}^m x_i$. Hence, $\sum_{i=0}^m x_i = 0$. We will crucially use this result in the next section.

3. The elementary abelian case

We consider in what follows equidistant liftings of local actions of finite p-groups. By Proposition 1.5, it is not restrictive to suppose $G = (\mathbb{Z}/p\mathbb{Z})^n$. It is known that local actions of such groups do not lift to characteristic zero in general. Nevertheless, there are examples of actions that lift, and the question of giving a criterium to determine if a local action of an elementary abelian p-group lifts to characteristic zero is still widely open.

In this section we discuss the existence of equidistant liftings. In a first part, following Pagot (which in his turn generalizes results of Raynaud ([Ray90]) and Green-Matignon ([GM98])), we relate the existence of those liftings with the existence of compatibility conditions between good deformation data defined in section 2. Then, we make further assumptions in order to establish additional constraints on the structure of the liftings: first of all, we work with $n=2$. We

recall the known cases ($m+1=p$, $m+1=2p$ and $m+1=3p$) and we go further, by finding additional conditions on the residues of multiplicative good deformation data.
In the case $p=3$ we can carry out explicitly the computation that permit to show that, when $m+1=15$, a lifting can not occur. We do this by constructing polynomials whose zeroes are poles of the logarithmic differential forms involved. One can apply the combinatorial conditions on the poles found previously to find algebraic relations between the coefficients of such polynomials, using Newton formulae. Combining together such relations, we conclude by finding a contradiction, and by showing that there is no equidistant lifting of $(\mathbb{Z}/3\mathbb{Z})^2$-actions, when $m+1=15$. Subsection 3.1 is a survey of known results, whose proofs may differ from the original one, as the content of the rest of the section is original work of the author.

3.1. **$\mathbb{F}_p$-vector spaces of multiplicative good deformation data.** Consider a lifting to characteristic zero of an action of $G=(\mathbb{Z}/p\mathbb{Z})^n$, and let $\{\sigma_1,\dots,\sigma_n\}$ be a set of generators of G. The action of $<\sigma_i>$ lifts for every $1\le i\le n$, yielding the existence of multiplicative good deformation data $\{\omega_1,\dots,\omega_n\}$. When the lifting is equidistant, the ω_i form an n-th dimensional vector space over $\mathbb{F}_p$. One can show that the existence of such vector spaces is also sufficient to have an equidistant lifting (Section 2.4. in [Pag02a]).
Following Pagot, we say that a vector space is $L_{m+1,n}$, if it is a n-dimensional $\mathbb{F}_p$-vector space of multiplicative good deformation data with $m+1$ simple poles each. In [Pag02a], Lemme 1.2, one finds the following restriction on the poles:

Lemma 3.1. *Let there exist a vector space $L_{m+1,n}$. Then $m+1=\lambda p^{n-1}$, with $\lambda\in\mathbb{N}^*$. If $\{\omega_1,\dots,\omega_n\}\in L_{m+1,n}$ is a basis for such vector space, then any pair (ω_i,ω_j) with $i\neq j$ has exactly $\lambda(p-1)^{n-1}$ poles in common.*

From now on, we consider equidistant liftings of actions of $G=(\mathbb{Z}/p\mathbb{Z})^2$. Suppose to have a $\mathbb{F}_p$-vector space $L_{m+1,2}$ generated by two forms ω_1 and ω_2. Then $m+1=\lambda p$, and we can partition the set of poles of these forms in such a way that $\omega_1+j\omega_2$ has its poles in all but the set $X^{(j)}:=\{x_1^{(j)},\dots,x_\lambda^{(j)}\}$ for $j=0,\dots,p-1$, and that ω_2 has its poles in all but the set $X^{(p)}:=\{x_1^{(p)},\dots,x_\lambda^{(p)}\}$. We can then write

$$\omega_1:=\frac{uz^{m-1}}{\prod_{j=1}^{p}\prod_{i=1}^{\lambda}(1-x_i^{(j)}z)}dz\quad\text{and}\quad\omega_2:=\frac{vz^{m-1}}{\prod_{j=0}^{p-1}\prod_{i=1}^{\lambda}(1-x_i^{(j)}z)}dz,\quad u,v\in k^\times. \tag{3.1}$$

Then, we consider the $p+1$ polynomials $P^{(j)}$, for $j\in\{0,\dots,p\}$, defined by

$$\begin{cases}P^{(0)}=\prod_{i=1}^{\lambda}(X-x_i^{(0)})\\ \dots\\ P^{(p)}=\prod_{i=1}^{\lambda}(X-x_i^{(p)})\end{cases}$$

Our aim is to express the equations (2.1) in terms of the coefficients of such $P^{(j)}$, which are symmetric functions in the variables given by the poles. We set $p_k(X^{(j)})=\sum_{i=1}^{\lambda}{x_i^{(j)}}^k$ the k-th symmetric power sum and

$$S_k(X^{(j)})=\begin{cases}\sum_{i_1<\dots<i_k}x_{i_1}^{(j)}\dots x_{i_k}^{(j)} & \text{if}\quad 1\le k\le\lambda\\ 1 & \text{if}\quad k=0\\ 0 & \text{if}\quad k>\lambda\end{cases}$$

the k-th elementary symmetric polynomial. The polynomials $P^{(j)}$ are defined in such a way to get the following conditions on the $S_k(X^{(j)})$:

Lemma 3.2. *Let u, v be as in the Equation (3.1) and set $a=\frac{u}{v}$. Then, for $j\in\{0,\dots,p-1\}$, we have $a+j\neq 0$ and*

$$S_i(X^{(j)})=\frac{aS_i(X^{(0)})+jS_i(X^{(p)})}{a+j}.$$

Proof. Since for $j=0$ both conditions are trivially satisfied, let $j \in \{1,\dots,p-1\}$. We have

$$\omega_1 = \frac{uP^{(0)}(x)dx}{\prod_{k=0}^{p} P^{(k)}(x)} \quad , \quad \omega_2 = \frac{vP^{(p)}(x)dx}{\prod_{k=0}^{p} P^{(k)}(x)} \quad \text{and} \quad \omega_1 + j\omega_2 = \frac{w_j P^{(j)}(x)}{\prod_{k=0}^{p} P^{(k)}(x)}$$

which implies that $w_j P^{(j)}(x) = uP^{(0)}(x) + jvP^{(p)}(x)$. Since the $P^{(j)}$s are monic and of the same degree, one gets $w_j = u + jv \neq 0$. □

Corollary 3.3. *We have $S_1(X^{(j)}) = S_1(X^{(j')})$ for every $j, j' \in \{0,\dots,p\}$.*

Proof. By remark 2.5, we get $\sum_{j=1}^{p} S_1(X^{(j)}) = 0$ and $\sum_{j=0}^{p-1} S_1(X^{(j)}) = 0$. We then have $S_1(X^{(0)}) = S_1(X^{(p)})$. Then, applying Lemma 3.2, we find $S_1(X^{(j)}) = S_1(X^{(0)})$ for every j. □

3.2. **Actions of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.** From now on, we suppose that $n = 2$, and that $p = 3$. As pointed out in Remark 2.3, in this case the characterizing data for ω_1 and ω_2 satisfy the partition condition of Proposition 2.2. Most results contained in this subsection stay true when assuming that the partition condition is satisfied, even for $p \neq 3$.

We investigate here the algebraic restrictions that a set $\{h_0,\dots,h_m\}$ of elements of $\mathbb{F}_3^{\times}$ shall satisfy in order to appear in the characterizing datum of a multiplicative good deformation datum. The main tool that we use is the combinatorics of the poles of the differential forms ω_1 and ω_2. The features of such combinatorics are expressed via the set of variables $X^{(j)} = (x_1^{(j)},\dots,x_\lambda^{(j)})$, and their symmetric functions $p_k(X^{(j)})$ and $S_k(X^{(j)})$.

Notice that we have chosen the $P^{(j)}$ in such a way that the common poles of ω_1 and ω_2 in $X^{(2)}$ have the same residues, and those in $X^{(1)}$ have opposite residues. Then one can recover all the residues using the following notation: we call $h_i^{(j)}$ the residue of ω_1 in $x_i^{(j)}$ if $j \in \{1,\dots,p\}$ and $h_i^{(0)}$ the residue of ω_2 in $x_i^{(0)}$. They all are element of $\mathbb{F}_3^{\times}$ and, by Proposition 2.4, the cardinality of the set $\{x_i^{(j)} : h_i^{(j)} = 1,\ j \in \{1,2,3\}\}$ is divided by 3.

If we define $q_k(X^{(j)}) = \sum_{i=1}^{\lambda} h_i^{(j)} {x_i^{(j)}}^k$, the equations (2.1) give, for $0 \leq k \leq 3\lambda - 2$,

$$\begin{cases} q_k(X^{(1)}) + q_k(X^{(2)}) + q_k(X^{(3)}) = 0 \\ q_k(X^{(0)}) - q_k(X^{(1)}) + q_k(X^{(2)}) = 0 \end{cases} .$$

3.2.1. *Computation of the residues.* In this section we use the previous relations to compute the residues of ω_1 and ω_2 at the respective poles. We establish a formula that determines precisely their values up to replacing ω_i with $-\omega_i$ for $i = 1,2$.

Definition 3.4. Let S be a set of poles for a differential form $\omega \in L_{m+1,2}$. Then S is said of type $(n_1,\dots,n_{p-1})$ if there are exactly n_i poles in S with residue equal to i for every $i \in \mathbb{F}_p^{\times}$.

Example 3.5. The fact that $q_0(X^{(1)}) + q_0(X^{(2)}) + q_0(X^{(3)}) = 0$ and $q_0(X^{(0)}) - q_0(X^{(1)}) + q_0(X^{(2)}) = 0$ implies that there exists at least a j with $q_0(X^{(j)}) = 0$. Following the definition, such $X^{(j)}$ is a set of poles of type (n_1, n_2) with $n_1 \equiv n_2 \bmod 3$.

Example 3.5 excludes some configurations of residues that cannot occur in spaces of good deformation data. Another restriction that can be given is the following.

Proposition 3.6. *Let $\lambda = 5$. Then there are at least two values of $j \in \{0,1,2,3\}$, such that $X^{(j)}$ is not of type $(5,0)$ or $(0,5)$.*

Proof. With a proper choice of a basis for $L_{15,2}$, we may suppose that $X^{(0)}$ is of type $(4,1)$ or $(1,4)$, and that $x_5^{(0)}$ is the pole having residue different from the others. If the claim is false, we can pick ω_1 in such a way that $X^{(1)}, X^{(2)}$ and $X^{(3)}$ are of type $(5,0)$. This means in particular that $\sum_{j=1}^{3} p_k(X^{(j)}) = 0$ for every $0 \leq k \leq 13$. For $k = 2$, this gives

$$0 = \sum_{j=1}^{3} p_2(X^{(j)}) = \sum_{j=1}^{3} S_2(X^{(j)}) + \sum_{j=1}^{3} S_1(X^{(j)})^2 = \sum_{j=1}^{3} S_2(X^{(j)}),$$

the last equality being obtained by Corollary 3.3. Now, by Lemma 3.2, we have

$$\begin{aligned}\sum_{j=1}^{3} S_2(X^{(j)}) &= \frac{aS_2(X^{(0)}) + S_2(X^{(3)})}{a+1} + \frac{aS_2(X^{(0)}) - S_2(X^{(3)})}{a-1} + S_2(X^{(3)}) \\ &= \frac{-a^2S_2(X^{(0)}) + a^2S_2(X^{(3)})}{a^2-1}.\end{aligned}$$

Combining the two relations, we get that $S_2(X^{(0)}) = S_2(X^{(3)})$, that is, $S_2(X^{(j)})$, and hence $p_2(X^{(j)})$, are constant. After possible translation on the set of poles, we may take $S_1(X^{(j)}) = 0$. In this way, Newton identities give $S_4(X^{(j)}) = -S_2(X^{(j)})^2 - p_4(X^{(j)})$, and we can apply the same argument above with $k = 4$ to show that $S_4(X^{(j)})$, and hence $p_4(X^{(j)})$, are constant.
Now, since $X^{(0)}$ is of type $(4,1)$ or $(1,4)$, the relations $\sum_{j=0}^{2} q_k(X^{(j)}) = 0$ for the first values of k give:

$$\begin{aligned}0 = & x_5^{(0)} = x_1^{(0)} + x_2^{(0)} + x_3^{(0)} + x_4^{(0)} \\ 0 = & {x_5^{(0)}}^2 = {x_1^{(0)}}^2 + {x_2^{(0)}}^2 + {x_3^{(0)}}^2 + {x_4^{(0)}}^2 \\ 0 = & {x_5^{(0)}}^4 = {x_1^{(0)}}^4 + {x_2^{(0)}}^4 + {x_3^{(0)}}^4 + {x_4^{(0)}}^4.\end{aligned}$$

Using Newton identities once again, we get $S_4(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = 0 = x_5^{(0)}$, which is a contradiction since all the poles are distinct, by condition (2.2). $\square$

Remark 3.7. With the same argument, we can show that Proposition 3.6 holds more in general when $\lambda \equiv -1 \bmod 3$, assuming that $X^{(0)}$ is of type $(\lambda - 1, 1)$. This leads to show that, if $\lambda = 2$, $X^{(j)}$ are of type $(1,1)$ for every j, which gives a simpler proof of [Pag02a, Theorem 9, (ii)] for $p = 3$.

We may suppose without loss of generality that $X^{(0)}$ is a set of poles of type (n_1, n_2) with $n_1 \equiv n_2 \bmod 3$, (i.e. it is of type $(4,1)$ or $(1,4)$). We then make this assumption, and we study the possible types of $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$.

Proposition 3.8. *Let $\lambda = 5$. Then the sets of poles $(X^{(j)})$ are all of type $(4,1)$ or $(1,4)$ for $j \in \{0,1,2,3\}$.*

Proof. Recall from Section 3.1 that we denoted by S_k the k-th elementary symmetric polynomials. These satisfy the relation

$$\sum_{i=1}^{l} \frac{h_i}{X - x_i} = \frac{\sum_{j=0}^{l-1} \sum_i (-1)^j h_i S_j(x_1, \ldots, \hat{x}_i, \ldots, x_l) X^{l-1-j}}{\prod_i (X - x_i)}. \tag{3.2}$$

We consider then the set of differential forms $\{\omega_j^*\}_{j=0,\ldots,3}$ defined by $\omega_j^* = \sum_{i=1}^{5} \frac{h_i^{(j)}}{X - x_i^{(j)}} dX$, and the polynomials $Q^{(j)}$ such that $\omega_j^* = \frac{Q^{(j)}(X)}{\prod_i (X - x_i^{(j)})} dX = \frac{Q^{(j)}(X)}{P^{(j)}(X)}\mathrm{dX}$. Using formula (3.2) with $l = 5$, plus the relation

$$S_n(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \sum_{k=0}^{n} (-1)^k x_i^k S_{n-k}(x_1, \ldots, x_n)$$

we get

$$Q^{(j)}(X) = \sum_{i=0}^{4} q_i(X^{(j)}) \hat{P}_{4-i}^{(j)}(X),$$

where the "hatted" polynomials are defined by

$$\hat{P}_n^{(j)}(X) = \sum_{i=0}^{n} (-1)^{n-i} S_{n-i}(X^{(j)}) X^i$$

for every $n \leq 4$. Notice that the polynomials $Q^{(1)}$ and $Q^{(2)}$ are related to $Q^{(0)}$ and $Q^{(3)}$. In fact, from Lemma 3.2, one gets $(a+j)\hat{P}_n^{(j)}(X) = a\hat{P}_n^{(0)}(X) + j\hat{P}_n^{(3)}(X)$ for $j \in \{1,2\}$. Moreover, for $k \leq 13$ we have $q_k(X^{(j)}) = q_k(X^{(3)}) - jq_k(X^{(0)})$. From this we get

$$(a+j)Q^{(j)}(X) = jQ^{(3)}(X) - ajQ^{(0)}(X) + R(X),\ j \in \{1,2\}, \tag{3.3}$$

where $R(X) := a\sum_{i=0}^4 q_i(X^{(3)})\hat{P}_{4-i}^{(0)}(X) - \sum_{i=0}^4 q_i(X^{(0)})\hat{P}_{4-i}^{(3)}(X)$, and does not depend on j. From now on, we omit the variable X in the polynomials of the following equations, for the sake of readability.
The relations between $Q^{(j)}$s and $P^{(j)}$s are made explicit by the relations $\omega_2 = \omega_0^* - \omega_1^* + \omega_2^*$ and $\omega_1 = \omega_1^* + \omega_2^* + \omega_3^*$. These yield

$$\begin{cases} Q^{(0)}P^{(1)}P^{(2)} - Q^{(1)}P^{(0)}P^{(2)} + Q^{(2)}P^{(0)}P^{(1)} = v \\ Q^{(1)}P^{(2)}P^{(3)} + Q^{(2)}P^{(1)}P^{(3)} + Q^{(3)}P^{(1)}P^{(2)} = u \end{cases}$$

Now, after Lemma 3.2 and Corollary 3.3 one has the relations $(a+1)P^{(1)} = aP^{(0)} + P^{(3)}$ and $(a-1)P^{(2)} = aP^{(0)} - P^{(3)}$ that turn the previous conditions into

$$[a^2Q^{(0)} - a(a+1)Q^{(1)} + a(a-1)Q^{(2)}]{P^{(0)}}^2 + [(a-1)Q^{(2)} + (a+1)Q^{(1)}]P^{(0)}P^{(3)} - \\ Q^{(0)}{P^{(3)}}^2 = v(a^2-1)$$

and

$$[-(a+1)Q^{(1)} + (a-1)Q^{(2)} - Q^{(3)}]{P^{(3)}}^2 + [a(a+1)Q^{(1)} + a(a-1)Q^{(2)}]P^{(0)}P^{(3)} + \\ a^2Q^{(3)}{P^{(0)}}^2 = u(a^2-1).$$

We finally plug in equation (3.3), to get the formula

$$aQ^{(3)}{P^{(0)}}^2 - RP^{(0)}P^{(3)} - Q^{(0)}{P^{(3)}}^2 = v(a^2-1).$$

Now, if $q_0(X^{(3)}) \neq 0$, by comparing the coefficients of the polynomials in the last equation we get that $S_k(X^{(3)}) = S_k(X^{(0)})$ for every $k = 1, \ldots 5$. But this is not possible since the zeroes of $P^{(0)}$ must be distinct from those of $P^{(3)}$. Then $q_0(X^{(3)}) = \sum_{i=0}^5 h_i^{(j)}$ vanishes for every j, and the proposition is proved. □

This is a rather strong result: in this way we have determined uniquely the possible values of residues $h_i^{(j)}$. After possibly renumbering the poles (but still assuming that $X^{(j)}$ is a set of poles outside those of a given differential form in $L_{m+1,2}$), we may assume without loss of generality that $h_i^{(j)} = 1$ if $2 \leq i \leq 5$, and $h_1^{(j)} = -1$ for every $j \in \{0,1,2,3\}$, and study the equations (2.1) of Section 2.1 with only the x_i as variables.

3.2.2. *Proof of the main theorem.* In this section, we prove that there are no spaces $L_{m+1,2}$ for $p = 3$ and $m+1 = 15$. The proof goes as follows: first we use the results of the preceding section in order to get the precise set of equations 3.4, a solution to which is equivalent to the existence of a space $L_{m+1,2}$. Then we suppose that such space exists, and we look for a contradiction at the level of equations. In Proposition 3.9, a relation between the symmetric power sums of the poles is given, developing the results of Section 3.1. Combining this with Newton identities, we can subsequently establish formulae involving such power sums and other symmetric polynomials involving the same poles. By inserting such identities in the equations (3.4), we can express all the symmetric functions in terms of a single pole and of the invertible elements u and v appearing in the presentation 3.1. We finally show, using an analogous of Proposition 3.9 in higher degree, that one of the poles must be given by $\frac{u-v}{v}$, and that this yields in turn the expected contradiction.

Let us start by setting the notations we use thorough the proof. Proposition 3.8 allows us to suppose, without loss of generality, that $h_1^{(j)} = -1$ for every $j \in \{0,1,2,3\}$ and that $h_i^{(j)} = 1$ for

$i \in \{2,3,4,5\}$ and $j \in \{0,1,2,3\}$. Recall that we set $p_\ell(X^{(j)}) = {x_1^{(j)}}^\ell + \cdots + {x_5^{(j)}}^\ell$, the ℓ-th power sum symmetric polynomial. Then one can reformulate the set of equations (2.1) (respectively for ω_1 and ω_2) as

$$\begin{cases} p_\ell(X^{(1)}) + p_\ell(X^{(2)}) + p_\ell(X^{(3)}) = -{x_1^{(1)}}^\ell - {x_1^{(2)}}^\ell - {x_1^{(3)}}^\ell \\ p_\ell(X^{(0)}) - p_\ell(X^{(1)}) + p_\ell(X^{(2)}) = -{x_1^{(0)}}^\ell + {x_1^{(1)}}^\ell - {x_1^{(2)}}^\ell \end{cases} \tag{3.4}$$

Note that these equations are invariant by translation and homothetic transformations. Then we can suppose that the quantity $S_1(X^{(j)})$, which is constant for every j, by Corollary 3.3, vanishes. Moreover we can set $x_1^{(0)} = 1$. For this last change to be admissible, one has only to show that $x_1^{(0)} \neq 0$ when $S_1(X^{(j)}) = 0$. This is true, since $x_1^{(0)} - x_1^{(1)} + x_1^{(2)} = -S_1(X^{(j)}) = 0$ and the poles are distinct. Hence, from (3.4), with $\ell = 1$, one gets the following linear conditions on poles of ω_1:

$$\begin{cases} x_1^{(1)} + x_1^{(2)} + x_1^{(3)} = 0 \\ x_1^{(1)} - x_1^{(2)} = 1 \end{cases} \tag{3.5}$$

which gives $x_1^{(1)} = x_1^{(3)} - 1$ and $x_1^{(2)} = x_1^{(3)} + 1$. Once that one has these relations, it is not hard to find the right hand side of the equations in terms of the sole number $x_1^{(3)}$. To get the same for the left hand side, we apply Newton identities, combined with the relations of Lemma 3.2, to reduce the number of variables.

Proposition 3.9. *There are the following relations between $p_\ell(X^{(j)})_{j=1,2}$, $p_\ell(X^{(0)})$ and $p_\ell(X^{(3)})$:*

- $(a+j)p_2(X^{(j)}) = ap_2(X^{(0)}) + jp_2(X^{(3)})$
- $(a+j)^2 p_4(X^{(j)}) = (a+j)(ap_4(X^{(0)}) + jp_4(X^{(3)})) + aj(p_2(X^{(0)}) - p_2(X^{(3)}))^2$
- $(a+j)^2 p_5(X^{(j)}) = (a+j)(ap_5(X^{(0)}) + jp_5(X^{(3)})) - aj((p_2(X^{(0)}) - p_2(X^{(3)}))(S_3(X^{(0)}) - S_3(X^{(3)})))$

Proof. The first equation is proved by observing that $S_1(X^{(j)}) = 0$ entails $p_2(X^{(j)}) = S_2(X^{(j)})$. Let us prove the relation in degree 4: Newton identities give

$$p_4(X^{(j)}) = -S_2(X^{(j)})p_2(X^{(j)}) - S_4(X^{(j)}) \text{ for every } j \in \{0,1,2,3\}.$$

Then, using Lemma 3.2 and the relation in degree 2, we can write

$$\begin{aligned}(a+j)^2 p_4(X^{(j)}) &= -(aS_2(X^{(0)}) + jS_2(X^{(3)})(ap_2(X^{(0)}) + jp_2(X^{(3)})) - (a+j)(aS_4(X^{(0)}) + jS_4(X^{(3)})) \\ &= a(a+j)(-S_2(X^{(0)})p_2(X^{(0)}) - S_4(X^{(0)})) + j(a+j)(-S_2(X^{(3)})p_2(X^{(3)}) - S_4(X^{(3)})) + \\ &\quad aj(S_2(X^{(0)}) - S_2(X^{(3)}))(p_2(X^{(0)}) - p_2(X^{(3)})) \\ &= (a+j)(ap_4(X^{(0)}) + jp_4(X^{(3)})) + aj(p_2(X^{(0)}) - p_2(X^{(3)}))^2.\end{aligned}$$

With the same strategy we can compute the relation in the degree 5:

$$\begin{aligned}(a+j)^2 p_5(X^{(j)}) &= (aS_3(X^{(0)}) + jS_3(X^{(3)})(ap_2(X^{(0)}) + jp_2(X^{(3)})) - (a+j)(aS_5(X^{(0)}) + jS_5(X^{(3)})) \\ &= a(a+j)(-S_3(X^{(0)})p_2(X^{(0)}) - S_5(X^{(0)})) + j(a+j)(-S_3(X^{(3)})p_2(X^{(3)}) - S_5(X^{(3)})) - \\ &\quad aj(S_3(X^{(0)}) - S_3(X^{(3)}))(p_2(X^{(0)}) - p_2(X^{(3)})) \\ &= (a+j)(ap_5(X^{(0)}) + jp_5(X^{(3)})) - aj((p_2(X^{(0)}) - p_2(X^{(3)}))(S_3(X^{(0)}) - S_3(X^{(3)}))).\end{aligned}$$

□

Proposition 3.9 is the key fact to perform our strategy. By expressing $p_\ell(X^{(j)})$ in terms of $p_\ell(X^{(0)})$ and $p_\ell(X^{(3)})$ we obtain the left hand side of equations (3.4) uniquely in terms of power sums in the variables of $X^{(0)}$ and $X^{(3)}$. Since the right hand side is expressed in terms of $x_1^{(3)}$, we can explicit the $x_i^{(j)}$ in terms only of $x_1^{(3)}$. This proceeding, despite its elementary nature, is computationally rather complex. We introduce then the following notations to help us simplify the formulae.

Define $\alpha_i = p_i(X^{(0)})$, $\beta_i = (p_i(X^{(0)}) - p_i(X^{(3)}))$, $\gamma_i = (S_i(X^{(0)}) - S_i(X^{(3)}))$, $\delta_i = S_i(X^{(0)})$. Note that $S_1(X^{(j)}) = 0$ entails that $\alpha_2 = \delta_2$. If one wants to extend the results of Proposition 3.9 to homogeneous polynomial of higher degree the following lemma turns out to be useful:

Lemma 3.10. *For every pair of natural numbers $i, k > 0$ and every $j \in \{1, 2\}$ we have*
$(aS_i(X^{(0)})+jS_i(X^{(3)}))(ap_k(X^{(0)})+jp_k(X^{(3)}))-(a+j)[aS_i(X^{(0)})p_k(X^{(0)})+jS_i(X^{(3)})p_k(X^{(3)})] = -aj\gamma_i\beta_k.$

Proof.

$$(aS_i(X^{(0)}) + jS_i(X^{(3)}))(ap_k(X^{(0)}) + jp_k(X^{(3)})) - (a + j)[aS_i(X^{(0)})p_k(X^{(0)}) + jS_i(X^{(3)})p_k(X^{(3)})] =$$
$$aj(S_i(X^{(0)})p_k(X^{(3)}) + S_i(X^{(3)})p_k(X^{(0)}) - S_i(X^{(0)})p_k(X^{(0)}) - S_i(X^{(3)})p_k(X^{(3)})) =$$
$$aj(S_i(X^{(0)}) - S_i(X^{(3)}))(p_k(X^{(3)}) - p_k(X^{(0)})) = -aj\gamma_i\beta_k.$$

□

Once the notations introduced, Proposition 3.9 transforms formulae (3.3) for $\ell = 2, 4, 5$ into

$$\begin{cases} {x_1^{(1)}}^2 + {x_1^{(2)}}^2 + {x_1^{(3)}}^2 &= \frac{a^2}{a^2-1}\beta_2 \\ {x_1^{(1)}}^2 - {x_1^{(2)}}^2 - {x_1^{(0)}}^2 &= \alpha_2 - \frac{a}{a^2-1}\beta_2 \end{cases}$$
$$\begin{cases} {x_1^{(1)}}^4 + {x_1^{(2)}}^4 + {x_1^{(3)}}^4 &= \frac{a^2}{a^2-1}\beta_4 + \frac{a^2}{(a^2-1)^2}\beta_2^2 \\ {x_1^{(1)}}^4 - {x_1^{(2)}}^4 - {x_1^{(0)}}^4 &= \alpha_4 - \frac{a}{a^2-1}\beta_4 + \frac{a^3+a}{(a^2-1)^2}\beta_2^2 \end{cases}$$
$$\begin{cases} {x_1^{(1)}}^5 + {x_1^{(2)}}^5 + {x_1^{(3)}}^5 &= \frac{a^2}{a^2-1}\beta_5 - \frac{a^2}{(a^2-1)^2}\beta_2\gamma_3 \\ {x_1^{(1)}}^5 - {x_1^{(2)}}^5 - {x_1^{(0)}}^5 &= \alpha_5 - \frac{a}{a^2-1}\beta_5 - \frac{a^3+a}{(a^2-1)^2}\beta_2\gamma_3. \end{cases}$$

The left hand sides can be easily calculated, using equalities (3.5). We get:

$$\begin{aligned}
{x_1^{(1)}}^2 + {x_1^{(2)}}^2 + {x_1^{(3)}}^2 &= -1 \\
{x_1^{(1)}}^2 - {x_1^{(2)}}^2 - {x_1^{(0)}}^2 &= -x_1^{(3)} - 1 \\
{x_1^{(1)}}^4 + {x_1^{(2)}}^4 + {x_1^{(3)}}^4 &= -1 \\
{x_1^{(1)}}^4 - {x_1^{(2)}}^4 - {x_1^{(0)}}^4 &= {x_1^{(3)}}^3 + x_1^{(3)} - 1 \\
{x_1^{(1)}}^5 + {x_1^{(2)}}^5 + {x_1^{(3)}}^5 &= x_1^{(3)}(1 - {x_1^{(3)}}^2) \\
{x_1^{(1)}}^5 - {x_1^{(2)}}^5 - {x_1^{(0)}}^5 &= {x_1^{(3)}}^2(1 - {x_1^{(3)}}^2) \\
{x_1^{(1)}}^7 + {x_1^{(2)}}^7 + {x_1^{(3)}}^7 &= {x_1^{(3)}}^3 - x_1^{(3)} \\
{x_1^{(1)}}^7 - {x_1^{(2)}}^7 - {x_1^{(0)}}^7 &= {x_1^{(3)}}^6 - {x_1^{(3)}}^4.
\end{aligned}$$

With this, we write the α_i and β_i in terms of a, $x_1^{(3)}$ and γ_3:

$$\begin{aligned}
\alpha_2 &= -\frac{1}{a} - 1 - x_1^{(3)} \\
\beta_2 &= \frac{1 - a^2}{a^2} \\
\alpha_4 &= {x_1^{(3)}}^3 + x_1^{(3)} - 1 + \frac{a^2 + 1}{a^3} \\
\beta_4 &= \frac{1 - a^4}{a^4} \\
\alpha_5 &= (x_1^{(3)} + \frac{1}{a})(x_1^{(3)} - {x_1^{(3)}}^3) - \frac{\gamma_3}{a}
\end{aligned}$$

$$\beta_5 = \frac{a^2-1}{a^2} x_1^{(3)} (1 - {x_1^{(3)}}^2) - \frac{\gamma_3}{a^2}.$$

Remark 3.11. Since we are dealing with power sums in characteristic 3, when $3|i$ the computation of α_i and β_i gives a tautological condition.

Now we look for the values of δ_i and γ_i, in terms of the variables a and $x_1^{(3)}$. In order to do this, we relate the γ_i and δ_i with α_i and β_i, using Newton identities.
We get $\gamma_4 + \beta_4 = \alpha_2\beta_2 + \beta_2^2$, $\delta_4 = -\alpha_4 - \alpha_2^2$, $\alpha_5 = \alpha_2\delta_3 - \delta_5$, and $\gamma_5 + \beta_5 = \delta_3\beta_2 + (\alpha_2 - \beta_2)\gamma_3$. Then

$$\begin{aligned}
\gamma_4 &= \frac{(a^2-1)(ax_1^{(3)}+1)}{a^3} \\
\delta_4 &= -\frac{(ax_1^{(3)}+1)(a^2{x_1^{(3)}}^2 + a^2x_1^{(3)} - ax_1^{(3)} + a + 1)}{a^3}.
\end{aligned}$$

Lemma 3.12. *We have* $ax_1^{(3)} + 1 \neq 0$.

Proof. Suppose that $ax_1^{(3)} + 1 = 0$. Then $\alpha_2 = -1$ and $\delta_4 = 0$. Evaluating the polynomial $P^{(0)}$ in $x_1^{(0)} = 1$, we have that $\delta_5 = S_4(x_2^{(0)}, \dots, x_5^{(0)}) = \delta_4 - \delta_3 + \delta_2 + 1$, then that $\delta_5 = -\delta_3$. We derive that

$$P^{(0)}(X) = X^5 - X^3 - \delta_3 X^2 + \delta_3 = (X+1)(X-1)(X^3 - \delta_3)$$

and we get a contradiction, because $P^{(0)}$ can not have multiple roots. $\square$

Lemma 3.12 allows us to get formulae for $\gamma_3, \delta_3, \gamma_5$ and δ_5 in a useful form.
We have $\delta_5 = -\delta_3 - \frac{a^3({x_1^{(3)}}^3 + {x_1^{(3)}}^2 + x_1^{(3)}) + a^2(-x_1^{(3)}+1) + a + 1}{a^3}$, and we recall that $\delta_5 = \alpha_2\delta_3 - \alpha_5$. The two conditions give

$$\delta_3 = \frac{\gamma_3}{1 + ax_1^{(3)}} + \frac{a^2({x_1^{(3)}}^3 + {x_1^{(3)}}^2 + 1) + a(-x_1^{(3)} + 1) + 1}{a^2} \tag{3.6}$$

$$\delta_5 = -\frac{\gamma_3}{1 + ax_1^{(3)}} + \frac{a^3({x_1^{(3)}}^3 + {x_1^{(3)}}^2 - x_1^{(3)} - 1) + a^2(1 - x_1^{(3)}) + a - 1}{a^3}. \tag{3.7}$$

In the same spirit, evaluating $P^{(3)}$ in $x_1^{(3)}$, one finds

$$\delta_5 - \gamma_5 = x_1^{(3)}(\delta_4 - \gamma_4) - {x_1^{(3)}}^2(\delta_3 - \gamma_3) + {x_1^{(3)}}^3(\delta_2 - \gamma_2) + {x_1^{(3)}}^5.$$

This can be used to obtain a linear system in the variables γ_3 and γ_5:

$$\begin{cases}
\gamma_5 = -\frac{1 + a{x_1^{(3)}}^3}{1 + ax_1^{(3)}}\gamma_3 + \frac{(a-a^3){x_1^{(3)}}^3 + (a-a^3)x_1^{(3)} - a^3 + a^2 + a - 1}{a^3} \\
-\gamma_5 = \frac{(a^3{x_1^{(3)}}^2 - a^2x_1^{(3)} + a^2 + a - 1)}{a^2(1 + ax_1^{(3)})}\gamma_3 + \frac{(a^4-a^2){x_1^{(3)}}^3 - (a^3-a)x_1^{(3)} - a^4 + a^3 - a^2 - a - 1}{a^4}.
\end{cases} \tag{3.8}$$

Theorem 3.13. *There are no vector spaces $L_{15,2}$ for $p = 3$. As a result, there exists no equidistant lifting Λ of a local action of $(\mathbb{Z}/3\mathbb{Z})^2$ such that $|\mathfrak{R}_\Lambda| = 20$.*

Proof. To prove the theorem we put together the results of this section to show that the conditions satisfied by the pole $x_1^{(3)}$ are not compatible. By solving the linear system (3.8), we get the following formulae:

$$\begin{cases}
a^2(1 + ax_1^{(3)})(-ax_1^{(3)} + a - 1)\gamma_3 &= (a+1)^2(a-1)(ax_1^{(3)} - a - 1) \\
(a^3(1 + ax_1^{(3)})^2(-ax_1^{(3)} + a - 1))\gamma_5 &= (a^6 - a^4){x_1^{(3)}}^6 + (a^4 - a^6){x_1^{(3)}}^5 + (a^6 + a^5 - a^4 - a){x_1^{(3)}}^3 + \\
&\quad + (-a^6 - a^5 + a^4 + a^3){x_1^{(3)}}^2 + (a^3 - a)x_1^{(3)} + a^5 + a^4 - a^3 + a^2 + 1.
\end{cases}$$

Notice that $-ax_1^{(3)} + a - 1 \neq 0$, otherwise the right hand side of the first equation would also vanish, yielding $ax_1^{(3)} - a - 1 = 0 = ax_1^{(3)} - a + 1$. Then $\gamma_3 = \frac{(a+1)^2(a-1)(ax_1^{(3)}-a-1)}{a^2(1+ax_1^{(3)})(-ax_1^{(3)}+a-1)}$. Substituting this expression for γ_3 in (3.6) and (3.7), one finds

$$\begin{cases} \delta_3 = & \frac{-a^4 x_1^{(3)^6} + a^4 x_1^{(3)^4} + (-a^4 - a) x_1^{(3)^3} + (a^4 + a^3 - a) x_1^{(3)^2} + a^3 + a^2 + 1}{a(1+ax_1^{(3)})^2(-ax+a-1))} \\ \delta_5 = & \frac{a^6 x_1^{(3)^6} + a^6 x_1^{(3)^4} + (a^6 + a^5 - a^3) x_1^{(3)^2} - a^5 - a^4 + a^3 - a^2 - 1}{a^3(1+ax_1^{(3)})^2(-ax+a-1)}. \end{cases}$$

Now, calculations analogous to those of Proposition 3.9 for $\ell = 7$ lead to the following system of equations:

$$\begin{cases} x_1^{(3)}(x_1^{(3)^2} - 1) = \frac{a^2}{a^2-1}\beta_7 + \frac{a^2}{(a^2-1)^2}(\beta_2\beta_5 - \beta_2\gamma_5 + \gamma_4\gamma_3 + \beta_2^2\delta_3) + \frac{a^2}{(a^2-1)^3}\beta_2^2\gamma_3 \\ x_1^{(3)^4}(x_1^{(3)^2} - 1) = \alpha_7 - \frac{a}{a^2-1}\beta_7 + \frac{a^3+a}{(a^2-1)^2}(\beta_2\beta_5 - \beta_2\gamma_5 + \gamma_4\gamma_3 + \beta_2^2\delta_3) - \frac{a^5}{(a^2-1)^3}\beta_2^2\gamma_3. \end{cases}$$

After performing all due substitutions, we find that $x_1^{(3)}$ is a root of the polynomial

$$\begin{aligned} -a^2X^5 + (-a^2 + a)X^4 - X^3 + (a^2 + a + 1)X^2 + (a^2 + 1)X + a - 1 = \\ (X - 1)^2(X + 1)(aX + 1)(-aX + (a - 1)). \end{aligned}$$

Now, $x_1^{(3)} \notin \mathbb{F}_3$, then it is either $x_1^{(3)} = -\frac{1}{a}$ or $x_1^{(3)} = \frac{a-1}{a}$. The first scenario is not possible by Lemma 3.12, and the second one can not yield as well, as we remarked at the beginning of the proof. Thus we find a contradiction that shows that a vector space $L_{15,2}$ can not exist. □

4. An algorithm for finding good deformation data

As the previous section shows, the calculations involved in finding a good deformation datum can be rather complex to hand, and a generalization of the techniques used in this paper requires the introduction of computational tools. To this scope, we present in this section an algorithm to determine the existence of spaces $L_{m+1,2}$. In particular, one can implement the algorithm in Singular to get a proof of Theorem 3.13 without the use of Proposition 3.8. The first main step in the algorithm is to isolate a finite number of possible types for poles configurations of a differential form in a space $L_{m+1,2}$. As before, we divide the set of poles in $p+1$ packets $X^{(0)}, \dots, X^{(p)}$, each consisting of λ poles, in such a way that $q_0(X^{(0)}) = 0$ (this is always possible by reasoning as in Example 3.5). The possible types of $X^{(0)}$ are then all $(n_1, \dots, n_{p-1})$ such that p divides $\sum_{i=1}^{p-1} in_i$. Then we can isolate all possible types for the packets $X^{(i)}$ with $i > 0$. This is done using Proposition 2.4 and a generalization of Proposition 3.6. The second step is to determine the existence of a solution for the system given by (2.1) and (2.2). This could be done in principle using any algorithm for checking inclusion of ideals, but in most cases it would not be very effective, since computing a Gröbner basis for the ideal associated with the polynomial system (2.1) does not give an answer in a reasonable timeframe. We then modify this ideal via a suitable automorphism $\sigma \in \mathrm{Aut}(\mathbb{F}_p[x_1, \dots, x_{\lambda(p+1)}])$ that acts on the variables as follows. If $x_{j_1}, \dots, x_{j_k} \in X^{(i)}$ are a maximal set of variables in $X^{(i)}$ corresponding to poles with the same residue, $k < p$, and $\zeta \in \mathbb{F}_p$ is a primitive k-th root of unity, then we set

$$\begin{cases} \sigma(x_{j_1}) = x_{j_1} + \dots + x_{j_k} \\ \sigma(x_{j_2}) = x_{j_2} + \zeta x_{j_3} + \dots + \zeta^{k-1} x_{j_1} \\ \dots \\ \sigma(x_{j_k}) = x_{j_k} + \zeta^{k-1} x_{j_1} + \dots + (\zeta^{(k-1)})^{(k-1)} x_{j_{k-1}}. \end{cases}$$

Whenever $k \geq p$, we partition the set $\{x_{j_1}, \dots, x_{j_k}\}$ into smaller sets of cardinality smaller than p and apply the definition above. This condition on the variables defines a unique automorphism $\sigma \in \mathrm{Aut}(\mathbb{F}_p[x_1, \dots, x_{\lambda(p+1)}])$. By applying σ to the polynomial system we wish to solve, we find that the Gröbner basis computation run much faster, significantly accelerating the whole

process, and resulting in a computational proof of Theorem 3.13.
The algorithm is as follows:

Algorithm 1 Existence of Good Deformation Data for $n = 2$

Input: $p \in \mathbb{N}$ prime and $\lambda \in \mathbb{N}$
Output: returns 1 if there is a space $L_{\lambda p,2}$, otherwise 0.

Compute possible types of the sets $X^{(0)}$ and $X^{(p)}$;
If $q_0(X^{(p)}) = 0$, then $q_0(X^{(i)}) = 0$;
If $q_0(X^{(p)}) \neq 0$, then compute possible types in $X^{(i)}$;
Index types t_i with $i \in \{0, \ldots, k\}$;
$S = \mathbb{F}_p[x_1, \ldots, x_{\lambda(p+1)}]$;
ideal $J \subset S$ generated by $\prod_{i \neq j}(x_i - x_j)$;
for $0 \leq i \leq k$ **do**
  ideal $I_i \subset S$ generated by polynomials in (2.1);
  Compute $\sigma \in \mathrm{Aut}(S)$;
  $K_i = \sigma(I_i)$;
  $L = \sigma(J)$;
  if $L \subset K_i$ **then**
    $s_i = 1$
  else
    $s_i = 0$;
  $i = i + 1$;
$s = \prod_{i=1}^{k}(s_i)$
if s=1 **then**
  return 0
else
  return 1

For λ and p fixed, the possible types for poles configurations of differential forms in a space $L_{m+1,2}$ are in finite number. Together with the fact that the procedure for computing inclusion of ideals ends, this proves that Alogrithm 1 ends as well. It gives the desired output because inclusion of ideals is respected by automorphisms of a polynomial ring. The algorithm can be implemented in *Singular*.

5. Consequences and perspectives

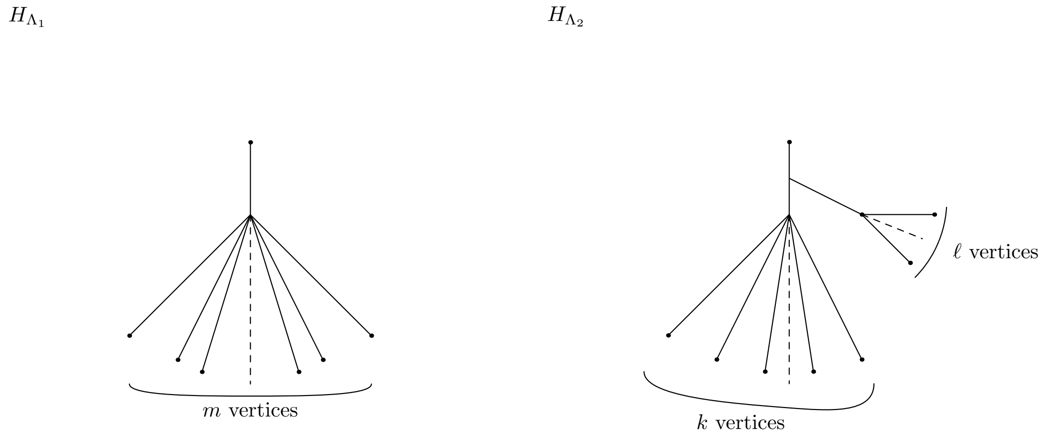
In this section we briefly regard how the results of this paper are connected to the questions presented in the introduction. We also outline a strategy for future work that would help tackle the computational difficulties of the approach used so far.

Other equidistant cases. As remarked in Section 3.2, the combinatorial results on the residues remain true also when $p \neq 3$, provided that the partition condition is satisfied. In this case, the arguments of proof of Theorem 3.13 can be carried on as well. Nevertheless, the computations differ for each p, and their complexity rises considerably. In order to get more general obstructions, we plan to combine geometric methods (e.g. formal patching) with more involved arguments of enumerative combinatorics.

Rigid ramification locus and Hurwitz trees. The result of Theorem 3.13 gives an obstruction to the existence of lifting of local actions in terms of their ramification locus: while it is still an open problem to show that there are no liftings of actions of $(\mathbb{Z}/3\mathbb{Z})^2$ with 20 ramification points, we have proved that the geometry of those points can not be equidistant. We remark that other configurations can be excluded thanks to Theorem 3.13, by studying the combinatorics of intermediate $\mathbb{Z}/3\mathbb{Z}$-covers.

Example 5.1. Let Λ be a lifting of an action of $(\mathbb{Z}/3\mathbb{Z})^2 = \langle \sigma_1, \sigma_2 \rangle$ with the following property: there exists a partition of the rigid ramification locus $\{R_1, R_2\} \subset \mathfrak{R}_\Lambda$ such that, for $i = 1, 2$, given $x \in R_i$, one has $|x - y| = |x - z|$ for every $y, z \in R_i \setminus \{x\}$. Consider now Λ_1 and Λ_2, the local actions giving rise to the intermediate extensions $R[[T]]^{\langle \sigma_1 \rangle}/R[[T]]^G$ and $R[[T]]^{\langle \sigma_2 \rangle}/R[[T]]^G$ respectively. Suppose that the intersections $\mathfrak{R}_{\Lambda i} \cap R_j$ are non-empty for every $1 \leq i, j \leq 2$, i.e. that both Λ_1 and Λ_2 are not equidistant. Then, by considerations on the good reduction at the boundary, we must have $\mathfrak{R}_{\Lambda 1} \cap R_j = \mathfrak{R}_{\Lambda 2} \cap R_j$ for $j = 1, 2$. This implies in particular that $|\mathfrak{R}_{\Lambda 1}| = |\mathfrak{R}_{\Lambda 2}|$, and that this cardinality can be deduced from the global number of ramification points. For example, if $|\mathfrak{R}_\Lambda| = 20$, then $|\mathfrak{R}_{\Lambda i}| = 15$ (because $|\mathfrak{R}_{\Lambda 1} \cap \mathfrak{R}_{\Lambda 2}| = 10$). Now, either $|\mathfrak{R}_{\Lambda 1} \cap R_1|$ or $|\mathfrak{R}_{\Lambda 1} \cap R_2|$ is an odd number smaller than 15, yielding the existence of a space $L_{3q,2}$ with q odd. Theorem 3.13 and previous results of Pagot show that this is not possible. Hence, the only case that can occur is when either Λ_1 or Λ_2 is equidistant.

With similar arguments, one can give several obstructions to lifting in the non-equidistant case, that can be formulated only in terms of the rigid ramification locus. A general formalism that is useful to formulate such obstructions is that of Hurwitz trees, introduced by Henrio in [Hen] to study local actions of $\mathbb{Z}/p\mathbb{Z}$ and partially generalized by Brewis and Wewers in [BW09]. In the case of the previous example, by fixing the cardinality of $\mathfrak{R}_\Lambda$ and by bounding the Hurwitz tree of Λ, we show that this gives restrictive conditions on the shape of Hurwitz trees of Λ_1 and Λ_2. With a slightly deeper use of known results about Hurwitz trees, one can show that when $|\mathfrak{R}_\Lambda| = 20$ and the Hurwitz tree of Λ has at most 4 non-terminal edges, we can have only the following Hurwitz trees for Λ_i



where $(12, 8, 8)$, $(9, 6, 11)$ and $(3, 2, 17)$ are the only possible triples for (m, k, ℓ).

Future work. In the PhD thesis [Tur14], we introduced a non-Archimedean analytic formalism to study Hurwitz trees, based on the theory of Berkovich analytic curves. Namely, we proved that Hurwitz trees are canonically embedded in the Berkovich unit disc over K, so that to each point of a Hurwitz tree one can associate a point of an analytic curve on which G acts upon. This opens the way to the use of Berkovich techniques, such as the theory of skeletons and piecewise monomial functions (as used in the work of Temkin [Tem14] and Cohen-Temkin-Trushin [CTT14]) and the theory of line bundles endowed with a metric (developed by Thuillier [Thu05] in the case of curves, and by Ducros and Chambert-Loir [CLD] in a whole generality), in the study of lifting problems. In the specific case of equidistant actions of elementary abelian groups, a characterization of good deformation data in these new terms would help us to reduce the computational complexity of the calculations involved in the proof of the existence of vector spaces $L_{m+1,n}$.

Another strategy that we deem useful to improve the results of this paper consists in deforming the cover in equicharacteristic p. This method is the core of Pop's approach to the Oort conjecture ([Pop14]), and has been used by Obus to prove lifting results for some dihedral and alternate groups (respectively in [Obu15] and [Obu16]). In these works, one uses such deformations to reduce the lifting problem to specific cases, where the ramification theory is somehow simpler. In the case of elementary abelian groups, we hope to use this approach to get two kinds of results: deduce the non-equidistant case from the equidistant one and reduce the lifting problem to the case of bounded conductor. In this way, it would be sufficient to prove or disprove the existence of spaces $L_{m+1,n}$ in a finite number of cases to get a general criterion for lifting. As a consequence, by combining this approach with Theorem 3.13, we would obtain a serious improvement of the results of the present paper.

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