Moduli spaces of curves via Schottky spaces over $\mathbf{Z}$

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Outline

1. Uniformization of curves
2. Berkovich spaces over $\mathbf{Z}$
3. Schottky space and universal Mumford curve over $\mathbf{Z}$

Outline

Schottky uniformization

Let $X^{an}/\mathbf{C}$ be a compact Riemann surface of genus g.

Theorem (Koebe *Rückkehrschnitt* theorem)

There exist $\Omega \subset \mathbf{P}^{1,\mathrm{an}}_{\mathbf{C}}$ *open dense subset, and* $\Gamma \subset \mathrm{PGL}_2(\mathbf{C})$ *a free group of rank* g *such that* $\Omega/\Gamma \cong X^{an}$.

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Let K be a complete non-archimedean valued field, and X/K a smooth projective curve of genus g with **split degenerate reduction** (i.e. all components of the stable reduction of $X_{\overline{K}}$ are $\mathbf{P}^1$).

Theorem (Mumford)

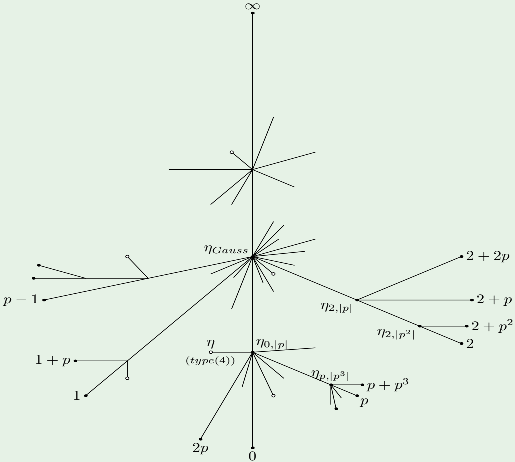
There exist $\Omega \subset \mathbf{P}^{1,\mathrm{an}}_{K}$ open dense subset, and $\Gamma \subset \mathrm{PGL}_2(K)$ a free group of rank g such that $\Omega/\Gamma \cong X^{\mathrm{an}}$.

Berkovich's $\mathbf{P}_K^{1,\mathrm{an}}$

$$\mathbf{P}_K^{1,\mathrm{an}} = \{x : K[T] \to \mathbf{R}_{\geq 0} \text{ multiplicative semi-norm : } x_{|K} = |\cdot|_K\} \cup \{\infty\}$$

Example

$\mathbf{P}_{\mathbf{C}_p}^{1,\mathrm{an}}, p \neq 2, 3$:



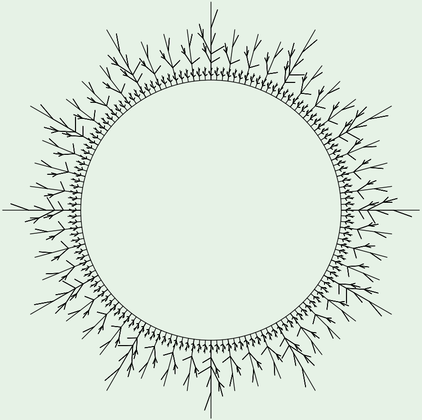
Berkovich's Tate curve

Let $\lambda \in K$, $A_\lambda = K[X,Y]/Y^2 - X(X-1)(X-\lambda)$, $E_\lambda := \mathrm{Spec}(A_\lambda)$

$E_\lambda{}^{an} = \{x : A_\lambda \to \mathbf{R}_{\geq 0} \text{ multiplicative semi-norm} : x_{|K} = |\cdot|_K\} \cup \{\infty\}$

Example

If $0 < |\lambda| < 1$, $E_\lambda{}^{an}$ looks like:



Schottky groups

Let $(k, |\cdot|)$ be a complete valued field. We denote by $\mathbf{P}_k^{1,\mathrm{an}}$

- the Berkovich projective line if k is non-archimedean;
- $\mathbf{P}^1_{\mathbf{C}}$ if $k = \mathbf{C}$;
- $\mathbf{P}^1_{\mathbf{C}}/\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ if $k = \mathbf{R}$.

Let Γ be a subgroup of $\mathrm{PGL}_2(k)$. It acts on $\mathbf{P}_k^{1,\mathrm{an}}$.

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Let Γ be a subgroup of $\mathrm{PGL}_2(k)$. It acts on $\mathbf{P}_k^{1,\mathrm{an}}$.

We say that Γ acts discontinuously at $x \in \mathbf{P}_k^{1,\mathrm{an}}$ if there exists a neighborhood U_x of x such that

$$\{\gamma \in \Gamma \mid \gamma(U_x) \cap U_x \neq \emptyset\} \text{ is finite.}$$

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A Schottky group over k is a finitely generated free subgroup of $PGL_2(k)$ containing only hyperbolic elements and with a nonempty discontinuity locus. The complement $\mathscr{L}$ of the discontinuity locus, called the limit set, is compact and contains only type 1 points.

Outline

The Berkovich analytic space $\mathbf{A}_A^{n,\mathrm{an}}$

Let $(A, \|\cdot\|)$ be a commutative Banach ring with unit. Let $n \in \mathbf{N}$.

The analytic space $\mathbf{A}_A^{n,\mathrm{an}}$ is the set of multiplicative semi-norms on $A[T_1, \ldots, T_n]$ bounded on A, *i.e.* maps

$$|.| : A[T_1, \ldots, T_n] \to \mathbf{R}_+$$

such that

1. $|0| = 0$ and $|1| = 1$;
2. $\forall f, g \in A[T_1, \ldots, T_n]$, $|f + g| \leqslant |f| + |g|$;
3. $\forall f, g \in A[T_1, \ldots, T_n]$, $|fg| = |f|\,|g|$;
4. $\forall f \in A$, $|f| \leqslant \|f\|$.

The topology on $\mathbf{A}_A^{n,\mathrm{an}}$

The set $\mathbf{A}_A^{n,\mathrm{an}}$ is endowed with the coarsest topology such that, for any f in $A[T_1, \ldots, T_n]$, the evaluation function

$$\begin{array}{ccc} \mathbf{A}_A^{n,\mathrm{an}} & \to & \mathbf{R}_+ \\ |.|_x & \mapsto & |f|_x \end{array}$$

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is continuous.

Theorem (Berkovich)

The space $\mathbf{A}_A^{n,\mathrm{an}}$ is Hausdorff and locally compact.

The structure sheaf on $\mathbf{A}_A^{n,\mathrm{an}}$

To each $x \in \mathbf{A}_A^{n,\mathrm{an}}$, we associate a residue field

$$\mathscr{H}(x) := \text{ completion of the fraction field of } A[T_1, \ldots, T_n]/\mathrm{Ker}(|\cdot|_x)$$

and an evaluation map

$$\chi_x \colon A[T_1, \ldots, T_n] \to \mathscr{H}(x).$$

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For every open subset U of $\mathbf{A}_A^{n,\mathrm{an}}$, $\mathscr{O}(U)$ is the set of maps

$$f \colon U \to \bigsqcup_{x \in U} \mathscr{H}(x)$$

such that

- $\forall x \in U,\ f(x) \in \mathscr{H}(x)$;
- f is locally a uniform limit of rational functions without poles.

Properties of $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$

Theorem (Lemanissier)

The space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ is locally path-connected.

Theorem (Poineau)

- *For every x in $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$, the local ring $\mathscr{O}_x$ is henselian, noetherian, regular, excellent.*
- *The structure sheaf of $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ is coherent.*

Theorem (Lemanissier - Poineau)

Relative closed and open discs over $\mathbf{Z}$ are Stein.

Outline

Koebe coordinates

To $\gamma \in \mathrm{PGL}_2(k)$ hyperbolic, we associate

- $\alpha \in \mathbf{P}^1(k)$ its attracting fixed point;
- $\alpha' \in \mathbf{P}^1(k)$ its repelling fixed point;
- $\beta \in k$ the quotient of its eigenvalues with absolute value < 1.

For $\alpha, \alpha', \beta \in k$ with $|\beta| \in (0, 1)$, we have

$$M(\alpha, \alpha', \beta) = \begin{pmatrix} \alpha - \beta\alpha' & (\beta - 1)\alpha\alpha' \\ 1 - \beta & \beta\alpha - \alpha' \end{pmatrix}.$$

Schottky space

Let $g \geqslant 2$. The Schottky space $\mathscr{S}_g$ is the subset of $\mathbf{A}_{\mathbf{Z}}^{3g-3,\mathrm{an}}$ consisting of the points

$$z = (x_3, \dots, x_g, x'_2, \dots, x'_g, y_1, \dots, y_g)$$

such that the subgroup of $\mathrm{PGL}_2(\mathscr{H}(z))$ defined by

$$\Gamma_z := \langle M(0, \infty, y_1), M(1, x'_2, y_2), M(x_3, x'_3, y_3), \dots, M(x_g, x'_g, y_g) \rangle$$

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is Schottky.

Theorem (Poineau - T.)

The Schottky space $\mathscr{S}_g$ is a connected open subset of $\mathbf{A}_{\mathbf{Z}}^{3g-3,\mathrm{an}}$.

Universal Mumford curve

Denote by $(X_3, \dots, X_g, X'_2, \dots, X'_g, Y_1, \dots, Y_g)$ the coordinates on $\mathbf{A}_{\mathbf{Z}}^{3g-3,\mathrm{an}}$ and consider the subgroup of $\mathrm{PGL}_2(\mathscr{O}(\mathscr{S}_g))$:

$$\Gamma = \langle M(0, \infty, Y_1), M(1, X'_2, Y_2), M(X_3, X'_3, Y_3), \dots, M(X_g, X'_g, Y_g)\rangle.$$

Theorem (Poineau - T.)

There exists a closed subset $\mathscr{L}$ of $\mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_g} := \mathscr{S}_g \times_{\mathcal{M}(\mathbf{Z})} \mathbf{P}^{1,\mathrm{an}}_{\mathbf{Z}}$ such that

1. *for each $z \in \mathscr{S}_g$, $\mathscr{L} \cap \mathrm{pr}_1^{-1}(z)$ is the limit set of Γ_z;*
2. *we have a commutative diagram of analytic morphisms*

$$\begin{tikzcd}
\mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_g} - \mathscr{L} \arrow[d] \arrow[dr] & \\
& (\mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_g} - \mathscr{L})/\Gamma =: \mathscr{X}_g \arrow[dl] \\
\mathscr{S}_g &
\end{tikzcd}$$

Connections with $\mathscr{M}_g$ and M_g^{trop}

The Schottky space $\mathscr{S}_g$ and the universal Mumford curve $\mathscr{X}_g$ fit in a commutative diagram:

$$
\begin{tikzcd}
\mathscr{U}_g \arrow[d] & \mathscr{X}_g \arrow[l] \arrow[d] \\
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Theorem (Poineau - T.)

The topological space $M_g^{\mathrm{trop}} := \sqcup \mathbf{R}_{>0}^{E(G)}$ can be embedded into $\mathscr{S}_g$. Moreover, there exists an open subset $\mathscr{B}_g \subset \mathscr{S}_g$ and a deformation retraction of $\mathscr{B}_g$ to M_g^{trop}.

What's next?

- Homotopy type of $\mathscr{S}_g$
  Teichmüller space over $\mathbf{Z}$ (cf. Culler-Vogtmann "Outer space")
- Steinness of $\mathscr{S}_g$
- Periods $(q_{i,j})_{1 \leqslant i,j \leqslant g}$ and Jacobians (Manin-Drinfeld, Myers)
- q-expansions of modular forms (Ichikawa)
  Schottky problem (= characterize the Torelli locus inside $\mathscr{A}_g$)
- Gauß-Manin connections
  Picard-Fuchs equations (Gerritzen):

$$\text{for } 1 \leqslant i \leqslant g, \begin{cases} \nabla\left(\dfrac{du_i}{u_i}\right) = \displaystyle\sum_{j=1}^{g} \beta_j \otimes \dfrac{dq_{i,j}}{q_{i,j}}; \\ \nabla(\beta_i) = 0. \end{cases}$$