Hypergeometric functions, part II

Special Functions Reading Group

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The Hypergeometric Series

Recall the (Euler) hypergeometric series:

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n. $$

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It is a solution of the hypergeometric equation

$$ E(a,b,c) : x(1-x)\frac{d^2y}{\mathrm{d}x^2} + [c-(a+b+1)x]\frac{dy}{\mathrm{d}x} - aby = 0, $$

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Fact

Every second-order ODE with three regular singularities can be transformed into a hypergeometric one.

Monodromy of ODE with regular singular points

The space of solutions of $E(a, b, c)$ around a point $x_0 \in \mathbf{C} - \{0, 1\}$ is a 2-dimensional vector space over $\mathbf{C}$. Any such solution can be analytically continued along every path γ in $\mathbf{C} - \{0, 1\}$.

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Let γ be a loop starting and ending in x_0, and y_1, y_2 two linearly independent solutions of $E(a,b,c)$ around x_0. We have:

$$\begin{pmatrix} \gamma_* y_1 \\ \gamma_* y_2 \end{pmatrix} = M_\gamma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The correspondence $\gamma \mapsto M_\gamma$ realizes a group representation

$$\rho : \pi_1(\mathbf{C} - \{0,1\}, x_0) \longrightarrow GL_2(\mathbf{C})$$

called monodromy representation. It is uniquely associated with $E(a,b,c)$ up to conjugation in $\mathrm{GL}_2(\mathbf{C})$.

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The (conjugacy class of) $\rho(\pi_1(\mathbf{C} - \{0, 1\}, x_0))$ is the monodromy group of $E(a, b, c)$.

Another solution of $E(a, b, c)$

Let D be the differential operator $y \mapsto x\frac{dy}{\mathrm{d}x}$.
Then $E(a, b, c)$ is

$$[(a + D)(b + D) - (c + D)(1 + D)\frac{1}{x}]y = 0.$$

The equality of differential operators

$$Dx^s = x^s(s + D)$$

yields

$$[(a+D)(b+D)-(c+D)(1+D)\frac{1}{x}]x^{1-c} = x^{1-c}[(a+1-c+D)(b+1-c+D)-(1+D)(2-c+D)\frac{1}{x}].$$

Then, $x^{1-c}{}_2F_1\left(\begin{matrix} a+1-c, b+1-c \\ 2-c \end{matrix}; x\right)$ is a second solution of $E(a, b, c)$ when $c \notin \mathbf{N}$.

Solutions around 1

The hypergeometric equation

$$E(a,b,c) : x(1-x)\frac{d^2y}{\mathrm{d}x^2} + [c-(a+b+1)x]\frac{dy}{\mathrm{d}x} - aby = 0$$

for $\xi = 1-x$ becomes

$$\xi(1-\xi)\frac{d^2y}{d\xi^2} + [a+b+1-c-(a+b+1)\xi)]\frac{dy}{d\xi} - aby = 0.$$

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As before, another solution can be found:

$$(1-x)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a, & c-b \\ c+1-a-b \end{matrix}; 1-x\right).$$

for $c - a - b \notin \mathbf{N}$

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We have

$$\begin{aligned} &[(a-D)(b-D) - (c-D)(1-D)\xi]\xi^a \\ &= (-a+D)(-b+D)\xi^a - (-c+D)(-1+D)\xi^{1+a} \\ &= -\xi^{1+a}[(1+a-c+D)(a+D) - (1+D)(a-b+1+D)\frac{1}{\xi}]. \end{aligned}$$

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Two solutions are:

$\xi^a {}_2F_1\left(\begin{matrix} a, & 1+a-c \\ a-b+1 & \end{matrix}; \xi\right)$ and $\xi^b {}_2F_1\left(\begin{matrix} b, & 1+b-c \\ b-a+1 & \end{matrix}; \xi\right)$
for $a-b \notin \mathbf{N}$.

Connnection matrices

We found six solutions of $E(a, b, c)$:

- f_{01}, f_{02} continuation in $\mathbf{C} \setminus \{(-\infty, 0] \cup [1, +\infty)\}$ of the solutions around 0
- f_{11}, f_{12} continuation in $\mathbf{C} \setminus (-\infty, 1]$ of the solutions around 1
- $f_{\infty 1}, f_{\infty 2}$ continuation in $\mathbf{C} \setminus [0, +\infty)$ of the solutions around ∞

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In the domain $\mathbb{H}_+ = \{x \in \mathbf{C} : \mathrm{Im}(x) > 0\}$, these solutions lie in a 2-dimensional $\mathbf{C}$-vector space. Hence there are matrices M_+^{10} and $M_+^{\infty 0}$ in $GL_2(\mathbf{C})$ such that

$$\begin{pmatrix} f_{01} \\ f_{02} \end{pmatrix} = M_+^{10} \begin{pmatrix} f_{11} \\ f_{12} \end{pmatrix}$$

$$\begin{pmatrix} f_{01} \\ f_{02} \end{pmatrix} = M_+^{\infty 0} \begin{pmatrix} f_{\infty 1} \\ f_{\infty 2} \end{pmatrix}$$

called connection matrices.

Schwarz triangles

Define continuous maps f_i:

$$f_i(x) := [f_{i1}(x) : f_{i2}(x)] \in \mathbf{P}^1(\mathbf{C}) \quad \text{for } i = 0, 1, \infty.$$

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Note:

- $f_0((0,1)) = (f_0(0), f_0(1))$
- $f_1((1,\infty)) = (f_1(1), f_1(\infty))$
- $f_\infty((-\infty,0)) = (f_\infty(-\infty), f_\infty(0))$

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In $\mathbb{H}_+$, f_0, f_1, f_∞ are related by linear fractional transformations (given by the connection matrices), that send lines to circles and lines: the boundary of $f_i(\mathbb{H}_+)$ is a "triangle with circular sides", a.k.a. a Schwarz triangle.

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The map f_0 can be extended to $\mathbb{H}_-$ through any of the three connected components of $\mathbf{R} \setminus \{0, 1\}$.

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The map f_0 can be extended to $\mathbb{H}_-$ through any of the three connected components of $\mathbf{R} \setminus \{0, 1\}$. The resulting image $f_0(\mathbb{H}_-)$ is found by applying the following:

Theorem (Schwarz Reflection Principle)

Let f be a holomorphic function on $\mathbb{H}_+ \cup (a, b) \cup \mathbb{H}_-$, and let $f((a, b))$ be a circle C. Then, $f(\mathbb{H}_-) = g^{-1}\left(\overline{g \circ f(\mathbb{H}_+)}\right)$ for any $g \in \mathrm{PGL}_2(\mathbf{C})$ sending C into $\mathbf{R} \cup \{\infty\}$.

$\implies f(\mathbb{H}_-)$ is the mirror image of $f(\mathbb{H}_+)$ with respect to C.

Analytic continuation along paths

Let γ be a loop, starting at $x_0 \in \mathbb{H}_+$, going around 0.

The image $f_0(\gamma)$ is a path in $\mathbf{P}^1(\mathbf{C})$, crossing the Schwarz triangles $f(\mathbb{H}_+)$, $f(\mathbb{H}_-)$ and a mirror image of $f(\mathbb{H}_-)$.

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The analytic continuation $\gamma_\star f_0$ is a fractional linear transformation: there is $M_\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{C})$ such that

$$\gamma_\star f = \frac{af + b}{cf + d}.$$

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The assignment $\gamma \mapsto M_\gamma$ describes the projective monodromy representation

$$\tilde{\rho} : \pi_1(\mathbf{C} - \{0, 1\}, x_0) \longrightarrow \mathrm{PGL}_2(\mathbf{C}).$$

The projective monodromy

Let's compute the projective monodromy groups $\tilde{\rho}(\pi_1(\mathbf{C} - \{0, 1\}, x_0))$.

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Theorem

The angles of the Schwarz triangle $f_0(\mathbb{H}_+)$ are:

- $|1 - c|\pi$ *at "$f_0(0)$"*
- $|c - a - b|\pi$ *at "$f_0(1)$"*
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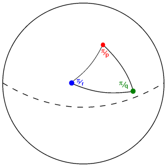
Suppose that the angles are integral quotients of π, and define $|1-c| = \frac{1}{p}$, $|c-a-b| = \frac{1}{q}$, $|a-b| = \frac{1}{r}$.

We are in one of three cases:

- $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ (Spherical)
- $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ (Euclidean)
- $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ (Hyperbolic)

Spherical monodromy



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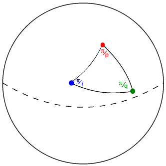

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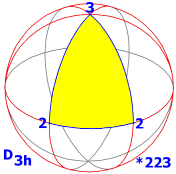
- $p = 2, q = 2 \implies$ Dihedral monodromy (D_{2r})



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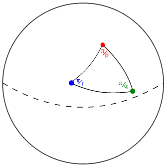

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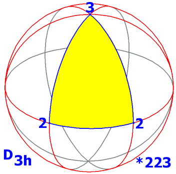

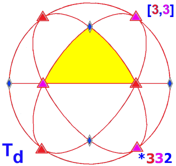

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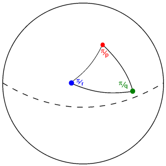

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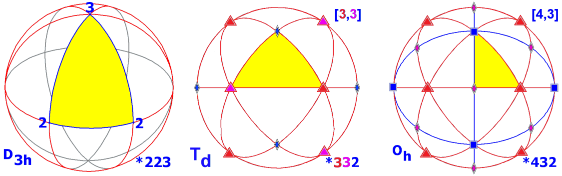

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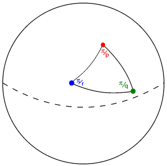

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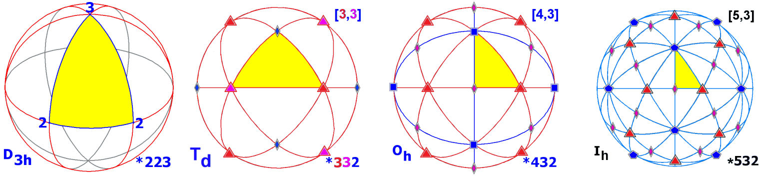
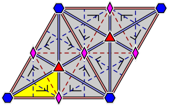
- $p = 2, q = 3, r = 5 \implies$ Icosahedral monodromy (A_5)



Euclidean monodromy

Infinite Schwarz triangles on the Euclidean plane, but only finite possibilities:

- $p = 2, q = 3, r = 6 \implies$ Hexagonal lattice



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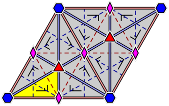

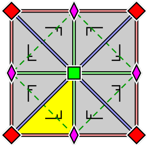

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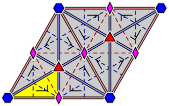

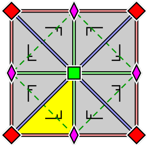

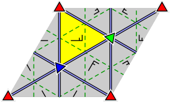

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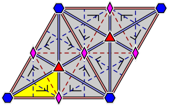
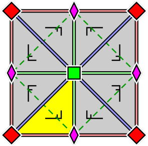
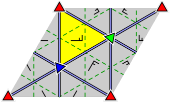
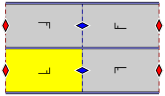
- $(p = 2, q = 2, r = \infty \implies$ Ruler$)$

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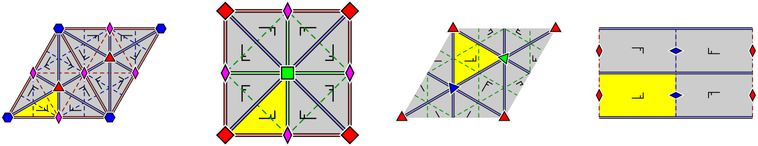


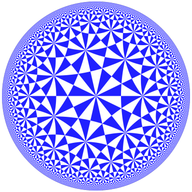
Note: the projective monodromy is a discrete subgroup of affine transformations ($f \mapsto af + b$), i.e. a Wallpaper group.

Hyperbolic monodromy

Infinite Schwarz triangles on the Hyperbolic plane, and infinite possibilities.

Some examples:

- $p = 2, q = 3, r = 7 \implies$ (2,3,7) triangular group

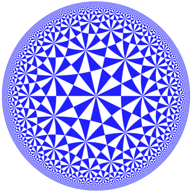


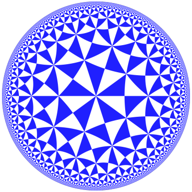
Hyperbolic monodromy

Infinite Schwarz triangles on the Hyperbolic plane, and infinite possibilities.

Some examples:

- $p = 2, q = 3, r = 7 \implies$ (2,3,7) triangular group
- $p = 2, q = 4, r = 5$





Hyperbolic monodromy

Infinite Schwarz triangles on the Hyperbolic plane, and infinite possibilities.

Some examples:

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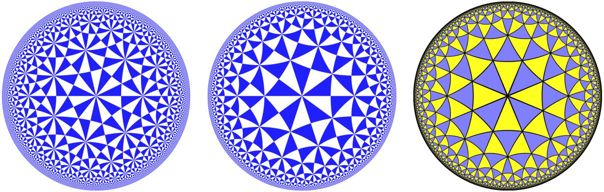

Hyperbolic monodromy

Infinite Schwarz triangles on the Hyperbolic plane, and infinite possibilities.

Some examples:

- $p = 2, q = 3, r = 7 \implies$ (2,3,7) triangular group
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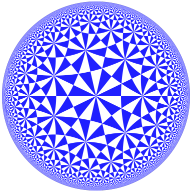
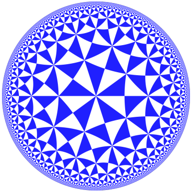
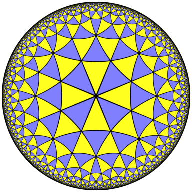
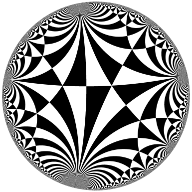
- $p = 3, q = 3, r = 4$
- ($p = 2, q = 3, r = \infty \implies$ conjugate to $\mathrm{PSL}_2(\mathbf{Z})$)

The End

. . . but in fact it's just the beginning!