SCHOTTKY SPACES AND UNIVERSAL MUMFORD CURVES OVER $\mathbb{Z}$

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Abstract. For every integer $g \geq 1$ we describe a construction of a universal Mumford curve of genus g in the framework of Berkovich spaces over $\mathbb{Z}$. This is achieved in two steps: first, we build an analytic space $\mathcal{S}_g$ that parametrizes marked Schottky groups over all valued fields at once. We show that $\mathcal{S}_g$ is an open, connected analytic space over $\mathbb{Z}$. Then, we prove that the Schottky uniformization of a given curve behaves well with respect to the topology of $\mathcal{S}_g$, both locally and globally. As a result, we can construct a relative curve over $\mathcal{S}_g$ whose fibers are Schottky-uniformized curves, and such that every Schottky uniformized curve can be retrieved in this way. Finally, by studying the action of the group $\mathrm{Out}(F_g)$ of outer automorphisms of a free group with g generators on $\mathcal{S}_g$, we show that the universal Mumford curve is universal also in the spirit of the theory of moduli spaces.

Contents

Introduction

In his celebrated paper [Mum72], Mumford introduces a class of p-adic curves with a uniformization property analogous to Schottky uniformization for Riemann surfaces. In this paper, we define a universal analytic family of Mumford curves of a given genus, in the framework of the theory of Berkovich spaces over the ring of integers of a number field. We show that such a family is a relative curve over a suitably defined Schottky space, that admits uniformization by a Schottky group, and we describe locally the structure of fundamental domains for the action of the universal Schottky group.

1. Berkovich spaces over $\mathbb{Z}$

1.1. **Analytic spaces over Banach rings.** Let $(A, \|\cdot\|)$ be a Banach ring. In this section, we recall Berkovich's definition of analytic spaces over A (see [Ber90, Section 1.5]).

We start with the affine analytic space of dimension n over A, denoted by $\mathbb{A}^{n,\mathrm{an}}_A$. It is a locally ringed space and we define it in three steps: underlying set, topology and structure sheaf.

The set underlying $\mathbb{A}^{n,\mathrm{an}}_A$ is the set of bounded multiplicative seminorms on $A[T_1, \dots, T_n]$ that are bounded on A, *i.e.* the set of maps

$$|\cdot| \colon A[T_1, \dots, T_n] \to \mathbb{R}_{\geqslant 0}$$

that satisfy the following properties:

(i) $|0| = 0$ and $|1| = 1$;
(ii) $\forall P, Q \in A[T_1, \dots, T_n, |P + Q| \leqslant |P| + |Q|$;
(iii) $\forall P, Q \in A[T_1, \dots, T_n, |P + Q| = |P|\,|Q|$;
(iv) $\forall a \in A, |a| \leqslant \|a\|$.

We set $\mathcal{M}(A) := \mathbb{A}^{0,\mathrm{an}}_A$ and call it the *spectrum* of A. Note that we have a projection map $\mathrm{pr}_A \colon \mathbb{A}^{n,\mathrm{an}}_A \to \mathcal{M}(A)$ induced by the morphism $A \to A[T_1, \dots, T_n]$.

Let x be a point of $\mathbb{A}^{n,\mathrm{an}}_A$. Denote by $|\cdot|_x$ the multiplicative seminorm associated to it. The ring $A[T_1, \dots, T_n]/\ker(|\cdot|_x)$ is a domain and we can consider its field of fractions. The seminorm $|\cdot|_x$ induces an absolute value on the later it and we can consider its completion, which we denote by $\mathcal{H}(x)$. We simply denote by $|\cdot|$ the absolute value on $\mathcal{H}(x)$ induced by $|\cdot|_x$ since no confusion may result.

We have a natural morphism $\chi_x \colon A[T_1, \dots, T_n] \to \mathcal{H}(x)$. For each $P \in A[T_1, \dots, T_n]$, we set $P(x) := \chi_x(P)$. Note that, by definition, we have $|P(x)| = |P|_x$.

The set $\mathbb{A}^{n,\mathrm{an}}_A$ is endowed with the coarsest topology such that, for each $P \in A[T_1, \dots, T_n]$, the map

$$x \in \mathbb{A}^{n,\mathrm{an}}_A \mapsto |P(x)| \in \mathbb{R}_{\geqslant 0}$$

is continuous. The resulting topological space is Hausdorff and locally compact. The spectrum $\mathcal{M}(A)$ is even compact. The projection map pr_A is continuous.

For each open subset V of $\mathbb{A}^{n,\mathrm{an}}_A$, we denote by S_V the set of element of $A[T_1, \dots, T_n]$ that do not vanish on V and set $K(V) := S_V^{-1} A[T_1, \dots, T_n]$.

Let U be an open subset of $\mathbb{A}^{n,\mathrm{an}}_A$. We define $\mathcal{O}(U)$ to be the set of maps

$$f \colon U \to \bigsqcup_{x \in U} \mathcal{H}(x)$$

such that

(i) for each $x \in U$, $f(x) \in \mathcal{H}(x)$;
(ii) each $x \in U$ has an open neighbourhood V on which f is a uniform limit of elements of $K(V)$.

One may now define arbitrary analytic spaces over A as locally ringed spaces that are locally isomorphic to some $(V(\mathcal{I}), \mathcal{O}_U/\mathcal{I})$, where U is an open subset of $\mathbb{A}^{n,\mathrm{an}}_A$ and $\mathcal{I}$ is a sheaf of ideals of $\mathcal{O}_U$.

A point x of an analytic space X over A is said to be archimedean or non-archimedean if the associated absolute valued on $\mathcal{H}(x)$ is. We denote by X^{a} (resp. X^{na}) the set of archimedean (resp. non-archimedean) points of x and call it the archimedean (resp. non-archimedean) part of X. It is well-known that an absolute value on a field is archimedean if, and only if, its restriction to the prime field is. It follows that we have

$$X^{\mathrm{a}} = \{x \in X \mid |2(x)| > 1\} \text{ and } X^{\mathrm{na}} = \{x \in X \mid |2(x)| \geqslant 1\}$$

(and 2 could be replaced by any integer bigger than 1). In particular, the archimedean and non-archimedean parts of X are respectively open and closed subsets of X.

To go further, one should define the category of analytic spaces over A. When A is a complete non-archimedean valued field, this has been achieved by V. Berkovich in [Ber90, Ber93] (with a more general notion of analytic space). In [Lem15], T. Lemanissier gave a definition over an arbitrary Banach ring. However, the category is shown to enjoy nice properties only under additional assumptions, for instance when A is a discrete valuation ring (with some mild extra hypotheses) or the ring of integers of a number field (see Section 1.3 for some definitions related to this setting). For future use, we note that, in those cases, fiber products exist.

1.2. **Relative projective line.** In the rest of the text, we will not only need affine spaces, but also projective spaces and, more precisely, relative projective lines over affine spaces. We explain here how to construct them in a down-to-earth way. Let $(A, \|\cdot\|)$ be a Banach ring. Let $n \in \mathbb{N}$ and denote by S the analytic space $\mathbb{A}^{n,\mathrm{an}}_A$ with coordinates $T_1, \dots, T_n$.

Let U (resp. V) be the affine space $\mathbb{A}^{n+1,\mathrm{an}}_A$ with coordinates $T_1, \dots, T_n, Z$ (resp. $T_1, \dots, T_n, Z'$) and denote by U_0 (resp. V_0) the open subset defined by the inequality $Z \neq 0$ (resp. $Z' \neq 0$). The morphism

$$\begin{array}{ccc} A[T_1, \dots, T_n, Z, Z^{-1}] & \to & A[T_1, \dots, T_n, Z', Z'^{-1}] \\ T_i & \mapsto & T_i \\ Z & \mapsto & Z'^{-1} \end{array}$$

induces an isomorphism $U_0 \xrightarrow{\sim} V_0$.

We denote by $\mathbb{P}^1_S$ the analytic space obtained by glueing U and V along U_0 and V_0 *via* the previous isomorphism. It comes with a natural projection morphism $\pi \colon \mathbb{P}^1_S \to S$. For any open subset S' of S, we denote by $\mathbb{P}^1_{S'}$ the analytic space $\pi^{-1}(S')$.

When $n = 0$, we will denote $\mathbb{P}^1_{\mathcal{M}(A)}$ by $\mathbb{P}^{1,\mathrm{an}}_A$. Note that, for each $s \in S$, the fiber $\pi^{-1}(s)$ identifies to $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(s)}$.

Let $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}(S))$. We may associate to it an endomorphism of $\mathbb{P}^1_S$ by the usual expression in coordinates

$$Z \mapsto \frac{aZ + b}{cZ + d}.$$

This way we get an action of $\mathrm{GL}_2(\mathcal{O}(S))$ on $\mathbb{P}^1_S$. It factors through $\mathrm{PGL}_2(\mathcal{O}(S))$. The image of M in $\mathrm{PGL}_2(\mathcal{O}(S))$ will be denoted by $[M] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that the action restricts to an action on each fiber of π, hence also on $\mathbb{P}^1_{S'}$ for any open subset S' of S.

1.3. **Berkovich spaces over $\mathbb{Z}$.** In this section, we consider the special case where $(A, \|\cdot\|) = (\mathbb{Z}, |\cdot|_\infty)$, where $|\cdot|_\infty$ denotes the usual absolute value. We refer to [Poi10], and especially Section 3.1 there, for more details.

The spectrum $\mathcal{M}(\mathbb{Z})$ is easily described using Ostrowski's theorem. It contains the following points:

- a point a_0, associated to the trivial absolute value $|\cdot|_0$, with residue field $\mathbb{Q}$;
- for each $\varepsilon \in (0, 1]$, a point a_∞^ε associated to the absolute value $|\cdot|_\infty^\varepsilon$, with residue field $\mathbb{R}$;
- for each prime number p and each $\varepsilon \in (0, +\infty)$, a point a_p^ε associated to the absolute value $|\cdot|_p^\varepsilon$, with residue field $\mathbb{Q}_p$;
- for each prime number p, a point $a_{p,0}$ associated to the seminorm on $\mathbb{Z}$ induced by the trivial absolute value on $\mathbb{Z}/p\mathbb{Z}$, with residue field $\mathbb{Z}/p\mathbb{Z}$.

Its archimedean part is the open subset $\mathcal{M}(\mathbb{Z})^a = \{a_\infty^\varepsilon \mid \varepsilon \in (0, 1]\}$.

The topology of $\mathcal{M}(\mathbb{Z})$ is quite simple. First, the branches are all homeomorphic to segments: for each prime number p, the map

$$b_p\colon \eta\in[0,1]\mapsto \begin{cases} a_{p,0} & \text{if } \eta=0;\\ a_p^{-\log(\eta)} & \text{if } \eta\in(0,1);\\ a_0 & \text{if } \eta=1\end{cases}$$

is a homeomorphism and the map

$$\beta_\infty\colon \varepsilon\in[0,1]\mapsto \begin{cases} a_0 & \text{if } \varepsilon=0;\\ a_p^{\varepsilon} & \text{if } \varepsilon\in(0,1]\end{cases}$$

is a homeomorphism too. Moreover, a subset U of $\mathcal{M}(\mathbb{Z})$ containg a_0 is open if, and only if, the intersection of U with each $b_p([0,1])$ and $\beta_\infty([0,1])$ is open and only finitely many of those sets are not contained entirely in U. In other words, $\mathcal{M}(\mathbb{Z})$ is homeomorphic to the Alexandroff one-point compactification of the disjoint union of the $b_p([0,1))$'s and $\beta_\infty((0,1])$, the point at infinity being a_0.

We will often think about an analytic space over $\mathbb{Z}$ as a family of analytic spaces over the different valued fields associated to the points of $\mathcal{M}(\mathbb{Z})$. The spaces over $\mathbb{Q}_p$, $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$ (the last two being endowed with the trivial absolute value) are then usual Berkovich spaces. Recall that the analytic spaces over $\mathbb{R}$ in the sense of Berkovich are the quotients of the corresponding usual analytic spaces over $\mathbb{C}$ by the complex conjugation.

Let us be more precise in the case of an affine space $\mathbb{A}^{n,\mathrm{an}}_{\mathbb{R},|\cdot|_\infty^{\varepsilon}}$, for some $\varepsilon\in(0,1]$. The complex conjugation induces an automorphism of $\mathbb{C}^n$ given by

$$c\colon z=(z_1,\dots,z_n)\in\mathbb{C}^n\mapsto(\bar z_1,\dots,\bar z_n)\in\mathbb{C}^n$$

and we have a homeomorphism

$$\rho_\varepsilon\colon z\in\mathbb{C}^n/\langle c\rangle\mapsto v_{z,\varepsilon}\in\mathbb{A}^{n,\mathrm{an}}_{\mathbb{R},|\cdot|_\infty^{\varepsilon}},$$

where $v_{z,\varepsilon}\colon P(T)\in\mathbb{R}[T]\mapsto|P(z)|_\infty^\varepsilon$. It follows that all the archimedean fibers are the same. More precisely, the map

$$\Phi\colon (v,\varepsilon)\in\mathbb{A}^{n,\mathrm{an}}_{\mathbb{R},|\cdot|_\infty}\times(0,1]\mapsto\rho_\varepsilon\circ\rho_1^{-1}(v)\in\left(\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}\right)^{\mathrm{a}}$$

is a homeomorphism. Note that $\Phi(v,\varepsilon)$ may also be defined explicitly as the seminorm defined by

$$\Phi(v,\varepsilon)\colon P(T)\in\mathbb{R}[T]\mapsto|P(v)|^\varepsilon.$$

In particular, the seminorms v and $\Phi(v,\varepsilon)$ are equivalent.

As regards topology, analytic spaces over $\mathbb{Z}$ are known to be locally path-connected thanks to [Lem15]. As one can expect, surprising phenomena occur when passing from archimedean to the non-archimedean part. We illustrate this by giving two examples of continuous sections of the projection $\mathrm{pr}_\mathbb{Z}\colon\mathbb{A}^{1,\mathrm{an}}_\mathbb{Z}\to\mathcal{M}(\mathbb{Z})$.

Example 1.3.1. Let α be an element of $\mathbb{C}$ that is transcendental over $\mathbb{Q}$. For each $a\in\mathcal{M}(\mathbb{Z})^{\mathrm{na}}$, denote by $\eta_{a,1}$ the Shilov boundary of the disc of center 0 and radius 1, *i.e.* the Gauß point, in the fiber $\mathrm{pr}_\mathbb{Z}^{-1}(a)$. The map

$$\sigma\colon a\in\mathcal{M}(\mathbb{Z})\mapsto\begin{cases}\eta_{a,1} & \text{if } a \text{ is non-archimedean};\\ \rho_\varepsilon(\alpha) & \text{if } a=a_\infty^\varepsilon \text{ with } \varepsilon\in(0,1].\end{cases}$$

is a continuous section of $\mathrm{pr}_{\mathbb{Z}} \colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Z}} \to \mathcal{M}(\mathbb{Z})$. For this it is enough to show that $\rho_\varepsilon(\alpha)$ tends to $\eta_{a_0,1}$ when ε goes to 0. Remark that the point $\eta_{a_0,1}$ corresponds to the trivial absolute value on $\mathbb{Z}[T]$ and that, for each $P \in \mathbb{Z}[T] - \{0\}$, we have

$$|P(\rho_\varepsilon(\alpha))| = |P(\alpha)|_\infty^\varepsilon \xrightarrow[\varepsilon \to 0]{} 1$$

since α is transcendental over $\mathbb{Q}$. The result follows.

Example 1.3.2. Let $r \in (0,1)$. For each $a \in \mathcal{M}(\mathbb{Z})^{\mathrm{na}}$, denote by $\eta_{a,r}$ the Shilov boundary of the disc of center 0 and radius r in the fiber $\varpi^{-1}(a)$. The map

$$\tau_r \colon a \in \mathcal{M}(\mathbb{Z}) \mapsto \begin{cases} \eta_{a,r} & \text{if } a \text{ is non-archimedean;} \\ \rho_\varepsilon(r^{1/\varepsilon}) & \text{if } a = a_\infty^\varepsilon \text{ with } \varepsilon \in (0,1]. \end{cases}$$

is a continuous section of $\mathrm{pr}_{\mathbb{Z}} \colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Z}} \to \mathcal{M}(\mathbb{Z})$. It is enough to show that $\rho_\varepsilon(r^{1/\varepsilon})$ tends to $\eta_{a_0,r}$ when ε goes to 0. This is clear since, for each $\varepsilon \in (0,1]$, we have

$$|T(\rho_\varepsilon(r^{1/\varepsilon})| = |r^{1/\varepsilon}|_\infty^\varepsilon = r$$

and $\eta_{a_0,r}$ is the only point of the fiber $\mathrm{pr}_{\mathbb{Z}}^{-1}(a_0)$ where T has absolute value r.

One can build a similar theory replacing $\mathbb{Z}$ by the ring of integers $\mathcal{O}_K$ of a number field K. To be more precise, let us denote by Σ_K the set of complex embeddings of K up to complex conjugation and endow $\mathcal{O}_K$ with the norm

$$\|\cdot\|_K := \max(|\sigma(\cdot)|_\infty, \sigma \in \Sigma_K).$$

Then, the spectrum $\mathcal{M}(\mathcal{O}_K)$ looks very similar to $\mathcal{M}(\mathbb{Z})$: it is a tree with one point associated to the trivial absolute value and, for each place of K, one branch emanating from it.

Remark that the restriction of seminorms induces a map $\mathcal{M}(\mathcal{O}_K) \to \mathcal{M}(\mathbb{Z})$, and more generally a map $\mathbb{A}^{n,\mathrm{an}}_{\mathcal{O}_K} \to \mathbb{A}^{n,\mathrm{an}}_{\mathbb{Z}}$. Those maps are continuous and open.

Note also that $\mathcal{M}(\mathcal{O}_K)$ is an analytic space over $\mathbb{Z}$ in the sense of Section 1.1. In particular, by [Lem15], it makes sense to consider the fiber product of an analytic space over $\mathbb{Z}$ by $\mathcal{M}(\mathcal{O}_K)$ over $\mathcal{M}(\mathbb{Z})$. We obtain canonical identifications $\mathbb{A}^{n,\mathrm{an}}_{\mathcal{O}_K} = \mathbb{A}^{n,\mathrm{an}}_{\mathbb{Z}} \times_{\mathcal{M}(\mathbb{Z})} \mathcal{M}(\mathcal{O}_K)$, $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{O}_K} = \mathbb{P}^{1,\mathrm{an}}_{\mathbb{Z}} \times_{\mathcal{M}(\mathbb{Z})} \mathcal{M}(\mathcal{O}_K)$, etc.

1.4. **Some useful inequalities.** In this section, we fix a complete valued field $(k, |\cdot|)$, archimedean or not. We state a few results that will be useful later.

Lemma 1.4.1. *Let $a, b \in k$. We have*

$$|a+b| \leqslant \max(|2|,1)\, \max(|a|,|b|).$$

If $|a| > \max(|2|,1)\, |b|$, then we have

$$|a+b| \geqslant \frac{|a|}{\max(|2|,1)}.$$

Proof. If $(k, |\cdot|)$ is non-archimedean, then $\max(|2|,1) = 1$, and those inequalities are well-known.

Assume that $(k, |\cdot|)$ is archimedean. Then $(k, |\cdot|)$ embeds isometrically into $(\mathbb{C}, |\cdot|_\infty^\varepsilon)$ for some $\varepsilon \in (0,1]$ and it is enough to prove the result for the latter. In this case, we have $\max(|2|,1) = 2^\varepsilon$. For any $a, b \in \mathbb{C}$, we have $|a+b|_\infty \leqslant 2\, \max(|a|_\infty, |b|_\infty)$ and the first result follows by raising the inequality to the power ε.

The inequality applied to $a+b$ and $-b$ gives $|a| \leqslant |2| \max(|a+b|, |b|)$. As a consequence, if we have $|a| > |2|\, |b|$, we must have $|a| \leqslant |2|\, |a+b|$. ∎

It will be useful to introduce a notation for discs. We consider here the Berkovich affine line $\mathbb{A}^{1,\mathrm{an}}_k$ over k with coordinate T.

Notation 1.4.2. For $a \in k$ and $r \in \mathbb{R}_{>0}$, we set

$$D^+(a,r) := \{x \in \mathbb{A}_k^{1,\mathrm{an}} \mid |T(x) - a| \leqslant r\},$$
$$D^-(a,r) := \{x \in \mathbb{A}_k^{1,\mathrm{an}} \mid |T(x) - a| < r\}.$$

Lemma 1.4.3. *Let $a, b \in k$ and $\rho_a, \rho_b \in \mathbb{R}_{>0}$. If $|a-b| > \max(|2|, 1)\max(\rho_a, \rho_b)$, then the closed discs $D^+(a,\rho_a)$ and $D^+(b,\rho_b)$ are disjoint.*

If $|\cdot|$ is non-archimedean, then the closed discs $D^+(a,\rho_a)$ and $D^+(b,\rho_b)$ are disjoint if, and only if, $|a-b| > \max(\rho_a,\rho_b)$.

Proof. If there exists a point x in $D^+(a,\rho_a) \cap D^+(b,\rho_b)$, then we have $|T(x)-a| \leqslant \rho_a$ and $|T(x)-b| \leqslant \rho_b$ in $\mathcal{H}(x)$, hence

$$|a-b| = |(a - T(x)) + (T(x) - b)| \leqslant \max(|2|, 1)\max(\rho_a, \rho_b)$$

by Lemma 1.4.1. The first part of the result follows.

The converse implication in the non-archimedean setting is well-known. ☐

Lemma 1.4.4. *Let $a, b \in k$. If $|a+b|^2 > \max(|4|, 1)\,|ab|$, then $|a| \neq |b|$.*

If $|\cdot|$ is non-archimedean, then we have $|a| \neq |b|$ if, and only if, $|ab| < |a+b|^2$.

Proof. If $|a| = |b|$, then, by Lemma 1.4.1, we have $|a+b| \leqslant \max(|2|,1)\,|a|$, hence

$$|a+b|^2 \leqslant \max(|2|^2, 1)\,|a|^2 = \max(|4|,1)\,|ab|.$$

The first part of the result follows.

Let us now assume that $|\cdot|$ is non-archimedean. Assume that $|a| \neq |b|$. Then, we have $|a+b| = \max(|a|,|b|) > \min(|a|,|b|)$, hence

$$|a+b|^2 = \max(|a|,|b|)^2 > \max(|a|,|b|)\min(|a|,|b|) = |a|\,|b|.$$

The converse implication follows directly from the first part of the statement. ☐

1.5. **Metric structure.** In this section, we fix a complete non-archimedean valued field $(k, |\cdot|)$. In the following, we will often encounter the projective line $\mathbb{P}_k^{1,\mathrm{an}}$ and we gather here a few metric properties.

First recall that $\mathbb{P}_k^{1,\mathrm{an}}$ has the structure of a real tree (see [Duc, (3.4.20)]). In particular, for any two distinct points $x, y \in \mathbb{P}_k^{1,\mathrm{an}}$, there exists a unique segment $[xy]$ joining x to y. Recall also that each segment consisting of points of type 2 or 3 carries a multiplicative length (or modulus) that is invariant under isomorphisms of $\mathbb{P}_k^{1,\mathrm{an}}$, *i.e.* under Möbius transformations (see [Duc, (3.6.23)]). To define the length of such a segment I, one may proceed as follows.

Notation 1.5.1. For $a \in k$ and $r \in \mathbb{R}_{>0}$, we denote by $\eta_{a,r}$ the unique point is the Shilov boundary of the closed disc $D^+(a,r)$.

There exist a finite extension k' of k, a coordinate T on $\mathbb{P}_k^{1,\mathrm{an}}$, $a \in k'$ and $r \leqslant s \in \mathbb{R}_{>0}$ such that I is the image of the segment $[\eta_{a,r}, \eta_{a,s}]$ by the projection map $\mathbb{P}_{k'}^{1,\mathrm{an}} \to \mathbb{P}_k^{1,\mathrm{an}}$. We then set

$$\ell(I) := \frac{s}{r} \in [1, +\infty).$$

It is independent of the choices made. It will convenient to set $\ell(\emptyset) := 1$.

Lemma 1.5.2. *Let a, b, c, d be distinct points of $\mathbb{P}^1(k)$ and denote their cross-ratio by $[a,b;c,d]$.*

Set $I := [ab] \cap [cd]$. It is either a segment consisting of points of type 2 or 3 or the empty set. If I is a non-trivial segment and if going from a to b and from c to d induces the same orientation on I, then we set $\varepsilon := -1$. In all other cases, we set $\varepsilon := 1$.

Then, we have

$$|[a,b;c,d]| = \ell(I)^{\varepsilon}.$$

Proof. Since the cross-ratio is invariant under Möbius transformations, we may assume that $b = 1$, $c = 0$ and $d = \infty$. Assume that $|a| < 1$. Then $[ab] \cap [cd] = [\eta_{0,|a|}, \eta_{0,1}]$ and going from a to b and from c to d induces the same orientation on it, hence $\varepsilon = -1$. We have

$$\ell([ab] \cap [cd])^{-1} = |a| = |[a, b; c, d]|$$

as desired. The other cases are dealt with similarly. □

2. Schottky groups

The notion of Schottky group is classical over $\mathbb{C}$ (see [MSW15]) and even over a complete valued non-archimedean field (see [GvdP80]). The definitions, results and proofs that appear in this section are adaptations of the standard ones to a relative setting.

2.1. **Geometric situation.** Let S be an analytic space over a Banach ring (archimedean or not). As in Section 1.2, consider the analytic space $\mathbb{P}^1_S$ and the projection morphism $\pi\colon \mathbb{P}^1_S \to S$. In this section, we describe geometric properties of the action of some groups of automorphisms of $\mathbb{P}^1_S$. It follows the strategy of [GvdP80, I, 4.1].

Definition 2.1.1. Let $(\gamma_1, \dots, \gamma_g) \in \mathrm{PGL}_2(\mathcal{O}(S))^g$. Let $\mathcal{B} = \big(B^+(\gamma_i^\varepsilon), 1 \leqslant i \leqslant g, \varepsilon = \pm 1\big)$ be a family of closed subsets of $\mathbb{P}^1_S$ that are disjoint. For each $i \in \{1, \dots, g\}$ and $\varepsilon \in \{-1, 1\}$, set

$$B^-(\gamma_i^\varepsilon) := \gamma_i^\varepsilon(\mathbb{P}^1_S - B^+(\gamma_i^{-\varepsilon})).$$

For each $s \in S$, $i \in \{1, \dots, g\}$, $\varepsilon \in \{-1, 1\}$ and $\sigma \in \{-, +\}$, set $B^\sigma_s(\gamma_i^\varepsilon) := B^\sigma(\gamma_i^\varepsilon) \cap \pi^{-1}(s)$.

We say that $\mathcal{B}$ is a *Schottky figure* adapted to $(\gamma_1, \dots, \gamma_g)$ if, for each $s \in S$, $i \in \{1, \dots, g\}$ and $\varepsilon \in \{-1, 1\}$, $B^+(\gamma_i^\varepsilon)$ is a closed disc in $\pi^{-1}(s) \simeq \mathbb{P}^1_{\mathcal{H}(s)}$ and $B^-(\gamma_i^\varepsilon)$ is a maximal open disc inside it.

In this section, we assume that we are in the situation of Definition 2.1.1. For $\sigma \in \{-, +\}$, we set

$$F^\sigma := \mathbb{P}^1_S - \bigcup_{\substack{1 \leqslant i \leqslant g \\ \varepsilon = \pm 1}} B^{-\sigma}(\gamma_i^\varepsilon).$$

Note that, for $\gamma_0 \in \{\gamma_1^{\pm 1}, \dots, \gamma_g^{\pm 1}\}$ and $\sigma \in \{-, +\}$, $B^\sigma(\gamma_0)$ is the unique disc among the $W^\sigma(\gamma)$'s containing $\gamma_0 F^\sigma$.

Set $\Delta := \{\gamma_1, \dots, \gamma_g\}$. Denote by F_g the free group over the alphabet Δ and by Γ the subgroup of $\mathrm{PGL}_2(\mathcal{O}(B))$ generated by Δ. We have a natural morphism $\varphi\colon F_g \to \Gamma$ sending each γ in Δ to γ. It induces an action of F_g on $\mathbb{P}^1_S$.

We now define subsets of $\mathbb{P}^1_S$ associated to the elements of F_g. As usual, we will identify those elements with the words over the alphabet $\Delta^\pm := \{\gamma_1^{\pm 1}, \dots, \gamma_g^{\pm 1}\}$.

Notation 2.1.2. For a non-empty reduced word $w = w'\gamma$ over Δ and $\sigma \in \{-, +\}$, we set

$$B^\sigma(w) := w'\, B^\sigma(\gamma).$$

Lemma 2.1.3. *Let u be a non-empty reduced word over $\Delta^\pm$. Then we have $uF^+ \subseteq B^+(u)$.*

Let v be a non-empty reduced word over $\Delta^\pm$. If there exists a word w over $\Delta^\pm$ such that $u = vw$, then we have $uF^+ \subseteq B^+(u) \subseteq B^+(v)$. If, moreover, $u \neq v$, then we have $B^+(u) \subseteq B^-(v)$.

Conversely, if we have $B^+(u) \subseteq B^+(v)$, then there exists a word w over $\Delta^\pm$ such that $u = vw$.

Proof. Write in a reduced form $u = u'\gamma$ with $\gamma \in \Delta^\pm$. We have $\gamma F^+ \subseteq B^+(\gamma)$, by definition. Applying u', it follows that $uF^+ \subseteq B^+(u)$.

Assume that there exists a word w such that $u = vw$ and let us prove that $B^+(u) \subseteq B^+(v)$. Be first assume that v is a single letter. We will argue by induction on the length $|u|$ of u. If $|u| = 1$, then $u = v$ and the result is trivial. If $|u| \geqslant 2$, denote by δ the first letter of w. By induction, we have $B^+(w) \subseteq B^+(\delta)$. Since $\delta \neq v^{-1}$, we also have $B^+(\delta) \subseteq \mathbb{P}^1_B - B^+(v^{-1})$. The result follows by applying v.

Let us now handle the general case. Write in a reduced form $v = v'\gamma$ with $\gamma \in \Delta^{\pm}$. By the former case, we have $B^+(\gamma w) \subseteq B^+(\gamma)$ and $B^+(\gamma w) \subseteq B^-(\gamma)$ if w is non-empty. The result follows by applying v'.

Assume that we have $B^+(u) \subseteq B^+(v)$. We will prove that there exists a word w such that $u = vw$ by induction on $|v|$. Write in reduced forms $u = \gamma u'$ and $v = \delta v'$. By the previous result, we have $B^+(u) \subseteq B^+(\gamma)$ and $B^+(v) \subseteq B^+(\delta)$, hence $\gamma = \delta$. If $|v| = 1$, this proves the result. If $|v| \geqslant 2$, then we deduce that we have $B^+(u') \subseteq B^+(v')$, hence, by induction, there exists a word w such that $u' = v'w$. It follows that $u = vw$. □

Corollary 2.1.4. *The morphism φ is an isomorphism and the group Γ is free on the generators $\gamma_1, \dots, \gamma_g$.*

Proof. If w is a non-empty word, then the previous lemma ensures that $wF^+ \neq F^+$. The result follows. □

As a consequence, we will now identify Γ with F_g and express the elements of Γ as words over the alphabet $\Delta^{\pm}$. In particular, we will allow us to speak of the length of an element γ of Γ. We will denote it by $|\gamma|$.

Set

$$O_n := \bigcup_{|\gamma| \leqslant n} \gamma F^+.$$

Since the complement of F^+ is the disjoint union of the open disks $B^-(\gamma)$ with $\gamma \in \Delta^{\pm}$, it follows from the description of the action that, for each $n \geqslant 0$, we have

$$\mathbb{P}^1_S - O_n = \bigsqcup_{|\gamma| = n+1} B^-(w).$$

It follows from Lemma 2.1.3 that, for each $n \geqslant 0$, O_n is contained in the interior of O_{n+1}. We set

$$O := \bigcup_{n \geqslant 0} O_n = \bigcup_{\gamma \in \Gamma} \gamma F^+.$$

2.2. **Over a valued field.**

Let $(k, |\cdot|)$ be a complete valued field. In this section, we will focus on the particular case $S = \mathcal{M}(k)$. In this setting, the material of this section is classical: see [MSW15, Project 4.5] and [GvdP80, I, 4.1.3] in the archimedean and non-archimedean case respectively.

We still assume that we are in the situation of Definition 2.1.1. Set $\iota := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{PGL}_2(k)$. It corresponds to the map $z \mapsto 1/z$ on $\mathbb{P}^{1,\mathrm{an}}_k$. The first result follows from an explicit computation.

Lemma 2.2.1. *Let $\alpha \in k^*$ and $\rho \in [0, |\alpha|)$. Then, we have*

$$\iota D^+(\alpha, \rho) = \begin{cases} D^+\left(\dfrac{\bar{\alpha}}{|\alpha|^2 - \rho^2}, \dfrac{\rho}{|\alpha|^2 - \rho^2}\right) & \text{if } k \text{ is archimedean;} \\ D^+\left(\dfrac{1}{\alpha}, \dfrac{\rho}{|\alpha|^2}\right) & \text{otherwise.} \end{cases}$$

□

Lemma 2.2.2. *Let $r > 0$ and let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{PGL}_2(k)$ such that $\gamma D^+(0,r) \subseteq \mathbb{A}_k^{1,\mathrm{an}}$. Then, we have $|d| > r|c|$ and*

$$\gamma D^+(0,r) = \begin{cases} D^+\left(\dfrac{b\bar{d} - a\bar{c}r^2}{|d|^2 - |c|^2\, r^2}, \dfrac{|ad-bc|\, r}{|d|^2 - |c|^2\, r^2}\right) & \text{if } k \text{ is archimedean;} \\ D^+\left(\dfrac{b}{d}, \dfrac{|ad-bc|\, r}{|d|^2}\right) & \text{otherwise.} \end{cases}$$

Proof. Let us first assume that $c = 0$. Then, we have $d \neq 0$, so the inequality $|d| > r|c|$ holds, and γ is affine with ratio a/d. The result follows.

Let us now assume that $c \neq 0$. In this case, we have $\gamma^{-1}(\infty) = -\frac{d}{c}$, which does not belong to $D(0,r)$ if, and only if, $|d| > r|c|$. Note that we have the following equality in $k(T)$:

$$\frac{aT+b}{cT+d} = \frac{a}{c} - \frac{ad-bc}{c^2}\,\frac{1}{T+\frac{d}{c}}.$$

By Lemma 2.2.1, there exist $\beta \in k$ and $\sigma > 0$ such that $\iota D^+(\frac{d}{c}, r) = D^+(\beta, \sigma)$. Then, we have $\gamma D^+(0,r) = D^+(\frac{a}{c} - \frac{ad-bc}{c^2}\,\beta, \left|\frac{ad-bc}{c^2}\right|\sigma)$ and the result follows from an explicit computation. □

Lemma 2.2.3. *Let $D' \subseteq D$ be closed concentric discs in $\mathbb{A}_k^{1,\mathrm{an}}$. Let $\gamma \in \mathrm{PGL}_2(k)$ such that $\gamma D' \subseteq \gamma D \subseteq \mathbb{A}_k^{1,\mathrm{an}}$. Then, we have*

$$\frac{\text{radius of } \gamma(D')}{\text{radius of } \gamma(D)} \leqslant \frac{\text{radius of } D'}{\text{radius of } D},$$

with an equality if k is non-archimedean.

Proof. Let p be the center of D and D' and let τ be the translation sending p to 0. Up to changing D into τD, D' into $\tau D'$, γ into $\gamma\tau^{-1}$ and γ' into $\gamma'\tau^{-1}$, we may assume that D and D' are centered at 0. The result then follows from Lemma 2.2.2. □

Proposition 2.2.4. *Assume that $\infty \in F^-$. Then, there exist $R > 0$ and $c \in (0,1)$ such that, for each $\gamma \in \Gamma - \{\mathrm{id}\}$, $B^+(\gamma)$ is a closed disc of radius at most $R\, c^{|\gamma|}$.*

Proof. Let $\delta, \delta' \in \Delta^{\pm}$. By Lemma 2.1.3, we have an inclusion of discs $B^+(\delta'\delta) \subseteq B^+(\delta')$. There exists $f_{\delta,\delta'} \in \mathrm{PGL}_2(k)$ that sends those discs to concentric disks inside $\mathbb{A}_k^{1,\mathrm{an}}$. In the non-archimedean case, the discs are already concentric, so one may take $f_{\delta,\delta'} = \mathrm{id}$ while, in the archimedean case, there is some work to be done, for which we refer to [MSW15, Project 3.4]. Set

$$c_{\delta,\delta'} := \frac{\text{radius of } f_{\delta,\delta'}(B^+(\delta'\delta))}{\text{radius of } f_{\delta,\delta'}(B^+(\delta'))} \in (0,1).$$

For each $\gamma \in \Gamma$ such that $\gamma\delta'$ is a reduced word, by Lemma 2.2.3, we have

$$\frac{\text{radius of } B^+(\gamma\delta'\delta)}{\text{radius of } B^+(\gamma\delta')} = \frac{\text{radius of } \gamma f_{\delta,\delta'}^{-1} f_{\delta,\delta'}(B^+(\delta'\delta))}{\text{radius of } \gamma f_{\delta,\delta'}^{-1} f_{\delta,\delta'}(B^+(\delta'))} \leqslant c_{\delta,\delta'}.$$

Set

$$R := \max(\{\text{radius of } B^+(\gamma) \mid \gamma \in \Delta^{\pm}\})$$

and

$$c := \max(\{c_{\gamma,\gamma'} \mid \gamma, \gamma' \in \Delta^{\pm}, \gamma' \neq \gamma^{-1}\}).$$

By induction, for each $\gamma \in \Gamma - \{\mathrm{id}\}$, we have

$$\text{radius of } B^+(\gamma) \leqslant R\, c^{|\gamma|}.$$

□

Corollary 2.2.5. *Let $w = (w_n)_{\neq 0}$ be a sequence of reduced words over $\Delta^{\pm}$ such that the associated sequence of discs $(B^+(w_n))_{n\geqslant 0}$ is strictly decreasing. Then, the intersection $\bigcap_{n\geqslant 0} B^+(w_n)$ is a single k-rational point p_w. Moreover, the discs $B^+(w_n)$ form a basis of neighbourhoods of p_w in $\mathbb{P}^{1,\mathrm{an}}_k$.*

Proof. Let k_0 be a finite extension of k such that $F^- \cap \mathbb{P}^{1,\mathrm{an}}_k(k_0) \neq \emptyset$. Consider the projection morphism $\pi_0 \colon \mathbb{P}^{1,\mathrm{an}}_{k_0} \to \mathbb{P}^{1,\mathrm{an}}_k$. For each $i \in \{1,\dots,g\}$, γ_i may be identified with an element $\gamma_{i,0}$ in $\mathrm{PGL}_2(k_0)$. The family $(\pi_0^{-1}(B^-(\gamma_i^{\pm 1}), 1 \leqslant i \leqslant g, \varepsilon = \pm 1)$ is a Schottky figure adapted to $(\gamma_{1,0},\dots,\gamma_{g,0})$. We will denote with a subscript 0 the associated sets: F_0^-, $B_0^+(w)$, etc. Note that these sets are all equal to the preimages of the corresponding sets by π_0.

Up to changing coordinates on $\mathbb{P}^{1,\mathrm{an}}_{k_0}$, we may assume that $\infty \in F_0^-$. The sequence of discs $(B_0^+(w_n))_{n\geqslant 0}$ is strictly decreasing, so by Lemma 2.1.3, the length of w_n tends to ∞ when n goes to ∞ and, by Proposition 2.2.4, the radius of $B_0^+(w_n)$ tends to 0 when n goes to ∞. It follows that $\bigcap_{n\geqslant 0} B_0^+(w_n)$ is a single point $p_{w,0}$ and that the discs $B_0^+(w_n)$ form a basis of neighbourhood of $p_{w,0}$ in $\mathbb{P}^{1,\mathrm{an}}_{k_0}$.

If k_0 is non-archimedean, then $p_{w,0}$ is a point of type 1, *i.e.* a point of $\mathbb{P}^{1,\mathrm{an}}_{k_0}(K)$, where K is the completion of an algebraic closure of k_0 (hence of k). If k_0 is archimedean, then, trivially, $p_{w,0}$ is a point of $\mathbb{P}^{1,\mathrm{an}}_{k_0}(K)$ with $K := \mathbb{C}$.

Set $p_w := \pi_0(p_{w,0}) \in \mathbb{P}^{1,\mathrm{an}}_k(K)$. It follows from the results over k_0 that $\bigcap_{n\geqslant 0} B^+(w_n) = \{p_w\}$ and that the discs $B^+(w_n)$ form a basis of neighbourhoods of p_w in $\mathbb{P}^{1,\mathrm{an}}_k$.

It remains to show that p_w is k-rational. Note that p_w belongs to the closure of $\mathbb{P}^{1,\mathrm{an}}_k(k)$, since it is the limit of the centers of the $B^+(w_n)$'s. Since k is complete, it is closed in K and the result follows. ☐

Corollary 2.2.6. *The set O is dense in $\mathbb{P}^{1,\mathrm{an}}_k$ and its complement is contained in $\mathbb{P}^{1,\mathrm{an}}_k(k)$.* ☐

2.3. **Limit sets.** We return to the general case of Definition 2.1.1 with an arbitrary analytic space S.

Definition 2.3.1. We say that a point $x \in \mathbb{P}^1_S$ is a *limit point* if there exist $x_0 \in \mathbb{P}^1_S$ and a sequence $(\gamma_n)_{n\geqslant 0}$ of distinct elements of Γ such that $\lim_{n\to\infty} \gamma_n(x_0) = x$.

The *limit set* L of Γ is the set of limit points of Γ.

Following [Bou71, III, §4, Définition 1], we say that the action of Γ on a subset E of $\mathbb{P}^1_S$ is *proper* if the map

$$\begin{array}{ccc} \Gamma \times E & \to & E \times E \\ (\gamma, x) & \mapsto & (x, \gamma\cdot x) \end{array}$$

is proper, where Γ is endowed with the discrete topology. By [Bou71, III, §4, Proposition 7], it is equivalent to requiring that, for every $x, y \in E$, there exist neighbourhoods U_x and U_y of x and y respectively such that the set $\{\gamma \in \Gamma \mid \gamma U_x \cap U_y \neq \emptyset\}$ is finite. By [Bou71, III, §4, Proposition 3], in this case, the quotient $\Gamma\backslash E$ is Hausdorff.

We denote by C the set of points $x \in \mathbb{P}^1_S$ that admit a neighbourhood U_x satisfying $\{\gamma \in \Gamma \mid \gamma U_x \cap U_x \neq \emptyset\} = \{\mathrm{id}\}$. Then C is an open subset of $\mathbb{P}^1_S$ and the quotient map $(\mathbb{P}^1_S - C) \to \Gamma\backslash(\mathbb{P}^1_S - C)$ is a local homeomorphism. In particular, the topological space $\Gamma\backslash(\mathbb{P}^1_S - C)$ is naturally endowed with a structure of analytic space *via* this map.

Proposition 2.3.2. *We have $O = C = \mathbb{P}^1_B - L$. Moreover, the action of Γ on O is free and proper and the quotient map $\Gamma\backslash O \to S$ is proper.*

Proof. Let $x \in L$. By definition, there exists $x_0 \in \mathbb{P}^1_B$ and a sequence $(\gamma_n)_{n\geqslant 0}$ of distinct elements of Γ such that $\lim_{n\to\infty} \gamma_n(x_0) = x$. Assume that $x \in F^+$. Since F^+ is contained in the interior of O_1, there exists $N \geqslant 0$ such that $\gamma_N(x_0) \in O_1$, hence we may assume that $x_0 \in O_1$. Lemma 2.1.3

then leads to a contradiction. It follows that L does not meet F^+, hence, by Γ-invariance, L is contained in $\mathbb{P}^1_S - O$.

Let $y \in \mathbb{P}^1_S - O$. By definition, there exists a sequence $(w_n)_{n\geqslant 0}$ of reduced words over $\Delta^\pm$ such that, for each $n \geqslant 0$, $|w_n| \geqslant n$ and $y \in B^-(w_n)$. Let $y_0 \in F^-$. By Lemma 2.1.3, for each $n \geqslant 0$, we have $w_n(y_0) \in B^-(w_n)$ and the sequence of discs $(B^+(w_n))_{n\geqslant 0}$ is strictly decreasing. By Corollary 2.2.5, $(w_n(y_0))_{n\geqslant 0}$ tends to y, hence $y \in L$. It follows that $\mathbb{P}^1_S - O = L$.

Set

$$U := F^+ \cup \bigcup_{\gamma\in\Delta^\pm} \gamma F^- = \mathbb{P}^1_S - \bigsqcup_{|\gamma|=2} B^+(\gamma).$$

It is an open subset of $\mathbb{P}^1_S$ and it follows from the properties of the action (see Lemma 2.1.3) that we have $\{\gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset\} = \{\mathrm{id}\} \cup \Delta^\pm$. Using the fact that the stabilizers of the points of U are trivial, we deduce that $U \subseteq C$. Letting Γ act, it follows that $O \subseteq C$. Since no limit point may belong to C, we deduce that this is actually an equality.

We have already seen that the action is free on O. Let us prove that it is proper. Let $x, y \in O$. There exists $n \geqslant 0$ such that x and y belong to the interior of O_n. By Lemma 2.1.3, the set $\{\gamma \in \Gamma \mid \gamma O_n \cap O_n \neq \emptyset\}$ is made of element of length at most $2n+1$. In particular, it is finite. We deduce that the action of Γ on O is proper.

Let K be a compact subset of S. Since $\pi\colon \mathbb{P}^1_S \to S$ is proper, $\pi^{-1}(K)$ is compact, hence its closed subset $F^+ \cap \pi^{-1}(K)$ is compact too. Since $F^+ \cap \pi^{-1}(K)$ contains a point of every orbit of every element of $\pi^{-1}(K)$, we deduce that $\Gamma\backslash(O \cap \pi^{-1}(K))$ is compact. □

2.4. **Koebe coordinates.** Let $(k, |\cdot|)$ be a complete valued field and and let γ be a loxodromic element of $\mathrm{PGL}_2(k)$. The eigenvalues of γ belong to a quadratic extension of k and have distinct absolute values. If k is archimedean, it follows immediately that they both belong to k, hence γ admits exactly two fixed points $\alpha, \alpha' \in \mathbb{P}^{1,\mathrm{an}}(k)$. If k is non-archimedean, then the result still holds by the same argument in characteristic different from 2 and, in general, as a consequence of Hensel's lemma (see [GvdP80, I, 1.4]).

We can choose α so that the associated eigenvalue has minimal absolute value. In this case, α and α' will be respectively the attracting and repelling fixed points of the Möbius transformation associated to γ. Denote by β the multiplier of γ, *i.e.* the ratio of the eigenvalues such that $|\beta| < 1$. For $\varepsilon \in \mathrm{PGL}_2(k)$ such that $\varepsilon(0) = \alpha$ and $\varepsilon(\infty) = \alpha'$, we have $\varepsilon^{-1}\gamma\varepsilon(z) = \beta z$. It follows that the parameters α, α' and β determine uniquely the transformation γ. They are called the *Koebe coordinates* of γ.

Conversely, given $(\alpha = [u\colon v], \alpha' = [u'\colon v'], \beta) \in (\mathbb{P}^{1,\mathrm{an}}_k)^3$ with $|\alpha| \neq |\alpha'|$ (*i.e.* $|uv'| \neq |u'v|$) and $0 < |\beta| < 1$, it is not difficult to determine explicitly the element of $\mathrm{PGL}_2(k)$ that has those Koebe coordinates. It is given by

$$M(\alpha, \alpha', \beta) = \begin{bmatrix} uv' - \beta u'v & (\beta - 1)uu' \\ (1-\beta)vv' & \beta uv' - u'v \end{bmatrix} \in \mathrm{PGL}_2(k).$$

In the rest of the paper, we will sometimes abuse notation and allow ourselves to use $M(\alpha, \alpha', \beta)$ in different contexts, for example when α, α', β belong to a ring (provided the conditions on the absolute values are satisified at each point of its spectrum). This should not cause any trouble.

The following interpretation of $|\beta|$ will be useful later.

Lemma 2.4.1. *Assume that k is non-archimedean. Let D be an open disc of $\mathbb{P}^{1,\mathrm{an}}_k$ containing α' and not α. Then $\gamma(D)$ is an open disc containing α' and the segment joining the boundary point of D to that of $\gamma(D)$ consists of points of type 2 or 3 and has length equal to $|\beta|^{-1}$.*

Proof. Möbius transformations preserve open discs, their boundary points, and the lenght of segments. Since $\varepsilon^{-1}\gamma\varepsilon(\varepsilon^{-1}(D)) = \varepsilon^{-1}(\gamma(D))$, it is enough to prove the result for $\varepsilon^{-1}\gamma\varepsilon$ and $\varepsilon^{-1}(D)$. In this case, it is clear. $\square$

We now check that the Koebe coordinates depend analytically on the entries of the corresponding matrix. In fact, this is true not only over a valued field, but even over $\mathbb{Z}$. To prove this, let us introduce some notation. Set

$$K_{\mathbb{Z}} := \{(\alpha, \alpha', \beta) \in (\mathbb{P}^{1,\mathrm{an}}_{\mathbb{Z}})^3 \mid |\alpha| \neq |\alpha'|, 0 < |\beta| < 1\}.$$

It is an open subset of $(\mathbb{P}^{1,\mathrm{an}}_{\mathbb{Z}})^3$. We also consider $\mathbb{P}^{4,\mathrm{an}}_{\mathbb{Z}}$ and write its elements in coordinates in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ instead of the usual $[a \colon b \colon c \colon d]$. Denote by $L_{\mathbb{Z}}$ the set of elements $x \in \mathbb{P}^{4,\mathrm{an}}_{\mathbb{Z}}$ such that the matrix

$$\begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} \in \mathrm{PGL}_2(\mathcal{H}(x))$$

is loxodromic.

Lemma 2.4.2. *The subset $L_{\mathbb{Z}}$ is open in $\mathbb{P}^{4,\mathrm{an}}_{\mathbb{Z}}$.*

Proof. Let us first consider the archimedean part $L^{\mathrm{a}}_{\mathbb{Z}}$ of $L_{\mathbb{Z}}$. By Section 1.3, it is enough to prove that its intersection with the fiber over the point a^1_∞, corresponding to the usual absolute value, is open. This allows to translate the statement into a statement about $\mathbb{P}^4(\mathbb{C})$ (since the set is clearly stable by complex conjugation), where it is a consequence of the continuity of the roots of a (degree 2) polynomial.

Let us now handle the non-archimedean part $L^{\mathrm{na}}_{\mathbb{Z}}$. By Lemma 1.4.4, we have

$$L^{\mathrm{na}}_{Z} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \left(\mathbb{P}^{4,\mathrm{an}}_{\mathbb{Z}}\right)^{\mathrm{na}} \;\middle|\; |ad - bc| < |a + d|^2 \right\}.$$

Let $x \in L^{\mathrm{na}}_{Z}$. There exists $r > 1$ such that $r\,|ad - bc| < |a + d|^2$. The open subset of $\mathbb{P}^{4,\mathrm{an}}_{\mathbb{Z}}$ defined by the inequality

$$r \, \max(|4|, 1)\, |ad - bc| < |a + d|^2$$

contains x and sits inside $L_{\mathbb{Z}}$, by Lemma 1.4.4 again. The result follows. $\square$

Proposition 2.4.3. *The morphism*

$$M \colon (\alpha, \alpha', \beta) \in K_{\mathbb{Z}} \mapsto M(\alpha, \alpha', \beta) \in L_{\mathbb{Z}}$$

is an isomorphism of analytic spaces over $\mathbb{Z}$. Its inverse is the map that associates to a loxodromic matrix its Koebe coordinates.

Proof. The map M is clearly analytic and it follows from the discussion above that it is a bijection. Let us prove that its inverse is also analytic.

Let $m \in L_{\mathbb{Z}}$. We may work in an affine chart of $\mathbb{P}^{4,\mathrm{an}}_{\mathbb{Z}}$ containing m and identify it to $\mathbb{A}^{4,\mathrm{an}}_{\mathbb{Z}}$. As a result, we may assume that the coefficients $a(m), b(m), c(m), d(m)$ of m are well-defined. Denote by $\lambda(m)$ and $\lambda'(m)$ the eigenvalues of the matrix associated to m, chosen so that $|\lambda(m)| < |\lambda'(m)|$. Remark that the inequality on the absolute values implies that $(a + d)(m) \neq 0$. The elements $\lambda(m)$ and $\lambda'(m)$ are then the two roots of the characteristic polynomial of the matrix:

$$X^2 - (a + d)(m)X + (ad - bc)(m) = (a + d)^2(m) \left(Y^2 - Y + \frac{ad - bc}{(a + d)^2}(m)\right),$$

where $Y = \frac{1}{(a+d)(m)}\, X$.

Note that the polynomial $P(Y) := Y^2 - Y + (ad-bc)/(a+d)^2$ (which is actually well-defined on the whole $L_{\mathbb{Z}}$) has analytic coefficients. We claim that λ and λ' are analytic functions of m. If m is archimedean and the discriminant $\Delta(m)$ of $P(m)(Y)$ is not real, this follows from the fact that there exists a determination of the square-root that is analytic in the neighbourhood of $\Delta(m)$. If m is archimedean and $\Delta(m)$ is real, then $\Delta(m) > 0$, since otherwise $\lambda(m)$ and $\lambda'(m)$ would be complex conjugates hence would have the same absolute value. The result then follows from the fact that there exists a determination of the square-root that is analytic in the neighborhood of $\Delta(m)$ and commutes with the complex conjugation.

Assume that m is non-archimedean. The stalk $\mathcal{O}_m$ of the structure sheaf is a local ring (whose maximal ideal is the set of elements that vanish at m). Denote its residue field by $\kappa(m)$ and set

$$\kappa(m)^\circ := \{f \in \kappa(m) \mid |f(m)| \leqslant 1\}$$

and

$$\kappa(m)^{\circ\circ} := \{f \in \kappa(m) \mid |f(m)| < 1\}.$$

The set $\kappa(m)^\circ$ is a local ring with maximal ideal $\kappa(m)^{\circ\circ}$. We denote its residue field by $\tilde{\kappa}(m)$. By Lemma 1.4.4, the image of $P(Y)$ in $\kappa(m)[Y]$ has coefficients in $\kappa(m)^\circ$ and its reduction is $Y^2 - Y$. The roots $\lambda(m)$ and $\lambda'(m)$ of $P(m)(Y)$ reduce respectively to the roots 0 and 1 of $Y^2 - Y$. By [Poi13, Corollaire 5.3] and [Poi10, Corollaire 2.5.2], $\kappa(m)^\circ$ and $\mathcal{O}_m$ are henselian, and it follows that λ and λ' are analytic in the neighbourhood of m.

It is now clear that $\beta = \lambda/\lambda'$ is analytic in the neighbourhood of m. Note that we can also recover α and α' from λ and λ' since they correspond to the associated eigenline. More precisely, we have $\alpha(m) = [b(m) \colon (\lambda - a)(m)]$ if $\lambda(m) \neq a(m)$ and $\alpha(m) = [(\lambda - d)(m) \colon c(m)]$ otherwise, and similarly for α'. It follows that α and α' are analytic in the neighbourhood of m. □

2.5. **Group theory.** Let $(k, |\cdot|)$ be a complete valued field. In this section, we give the general definition of Schottky group over k and explain how it relates to the geometric situation considered in Section 2.1.

Definition 2.5.1. A Schottky group over k is a subgroup of $\mathrm{PGL}_2(k)$ that is free, finitely generated, and discontinuous. As a consequence, every element of a Schottky group is loxodromic, *i.e.* is represented by a matrix whose eigenvalues have different absolute value.

Remark 2.5.2. The notion of Schottky group over k is left unchanged if one replaces the absolute value on k by an equivalent one.

Remark 2.5.3. Let $(k', |\cdot|')$ be an extension of $(k, |\cdot|)$. A subgroup Γ of $\mathrm{PGL}_2(k)$ is a Schottky group over k if, and only if, it is a Schottky group over k'.

Lemma 2.5.4. *Let Γ be a subgroup of $\mathrm{PGL}_2(k)$ generated by finitely many elements $\gamma_1, \dots, \gamma_g$. If there exists a Schottky figure adapted to $(\gamma_1, \dots, \gamma_g)$, then Γ is a Schottky group.*

Proof. □

Definition 2.5.5. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(k)$, with $c \neq 0$, be a loxodromic matrix and let $\lambda \in \mathbb{R}_{>0}$ be a positive real number. We call open and closed *twisted Ford discs* associated to (γ, λ) the sets

$$D^-_{(\gamma,\lambda)} := \left\{ z \in k \;\middle|\; \lambda|\gamma'(z)| = \lambda \frac{|ad-bc|}{|cz+d|^2} > 1 \right\}$$

and

$$D^+_{(\gamma,\lambda)} := \left\{ z \in k \;\middle|\; \lambda|\gamma'(z)| = \lambda \frac{|ad-bc|}{|cz+d|^2} \geqslant 1 \right\}.$$

Lemma 2.5.6. *Let $\alpha, \alpha', \beta \in k$ with $|\beta| < 1$ and let $\lambda \in \mathbb{R}_{>0}$. Set $\gamma = M(\alpha, \alpha', \beta) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The twisted Ford discs $D^-_{(\gamma,\lambda)}$ and $D^+_{(\gamma,\lambda)}$ have center*

$$\frac{\alpha' - \beta\alpha}{1-\beta} = -\frac{d}{c}$$

and radius

$$\rho = \frac{(\lambda|\beta|)^{1/2}|\alpha - \alpha'|}{|1-\beta|} = \frac{(\lambda\,|ad-bc|)^{1/2}}{|c|}.$$

The twisted Ford discs $D^-_{(\gamma^{-1},\lambda^{-1})}$ and $D^+_{(\gamma^{-1},\lambda^{-1})}$ have center

$$\frac{\alpha - \beta\alpha'}{1-\beta} = \frac{a}{c}$$

and radius $\rho' = \rho/\lambda$. □

Lemma 2.5.7. *For every loxodromic $\gamma \in \mathrm{PGL}_2(k)$ that does not fix ∞ and every $\lambda \in \mathbb{R}_{>0}$, we have $\gamma(D^+_{(\gamma,\lambda)}) = \mathbb{P}^{1,\mathrm{an}}_k - D^-_{(\gamma^{-1},\lambda^{-1})}$.*

Proof. Let us write $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since γ does not fix ∞, we have $c \neq 0$. Let K be a complete valued extension of k and let $z \in K$. We have $|-c\gamma(z)+a|\,|cz+d| = |ad-bc|$, hence

$$z \in D_{(\gamma,\lambda)} \iff \lambda\frac{|ad-bc|}{|cz+d|^2} \geq 1 \iff \lambda^{-1}\frac{|ad-bc|}{|-c\gamma(z)+a|^2} \leq 1.$$

Since we have $\gamma^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, and the latter condition describes precisely the complement of $D^-_{(\gamma^{-1},\lambda^{-1})}$.

□

Definition 2.5.8. Let Γ be a Schottky group of rank g. We say that a basis $B = (\gamma_1, \dots, \gamma_g)$ of Γ is a *Schottky basis* if there exists a Schottky figure that is adapted to it. The datum (Γ, B) of a Schottky group and a Schottky basis B of Γ is called a *marked Schottky group.*

If k is archimedean, it is a classical result of Marden [Mar74] that there exist Schottky groups with no Schottky bases. On the contrary, in the non-archimedean case, a theorem of Gerritzen ensures that Schottky bases always exist (see [Ger74, §2, Satz 1]).

Theorem 2.5.9. *Assume that k is non-archimedean. Let Γ be a Schottky group over k whose limit set does not contain ∞. Then, there exist a basis $(\delta_1, \dots, \delta_g)$ of Γ and positive real numbers $\lambda_1, \dots, \lambda_g \in \mathbb{R}_{>0}$ such that the family of twisted Ford discs $\big(D^-_{\delta_1,\lambda_1}, \dots, D^-_{\delta_g,\lambda_g}, D^-_{\delta_1^{-1},\lambda_1^{-1}}, \dots, D^-_{\delta_g^{-1},\lambda_g^{-1}}\big)$ is a Schottky figure adapted to $(\delta_1, \dots, \delta_g)$.* □

3. The Schottky space over $\mathbb{Z}$

In this section, we define a parameter space for marked Schottky groups of a given rank, where the marking is given by the choice of a basis. Already for Schottky groups of rank one, one gets an interesting construction, but most uniformization phenomena that are at the center of our interest become apparent only when the rank is at least two.

3.1. The space $\mathcal{S}_1$.

Let $\Gamma = \langle\gamma\rangle$ be a Schottky group of rank one over a valued field $(k, |\cdot|)$. Then, γ is conjugated in $PGL_2(k)$ to a unique matrix of the form $M(0, \infty, \beta)$ with $0 < |\beta| < 1$ (which corresponds to the multiplication by β as an endomorphism of $\mathbb{P}^{1,\mathrm{an}}_k$; see Section 2.4 for the notation).

Consider the affine line $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{Z}}$ with coordinate Y and set

$$\mathcal{S}_1 := \{x \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Z}} \mid 0 < |Y(x)| < 1\}.$$

With each point $x \in \mathcal{S}_1$, one can canonically associate a Schottky group of rank one

$$\Gamma_x := \langle M(0, \infty, Y(x))\rangle \subset \mathrm{PGL}_2(\mathcal{H}(x)).$$

The condition imposed on $\mathcal{S}_1$ ensures that $M(0, \infty, Y(x))$ is a loxodromic transformation of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ having 0 as attracting point and ∞ as repelling point. Given a Schottky group of rank one over $\mathbb{Q}_p$, we can retrieve it as a $M(0, \infty, Y(x))$ for a unique $x \in \mathcal{S}_1$ with $\mathcal{H}(x) = \mathbb{Q}_p$. For a general valued field $(k, |\cdot|)$ and an element $\beta \in k$ such that $0 < |\beta| < 1$, the group generated by $M(0, \infty, \beta)$ can be retrieved as above from a point of $\mathcal{S}_1 \times_{\mathbb{Z}} k$. This can be seen as a consequence of Lemma 3.2.5 in the special case where $g = 1$.

3.2. Construction of $\mathcal{S}_g$ and equivalent definitions.

In this section, we fix $g \geqslant 2$ and we consider the space $\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$ and denote its coordinates by $X_3, \dots, X_g, X'_2, \dots, X'_g, Y_1, \dots, Y_g$. For notational convenience, we set $X_1 := 0$, $X_2 := 1$ and $X'_1 := \infty$ (seen as morphisms from $\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$ to $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{Z}}$). We denote by $\mathrm{pr}_{\mathbb{Z}} \colon \mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}} \to \mathcal{M}(\mathbb{Z})$ the projection morphism. Let U_g be the open subset of $\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$ defined by the inequalities

$$\begin{cases} 0 < |Y_i| < 1 \text{ for } 1 \leqslant i \leqslant g; \\ X_i^{\sigma_i} \neq X_j^{\sigma_j} \text{ for } i, j \in \{1, \dots, g\} \text{ and } \sigma_i, \sigma_j \in \{\emptyset, '\}. \end{cases}$$

For $\in \{1, \dots, g\}$, consider the transformations

$$M_i := M(X_i, X'_i, Y_i) \in \mathrm{PGL}_2(\mathcal{O}(U_g)).$$

Definition 3.2.1. The *Schottky space of rank g* over $\mathbb{Z}$, denoted by $\mathcal{S}_g$, is the set of points $x \in U_g$ such that the subgroup Γ_x of $\mathrm{PGL}_2(\mathcal{H}(x))$ defined by

$$\langle M_1(x), M_2(x), \dots, M_g(x)\rangle$$

is a Schottky group of rank g.

Notation 3.2.2. Recall that a Schottky group over a valued field $(k, |\cdot|)$ gives rise to a k-analytic curve by means of uniformization (see in the archimedean case, and [Mum72] in the non-archimedean case). Given $x \in \mathcal{S}_g$, we denote by Γ_x the marked Schottky group of ordered basis $(M_1(x), M_2(x), \dots, M_g(x))$, and by $\mathcal{C}_x$ the $\mathcal{H}(x)$-analytic curve obtained via Schottky uniformization by Γ_x. The curve $\mathcal{C}_x$ has semi-stable reduction, and a theorem of Berkovich ([Ber90, 4.3.2]) asserts then that the dual graph of the stable model of $\mathcal{C}_x$ can be canonically realized as a subset of $\mathcal{C}_x$. Such a subset, denoted by Σ_x, is a graph of Betti number g, called the *skeleton* of $\mathcal{C}_x$. If $\mathcal{B} = \left(B^+_{i,\varepsilon} \middle| 1 \leq i \leq g, \varepsilon = \pm 1\right)$ is a Schottky figure adapted to a Schottky basis of Γ_x and F^+ is the associated "closure of a fundamental domain" (see Definition 2.1.1 and following paragraph), then there is an isomorphism $\mathcal{C}_x \cong F^+/\Gamma_x$ and the skeleton Σ_x of the Mumford curve $\mathcal{C}_x$ is obtained through pairwise identification, for every $i = 1, \dots, g$, of the points of the Shilov boundaries of $B^+_{i,1}, B^+_{i,-1}$ in the tree corresponding to the skeleton of F^+.

Note that the identification of $\mathcal{H}(x)$ with a valued extension of $\mathcal{H}(\mathrm{pr}_{\mathbb{Z}}(x))$ is not canonical, and to different immersions $\mathcal{H}(\mathrm{pr}_{\mathbb{Z}}(x)) \hookrightarrow \mathcal{H}(x)$ one associates different Schottky groups, yielding curves $\mathcal{C}_x$

that might not be isomorphic. To lift the ambiguity from this situation one has then to consider the points of the base-change $\mathcal{S}_g \times_{\mathbb{Z}} \mathcal{H}(x)$, as made more precise in the following remark.

Remark 3.2.3. Let $(A, \|\cdot\|)$ be a Banach ring. Starting with $\mathcal{M}(A)$ instead of $\mathcal{M}(\mathbb{Z})$, one can define a Schottky space $\mathcal{S}_{g,A}$ over A, that can be related to the Schottky space over $\mathbb{Z}$ in the following way. If we denote by $\pi_A \colon \mathbb{A}^{3g-3,\mathrm{an}}_A \to \mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$ the projection map, it follows from Remark 2.5.3 that we have $\mathcal{S}_{g,A} = \pi_A^{-1}(\mathcal{S}_g)$. In other words, when the suitable categories and fiber products are defined, we have $\mathcal{S}_{g,A} = \mathcal{S}_g \times_{\mathcal{M}(\mathbb{Z})} \mathcal{M}(A)$. Moreover, the projection map π_A respects all the data: for each $x \in \mathcal{S}_{g,A}$, the group Γ_x is the image of $\Gamma_{\pi(x)}$ by the inclusion $\mathrm{PGL}_2(\mathcal{H}(\pi(x)) \subseteq \mathrm{PGL}_2(\mathcal{H}(x))$, the limit set of $\Gamma_{\pi(x)}$ in $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(\pi(x))}$ is the preimage of the limit set of Γ_x in $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, and so on.

In the special case where $A = \mathbb{C}$, we obtain a subset $\mathcal{S}_{g,\mathbb{C}}$ of $\mathbb{C}^{3g-3}$. As one may expect, this is a classical object that has already been closely investigated. By [Hej75, Lemma 5.11], one has a covering map $\mathcal{T}_{g,\mathbb{C}} \longrightarrow \mathcal{S}_{g,\mathbb{C}}$ from the Teichmüller space to the complex Schottky space. One deduces that $\mathcal{S}_{g,\mathbb{C}}$ is a connected subset of $\mathbb{C}^{3g-3}$.

Recall that, for $\varepsilon \in (0,1]$, we denote by a_∞^ε the point of $\mathcal{M}(\mathbb{Z})$ associated with $|\cdot|_\infty^\varepsilon$ and that we have an isomorphism $\mathcal{H}(a_\infty^\varepsilon) \simeq \mathbb{R}$. We will use the canonical map $\rho_\varepsilon \colon \mathbb{C}^{3g-3} = \mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{C}} \to \mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{R}}$, where $\mathbb{R}$ and $\mathbb{C}$ are endowed with $|\cdot|_\infty^\varepsilon$.

Lemma 3.2.4. *For $\varepsilon \in (0,1]$, we have $\mathcal{S}_g \cap \mathrm{pr}_{\mathbb{Z}}^{-1}(a_\infty^\varepsilon) = \rho_\varepsilon(\mathcal{S}_{g,\mathbb{C}})$. The set $\mathcal{S}_g \cap \big(\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}\big)^{\mathrm{a}}$ is a connected open subset of $\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$.*

Proof. By Remarks 2.5.3 and 2.5.2, $\rho_\varepsilon^{-1}(\mathcal{S}_g \cap \mathrm{pr}_{\mathbb{Z}}^{-1}(a_\infty^\varepsilon))$ coincides with the usual complex Schottky space $\mathcal{S}_{g,\mathbb{C}}$. In other words, $\mathcal{S}_g \cap \mathrm{pr}_{\mathbb{Z}}^{-1}(a_\infty^\varepsilon)$ is the quotient of $\mathcal{S}_{g,\mathbb{C}}$ by the complex conjugation. In particular, it is a connected open subset of $\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{R}}$.

Recall the homeomorphism

$$\Phi \colon \mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{R}} \times (0,1] \to \big(\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}\big)^{\mathrm{a}}$$

from Section 1.3. It follows from Remark 2.5.2 that it induces a bijection between $\big(\mathcal{S}_g \cap \mathrm{pr}_{\mathbb{Z}}^{-1}(a_\infty^\varepsilon)\big) \times (0,1]$ and $\mathcal{S}_g^{\mathrm{a}}$. As a consequence, $\mathcal{S}_g^{\mathrm{a}}$ is connected and open, since $\big(\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}\big)^{\mathrm{a}}$ is open in $\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$. □

Let F_g be the free group of rank g with basis $e_1, \dots, e_g$. For each complete valued field k, we denote by $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2(k))$ the set of group morphisms $\varphi \colon F_g \to \mathrm{PGL}_2(k)$ that satisfy the following conditions:

(i) $\varphi(e_1)$ is loxodromic with attracting fixed point 0 and repelling fixed point ∞;
(ii) $\varphi(e_2)$ is loxodromic with attracting fixed point 1;
(iii) the image of φ is a Schottky group of rank g.

Each point x of $\mathcal{S}_g$ gives rise to an element φ_x of $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2(\mathcal{H}(x)))$ that sends e_i to $M_i(x)$.

To state a converse result, we need to introduce an equivalence relation similar to that of [Ber90, Remark 1.2.2 (ii)]. We say that two elements $\varphi_1 \in \mathrm{Hom}_S(F_g, \mathrm{PGL}_2(k_1))$ and $\varphi_2 \in \mathrm{Hom}_S(F_g, \mathrm{PGL}_2(k_2))$ are equivalent if there exists an element $\varphi \in \mathrm{Hom}_S(F_g, \mathrm{PGL}_2(k))$ and isometric embeddings $k \hookrightarrow k_1$ and $k \hookrightarrow k_2$ that make the following diagram commute:

$$
\begin{tikzcd}
& & \mathrm{PGL}_2(k_1) \\
F_g \arrow[urr, "\varphi_1"] \arrow[r, "\varphi"] \arrow[drr, "\varphi_2"] & \mathrm{PGL}_2(k) \arrow[ur, hook] \arrow[dr, hook] & \\
& & \mathrm{PGL}_2(k_2)
\end{tikzcd}
$$

We denote by $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2)$ the set of classes of this equivalence relation.

Lemma 3.2.5. *The map $x \mapsto \varphi_x$ is a bijection between the underlying set of $\mathcal{S}_g$ and $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2)$.*

Proof. If $\varphi_x = \varphi_y$ then $\mathcal{H}(x) = \mathcal{H}(y)$ and $\varphi_x(F_g) = \varphi_y(F_g)$ coincide as marked Schottky groups in $\mathrm{PGL}_2(\mathcal{H}(x))$. Hence $x = y$ in $\mathcal{S}_g$ and $x \mapsto \varphi_x$ is injective. Conversely, for a valued field k and a map $\varphi \in \mathrm{Hom}_S(F_g, \mathrm{PGL}_2(k))$, we can consider the point $y \in \mathbb{A}_k^{3g-3,\mathrm{an}}$ given by the Koebe coordinates of the marked Schottky group $\varphi(F_g)$. The image of y under the canonical projection $\mathbb{A}_k^{3g-3,\mathrm{an}} \to \mathbb{A}_{\mathbb{Z}}^{3g-3,\mathrm{an}}$ is a point $x \in \mathcal{S}_g$ and there is an isometric embedding $\mathcal{H}(x) \hookrightarrow k$ realizing φ_x as canonical representative of the class of φ in $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2)$. Hence $x \mapsto \varphi_x$ is surjective. $\square$

Remark 3.2.6. Every marked Schottky group of rank g over a complete valued field k is conjugated in $\mathrm{PGL}_2(k)$ to a unique marked Schottky group with the property that its Koebe coordinates are of the form $\{(0, \infty, \beta_1), (1, \alpha'_2, \beta_2), \dots, (\alpha_g, \alpha'_g, \beta_g)\}$. The combination of this observation with Lemma 3.2.5 implies that every Schottky group over k can be retrieved, up to conjugation, as a Γ_x for a suitable point $x \in \mathcal{S}_{g,k}$.

3.3. **Openness of $\mathcal{S}_g$.**

Definition 3.3.1. Let S be an analytic space. Consider the relative affine line $\mathbb{A}_S^1$ with coordinate Z. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{PGL}_2(\mathcal{O}(S))$ and $\lambda \in \mathbb{R}_{>0}$, we set

$$D^+_{(\gamma,\lambda)} := \{x \in \mathbb{A}_S^1 \mid |(cZ + d)(x)|^2 \leqslant \lambda |(ad - bc)(x)|\}$$

and

$$D^-_{(\gamma,\lambda)} := \{x \in \mathbb{A}_S^1 \mid |(cZ + d)(x)|^2 < \lambda |(ad - bc)(x)|\}.$$

We call such sets closed and open *relative twisted Ford discs* respectively.

We now generalize Gerritzen's theorem 2.5.9 to the relative setting.

Proposition 3.3.2. *Let x be a non-archimedean point of $\mathcal{S}_g$ such that ∞ is not a limit point of Γ_x. There exist an open neighbourhood W of x in U_g, an automorphism τ of F_g and $\lambda_1, \dots, \lambda_g \in \mathbb{R}_{>0}$ such that, denoting*

$$(N_1, \dots, N_g) := \tau \cdot (M_1, \dots, M_g) \in \mathrm{GL}_2(\mathcal{O}(U_g))^g,$$

the family of relative twisted Ford discs over W

$$(D_{N_1,\lambda_1}, D_{N_1^{-1},\lambda_1^{-1}}, \dots, D_{N_g,\lambda_g}, D_{N_g^{-1},\lambda_g^{-1}})$$

is a Schottky figure adapted to the family $(N_1, \dots, N_g)$ of $\mathrm{PGL}_2(\mathcal{O}(W))$.

Proof. By Theorem 2.5.9, we can find a basis $(\delta_1, \dots, \delta_g)$ of Γ_x and positive real numbers $\lambda_1, \dots, \lambda_g$ such that the family of twisted Ford discs $(D_{\delta_1,\lambda_1}, D_{\delta_1^{-1},\lambda_1^{-1}}, \dots, D_{\delta_g,\lambda_g}, D_{\delta_g^{-1},\lambda_g^{-1}})$ is a Schottky figure adapted to $(\delta_1, \dots, \delta_g)$.

Denote by τ the automorphism of Γ_x sending $M_i(x)$ to δ_i. Identify F_g with Γ_x by sending e_i to $M_i(x)$, we get an automorphism of F_g that we still denote by τ. Set

$$(N_1, \dots, N_g) := \tau \cdot (M_1, \dots, M_g) \in \mathrm{GL}_2(\mathcal{O}(U_g))^g$$

and write

$$N_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

for $i \in \{1, \dots, g\}$. Note that the coefficients of the N_i's are rational functions in the X_j's, X'_j's and Y_j's. Denote by U' the open subset of U_g where they are all defined.

Let V be the open subset of U' defined by

$$|\operatorname{tr}(N_i)|^2 > \max(|4|, 1)\,|\det(N_i)| \text{ for } 1 \leqslant i \leqslant g.$$

By Lemma 1.4.4, V contains x and, for each $y \in V$ and each $i \in \{1, \dots, g\}$, the matrix N_i is loxodromic.

Let $i \neq j \in \{1, \dots, g\}$. Since ∞ is not a limit point of Γ_x, it cannot be a fixed point of $N_i(x)$ or $N_j(x)$, hence $c_i(x)c_j(x) \neq 0$. There exists a neighbourhood $W_{i,j}$ of x in V such that $c_i c_j$ does not vanish on $W_{i,j}$. In this case, for each $y \in W_{i,j}$, ∞ is not a fixed point of $N_i(y)$ or $N_j(y)$ and, by Lemma 2.5.6, we have

$$D^+_{N_i(y),\lambda_i} = D^+\left(-\frac{d_i}{c_i}(y), \left|\frac{a_i d_i - b_i c_i}{c_i^2}(y)\right|^{1/2} \lambda_i^{1/2}\right)$$

and

$$D^+_{N_j(y),\lambda_j} = D^+\left(-\frac{d_j}{c_j}(y), \left|\frac{a_j d_j - b_j c_j}{c_j^2}(y)\right|^{1/2} \lambda_j^{1/2}\right).$$

By assumption, the discs at x are disjoint and Lemma 1.4.3 ensures that we have

$$\left|\frac{d_i}{c_i}(x) - \frac{d_j}{c_j}(x)\right| > \max\left(\left|\frac{a_i d_i - b_i c_i}{c_i^2}(x)\right|^{1/2} \lambda_i^{1/2}, \left|\frac{a_j d_j - b_j c_j}{c_j^2}(x)\right|^{1/2} \lambda_j^{1/2}\right).$$

Since x is non-archimedean, we have $\max(|2(x)|, 1) = 1$, hence, up to shrinking $W_{i,j}$, we may assume that, for each $y \in W$, we have

$$\left|\frac{d_i}{c_i}(y) - \frac{d_j}{c_j}(y)\right| > \max(|2(y)|, 1) \max\left(\left|\frac{a_i d_i - b_i c_i}{c_i^2}(y)\right|^{1/2} \lambda_i^{1/2}, \left|\frac{a_j d_j - b_j c_j}{c_j^2}(y)\right|^{1/2} \lambda_j^{1/2}\right),$$

which implies that $D^+_{N_i(y),\lambda_i}$ and $D^+_{N_j(y),\lambda_j}$ are disjoint, by Lemma 1.4.3. Similar arguments show that, up to shrinking $W_{i,j}$, we may ensure that the discs $D^+_{N_i(y),\lambda_i}$, $D^+_{N_i^{-1}(y),\lambda_i^{-1}}$, $D^+_{N_j(y),\lambda_j}$ and $D^+_{N_j^{-1}(y),\lambda_j^{-1}}$ are all disjoint.

The result now holds with $W := \bigcap_{i \neq j} W_{i,j}$. □

Lemma 3.3.3. *Let x be a non-archimedean point of $\mathcal{S}_g$. The separable closure of $\mathcal{H}(\mathrm{pr}_{\mathbb{Z}}(x))$ in $\mathcal{H}(x)$ is a finite extension of $\mathcal{H}(\mathrm{pr}_{\mathbb{Z}}(x))$. In particular, there exists an algebraic integer ω that does not belong to $\mathcal{H}(x)$. If $\mathrm{pr}_{\mathbb{Z}}(x) = a_0$, we may moreover assume that ω is totally real.*

Proof. It is enough to prove that, for each non-archimedean complete valued field k, each integer n and each $x \in \mathbb{A}_k^{n,\mathrm{an}}$, the separable closure of k in $\mathcal{H}(x)$ is a finite extension of k. By induction, we may assume that $n = 1$. If k is trivially valued, then we can conclude by the explicit description of the field $\mathcal{H}(x)$, so we may assume that this is not the case.

If x is a rigid point, then the result is clear. Otherwise, the point x corresponds to an absolute value $|\cdot|_x$ on the function field F of $\mathbb{A}^1_k$ and $\mathcal{H}(x)$ is the completion of F with respect to it. The local ring $\mathcal{O}_x$ being henselian, it contains the henselization of F with respect to $|\cdot|_x$, that is to say the separable closure of F in $\mathcal{H}(x)$. In particular, this separable closure is contained in the separable closure of F in the ring of functions of some strict affinoid neighbourhood of x, and it follows from Noether normalization lemma that the latter is a finite extension. $\square$

Corollary 3.3.4. *Let x be a non-archimedean point of $\mathcal{S}_g$. There exist an open neighbourhood W of x in U_g, an automorphism τ of F_g and a family of closed subsets of $\mathbb{P}^1_W$ that is a Schottky figure adapted to $\tau \cdot (M_1, \dots, M_g)$.*

Proof. Let ω be an algebraic integer as in Lemma 3.3.3. Let K be a number field containing ω.

We will work over the Schottky space $\mathcal{S}_{g,\mathcal{O}_K}$ over $\mathcal{M}(\mathcal{O}_K)$ defined in Remark 3.2.3. Denote by $\pi_K \colon \mathcal{S}_{g,\mathcal{O}_K} \to \mathcal{S}_g$ the projection morphism. Let $x_K \in \pi_K^{-1}(x)$ and set $U'_g := \pi_K^{-1}(U_g)$.

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & -\omega \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}(U'_g))$. Note that we have $A(\omega) = \infty$ and that, for each $z \in (U'_g)^{\mathrm{a}}$, $A(z) \in GL_2(\mathbb{R})$. For $i \in \{1, \dots, g\}$, set $M_{\infty,i} := A^{-1} M_i A$ in $\mathrm{PGL}_2(\mathcal{O}(U'_g))$. Denote by Γ_{∞,x_K} the subgroup of $\mathrm{PGL}_2(\mathcal{H}(x_K))$ generated by $M_{\infty,1}(x_K), \dots, M_{\infty,g}(x_K)$. By Corollary 2.2.6 and Remark 3.2.3, ω is not a limit point of Γ_{x_K}, hence ∞ is not a limit point of Γ_{∞,x_K}. By Proposition 3.3.2, there exists an open neighbourhood W' of x_K in U'_g, an automorphism τ of F_g and $\lambda_1, \dots, \lambda_g \in \mathbb{R}_{>0}$ such that, denoting

$$(N_{\infty,1}, \dots, N_{\infty,g}) := \tau \cdot (M_{\infty,1}, \dots, M_{\infty,g}) \in \mathrm{GL}_2(\mathcal{O}(U'_g))^g,$$

the family of twisted isometric discs

$$(D_{N_{\infty,1},\lambda_1}, D_{N_{\infty,1}^{-1},\lambda_1^{-1}}, \dots, D_{N_{\infty,g},\lambda_g}, D_{N_{\infty,g}^{-1},\lambda_g^{-1}})$$

is a Schottky figure adapted to the family $(N_{\infty,1}, \dots, N_{\infty,g})$ of $\mathrm{PGL}_2(\mathcal{O}(W'))$. If $p(x) \neq a_0$, then x belongs to the interior of the non-archimedean part of $\mathcal{S}_g$ and we may assume that $W' \subseteq \mathcal{S}_g^{\mathrm{na}}$.

For each $i \in \{1, \dots, g\}$, set $N_i := A\, N_{\infty,i}\, A^{-1}$. Note that we have

$$(N_1, \dots, N_g) := \tau \cdot (M_1, \dots, M_g) \in \mathrm{GL}_2(\mathcal{O}(U'_g))^g$$

and that the family $(A^{-1}(D_{N_{\infty,1},\lambda_1}), \dots, A^{-1}(D_{N_{\infty,g},\lambda_g}), A^{-1}(D_{N_{\infty,1}^{-1},\lambda_1^{-1}}), \dots, A^{-1}(D_{N_{\infty,g}^{-1},\lambda_g^{-1}}))$ is a Schottky figure adapted to the family $(N_1, \dots, N_g)$ of $\mathrm{PGL}_2(\mathcal{O}(W'))$. Set $W := \pi_K(W')$. It is an open subset of U. For $i \in \{1, \dots, g\}$ and $\varepsilon \in \{-1, 1\}$, set $B^\varepsilon(N_i^\varepsilon) := \pi_K(A^{-1}(D_{N_{\infty,i},\lambda_i^\varepsilon}))$. To prove that the family $(B^\varepsilon(N_i^\varepsilon), 1 \leqslant i \leqslant g, \varepsilon = \pm 1)$ is a Schottky figure adapted to the family $(N_1, \dots, N_g)$ of $\mathrm{PGL}_2(\mathcal{O}(W))$, it remains to prove that, for each $y \in W$, $i \in \{1, \dots, g\}$ and $\varepsilon \in \{-1, 1\}$, $B^-(N_i^\varepsilon) \cap \pi^{-1}(y)$ is an open disc.

Let $y \in W$, $i \in \{1, \dots, g\}$ and $\varepsilon \in \{-1, 1\}$. Let $y' \in \pi_K^{-1}(y) \cap W'$. Assume that y is non-archimedean. The set $A^{-1}(D_{N_{\infty,i},\lambda_i^\varepsilon}) \cap \pi^{-1}(y')$ is an open disc over $\mathcal{H}(y')$ that contains a fixed point of $N_i(y')$. Since $N_i(y')$ is defined over $\mathcal{H}(y)$, its fixed points come from $\mathcal{H}(y)$-rational points by base change to $\mathcal{H}(y')$ and we deduce that $B^-(N_i^\varepsilon) \cap \pi^{-1}(y)$ is an open disc. Assume that y is archimedean. Note that, in this case, we have $p(x) = a_0$, hence ω is totally real, by assumption. If $\mathcal{H}(y') = \mathcal{H}(y)$, then the result holds. Otherwise, we have $\mathcal{H}(y') = \mathbb{C}$ and $\mathcal{H}(y) = \mathbb{R}$ and the result follows from the fact that $A(y') \in \mathrm{GL}_2(\mathbb{R})$.

$\square$

We have now collected all the results necessary to the proof of the main theorem of this section.

Theorem 3.3.5. *The Schottky space $\mathcal{S}_g$ is an open subset of $\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$.*

Proof. Let $x \in \mathcal{S}_g$. We want to prove that there exists an open subset of $\mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$ containing x that is contained in $\mathcal{S}_g$. This follows from Lemma 3.2.4 when x is archimedean and from Corollary 3.3.4 and Lemma 2.5.4 when x is non-archimedean. $\square$

3.4. **The space of Schottky bases.** Let us assume $g \geq 2$ and fix a complete non-archimedean valued field $(k, |\cdot|)$. We denote by $\mathcal{SB}_{g,k}$ the subspace of $\mathcal{S}_{g,k}$ consisting of Schottky bases (see Remark 3.2.3 and Definition 2.5.8 respectively for these notions).

Notation 3.4.1. Let A be a finite subset of $\mathbb{P}^{1,\mathrm{an}}_k(k)$ with at least 2 elements. For each $\alpha \in A$, we denote by $D^-(\alpha, A)$ the biggest open disc with center α containing no other element of A and we denote by $p_{\alpha,A}$ its boundary point in $\mathbb{P}^{1,\mathrm{an}}_k$.

Note that, if A contains at least 3 elements, then all the discs $D^-(\alpha, A)$ are disjoint.

Proposition 3.4.2. *Let $\alpha_1, \alpha_1', \dots, \alpha_g, \alpha_g'$ be distinct elements of $\mathbb{P}^{1,\mathrm{an}}_k(k)$. Set $A := \{\alpha_1, \alpha_1', \dots, \alpha_g, \alpha_g'\}$. Let $\beta_1, \dots, \beta_g$ be elements of k with absolute values in $(0,1)$. For each $i \in \{1, \dots, g\}$, set $\gamma_i := M(\alpha_i, \alpha_i', \beta_i) \in \mathrm{PGL}_2(k)$. The following conditions are equivalent:*

(i) there exists a Schottky figure adapted to $(\gamma_1, \dots, \gamma_g)$;

(ii) for each $i \in \{1, \dots, g\}$, we have $\ell([p_{\alpha_i,A}\, p_{\alpha_i',A}]) < |\beta_i|^{-1}$;

(iii) for each $i, j, k \in \{1, \dots, g\}$ with $j \neq i$, $k \neq i$ and $\sigma_j, \sigma_k \in \{\emptyset, '\}$, we have

$$|\beta_i| \cdot |[\alpha_j^{\sigma_j}, \alpha_k^{\sigma_k}; \alpha_i, \alpha_i']| < 1.$$

Proof. $(i) \implies (ii)$ Let $\mathcal{B} = (B^+(\gamma_i^\varepsilon), 1 \leqslant i \leqslant g, \varepsilon = \pm 1)$ be a Schottky figure adapted to $(\gamma_1, \dots, \gamma_g)$. Let $i \in \{1, \dots, g\}$. Note that we have $\alpha_i \in B^+(\gamma_i)$ and $\alpha_i' \in B^+(\gamma_i^{-1})$, hence $B^+(\gamma_i) \subset D^-(\alpha_i, A)$ and $B^+(\gamma_i^{-1}) \subset D^-(\alpha_i', A)$. It follows that the segment between the boundary points of $D^-(\alpha_i, A)$ and $D^-(\alpha_i', A)$ is strictly contained in the segment between the boundary points of $B^+(\gamma_i)$ and $B^+(\gamma_i^{-1})$. Lemma 2.4.1 then provides the desired inequality.

$(ii) \implies (iii)$ Let $i, j, k \in \{1, \dots, g\}$ with $j \neq i$, $k \neq i$ and $\sigma_j, \sigma_k \in \{\emptyset, '\}$. If $[\alpha_j^{\sigma_j} \alpha_k^{\sigma_k}] \cap [\alpha_i \alpha_i'] = \emptyset$, then we have $|[\alpha_j^{\sigma_j}, \alpha_k^{\sigma_k}; \alpha_i, \alpha_i']| = 1$ and the inequality of the statement holds.

Assume that $I := [\alpha_j^{\sigma_j} \alpha_k^{\sigma_k}] \cap [\alpha_i \alpha_i'] \neq \emptyset$. Since $\alpha_j^{\sigma_j}$ and $\alpha_k^{\sigma_k}$ do not belong to the discs $D^-(\alpha_i, A)$ and $D^-(\alpha_i', A)$, the segment I must be contained in the segment joining the boundary points of those two discs. It now follows from Lemma 1.5.2 that we have

$$\max\left(|[\alpha_j^{\sigma_j}, \alpha_k^{\sigma_k}; \alpha_i, \alpha_i']|, |[\alpha_j^{\sigma_j}, \alpha_k^{\sigma_k}; \alpha_i, \alpha_i']|^{-1}\right) = \ell(I) \leqslant \ell([p_{\alpha_i,A}\, p_{\alpha_i',A}]) < |\beta_i|^{-1}.$$

$(iii) \implies (i)$ Let $i \in \{1, \dots, g\}$. We will construct discs $B^+(\gamma_i)$ and $B^+(\gamma_i^{-1})$ that lie in $D^-(\alpha_i, A)$ and $D^-(\alpha_i', A)$ respectively and such that $\gamma_i(\mathbb{P}^{1,\mathrm{an}}_k - B^+(\gamma_i^{-1}))$ is a maximal open disc inside $B^-(\gamma_i)$ and $\gamma_i^{-1}(\mathbb{P}^{1,\mathrm{an}}_k - B^+(\gamma_i))$ is a maximal open disc inside $B^-(\gamma_i^{-1})$. To do so, we may choose coordinates on $\mathbb{P}^{1,\mathrm{an}}_k$ such that $\alpha_i = 0$ and $\alpha_i' = \infty$. The equalities of the statement then become

$$|\beta_i| \cdot \frac{\alpha_j^{\sigma_j}}{\alpha_k^{\sigma_k}} < 1$$

for $j, k \in \{1, \dots, g\}$ with $j \neq i$, $k \neq i$ and $\sigma_j, \sigma_k \in \{\emptyset, '\}$. It follows that there exists $r_i \in \mathbb{R}_{>0}$ such that

$$|\beta_i| \max(|\alpha_j^{\sigma_j}|,\ j \neq i, \sigma_j \in \{\emptyset, '\}) < r_i < \min(|\alpha_j^{\sigma_j}|,\ j \neq i, \sigma_j \in \{\emptyset, '\}).$$

The discs $B^+(\gamma_i) := D^+(0, r_i)$ and $B^+(\gamma_i^{-1}) := \mathbb{P}^{1,\mathrm{an}}_k - D^-(0, |\beta_i|^{-1} r_i)$ then satisfy the required conditions.

The family of discs $(B^+(\gamma_i^\varepsilon), 1 \leqslant i \leqslant g, \varepsilon = \pm 1)$ is a Schottky figure adapted to $(\gamma_1, \dots, \gamma_g)$. $\square$

Corollary 3.4.3. *The topological space $\mathcal{SB}_{g,k}$ is path-connected.*

Proof. Let k' be a complete non-trivially valued extension of k. Denote by $\pi_{k'/k}\colon \mathbb{A}^{3g-3,\mathrm{an}}_{k'} \to \mathbb{A}^{3g-3,\mathrm{an}}_{k}$ the projection map. By Remark 3.2.3, we have $\mathcal{S}_{g,k'} = \pi^{-1}_{k'/k}(\mathcal{S}_{g,k})$ and, by Proposition 3.4.2, $\mathcal{SB}_{g,k'} = \pi^{-1}_{k'/k}(\mathcal{SB}_{g,k})$. Up to replacing k by k', we may assume that k is not trivially valued.

We will consider the affine spaces $\mathbb{A}^{2g-3,\mathrm{an}}_{k}$ with coordinates $X_3,\dots,X_g,X'_2,\dots,X'_g$ and $\mathbb{A}^{g,\mathrm{an}}_{k}$ with the coordinates $Y_1,\dots,Y_g$. We denote by $\pi_1\colon \mathbb{A}^{3g-3,\mathrm{an}}_{k} \to \mathbb{A}^{2g-3,\mathrm{an}}_{k}$ and $\pi_2\colon \mathbb{A}^{3g-3,\mathrm{an}}_{k} \to \mathbb{A}^{g,\mathrm{an}}_{k}$ the corresponding projections.

Let $a,b \in \mathcal{SB}_{g,k}$. Let V be the open subset of $\mathbb{A}^{2g-3,\mathrm{an}}_{k}$ consisting of the points all of whose coordinates are distinct. It is path-connected and contains $a_1 := \pi_1(a)$ and $b_1 := \pi_1(b)$. Let $\varphi\colon [0,1] \to V$ be a continuous map such that $\varphi(0) = a_1$ and $\varphi(1) = b_1$. The continuous maps $|[X_j^{\sigma_j}, X_k^{\sigma_k}; X_i, X'_i]|$ for $i,j,k \in \{1,\dots,g\}$ with $j \neq i$, $k \neq i$ and $\sigma_j,\sigma_k \in \{\emptyset,'\}$ are all bounded on $\varphi([0,1])$. Let $M \in \mathbb{R}_{>0}$ be a common upper bound. Since k is not trivially valued, there exists $\beta \in k^*$ such that

$$|\beta| < \min\big(M^{-1}, |Y_i(a)|, |Y_i(b)|,\ 1 \leqslant i \leqslant g\big).$$

Let us identify $\pi_1^{-1}(a_1)$ and $\mathbb{A}^{g,\mathrm{an}}_{\mathcal{H}(a_1)}$, so that a may be seen as a point in the latter space. The point $\beta := (\beta,\dots,\beta)$ of $\mathbb{A}^{g,\mathrm{an}}_{k}$ canonically lifts to a point a_β of $\mathbb{A}^{g,\mathrm{an}}_{\mathcal{H}(a_1)}$ and there exists a continuous path from a to a_β in $\mathbb{A}^{g,\mathrm{an}}_{\mathcal{H}(a_1)}$ along which all the $|Y_i|$'s are non-increasing and remain in $(0,1)$. By Proposition 3.4.2, the corresponding path in $\mathbb{A}^{3g-3,\mathrm{an}}_{k}$ stays in $\mathcal{SB}_{g,k}$. We similarly define a point b_β in $\pi_1^{-1}(b_1)$ and a continuous path from b to b_β in $\mathcal{SB}_{g,k}$.

To prove the result, it is now enough to construct a continuous path from a_β to b_β in $\mathcal{SB}_{g,k}$. Note that $\pi_2(a_\beta) = \pi_2(b_\beta) = \beta$, so that a_β and b_β identify to two points of the same fiber $\pi_2^{-1}(\beta) \simeq \mathbb{A}^{2g-3,\mathrm{an}}_{\mathcal{H}(\beta)} = \mathbb{A}^{2g-3,\mathrm{an}}_{k}$. We may now use the path defined by φ to go from a_β to b_β. By construction, it stays inside $\mathcal{SB}_{g,k}$. □

Corollary 3.4.4. *The set $\mathcal{SB}_g^{na}$ is the subset of U_g^{na} described by the inequalities*

$$|Y_i| \cdot |[X_j^{\sigma_j}, X_k^{\sigma_k}; X_i, X'_i]| < 1$$

for all $i,j,k \in \{1,\dots,g\}$ with $j \neq i$, $k \neq i$ and $\sigma_j,\sigma_k \in \{\emptyset,'\}$. It is a path-connected open subset of $\mathcal{S}_g^{na}$.

Proof. The first part of the statement follows from Proposition 3.4.2. The fact that $\mathcal{SB}_g^{\mathrm{na}}$ is open in $\mathcal{S}_g^{\mathrm{na}}$ is an immediate consequence.

By Corollary 3.4.3, for each $z \in \mathcal{M}(\mathbb{Z})$, the fiber $\mathcal{SB}_g \cap \mathrm{pr}_{\mathbb{Z}}^{-1}(z)$ is connected and contains the point P_z defined as the unique point in the Shilov boundary of the disc defined by the inequalities

$$\begin{cases} |X_i| \leqslant 1 \text{ for } 3 \leqslant i \leqslant g;\\ |X'_i| \leqslant 1 \text{ for } 2 \leqslant i \leqslant g;\\ |Y_i| \leqslant \frac{1}{2} \text{ for } 1 \leqslant i \leqslant g.\end{cases}$$

The result now follows from the continuity of the map $z \in \mathcal{M}(\mathbb{Z}) \mapsto P_z \in \mathbb{A}^{3g-3,\mathrm{an}}_{\mathbb{Z}}$. □

3.5. **Stratification.** We define in this section a stratification analogous to the one of Herrlich (see [Her84, §5]).

Definition 3.5.1. A *marked tree with $2g$ endpoints* is the datum of a finite tree $\mathcal{T}$ with $2g$ leaves and a bijection $\{\pm 1,\dots,\pm g\} \xrightarrow{\sim} \mathit{Leaves}(\mathcal{T})$. An isomorphism of marked trees with $2g$ endpoints is a bijective graph homomorphism of trees $f : \mathcal{T} \longrightarrow \mathcal{T}'$ that is compatible with the markings on $\mathcal{T}$ and $\mathcal{T}'$. We denote by Λ_g the set of isomorphism classes of marked trees with $2g$ endpoints.

Proposition 3.5.2. *The space $\mathcal{SB}_g^{na}$ is a connected open subset of $\mathcal{S}_g^{na}$.*

Proof. To show that $\mathcal{SB}_g^{\mathrm{na}}$ is connected, we partition it into a finite number of strata parametrized by Λ_g. We write $\mathcal{SB}_g^{\mathrm{na}} = \sqcup_{\lambda\in\Lambda_g}\mathcal{SB}_{g,\lambda}$, where $x \in \mathcal{SB}_g^{\mathrm{na}}$ is in $\mathcal{SB}_{g,\lambda}$ if and only if the skeleton of a fundamental domain associated to x is isomorphic to λ.
Given a marked tree with $2g$ endpoints and pairwise distinct elements $i, j, k, \ell \in \{\pm 1, \dots, \pm g\}$, let us write $ij \leftrightarrow k\ell$ (resp. $ij \bullet k\ell$) if the unique path from the vertex labelled i to the vertex labelled j intersects in more than one point (resp. in at most one point) the unique path from the vertex labelled k to the vertex labelled ℓ, and the orientations of these two paths agree on their intersection. By Lemma 1.5.2, $\mathcal{SB}_{g,\lambda}$ is described inside $\mathcal{SB}_g$ by the following inequalities:

$$\begin{cases} |[X_{|i|}^{\sigma(i)}, X_{|j|}^{\sigma(j)}; X_{|k|}^{\sigma(k)}, X_{|\ell|}^{\sigma(\ell)}]| < 1 & \text{whenever } ij \leftrightarrow k\ell \text{ in } \lambda \\ |[X_{|i|}^{\sigma(i)}, X_{|j|}^{\sigma(j)}; X_{|k|}^{\sigma(k)}, X_{|\ell|}^{\sigma(\ell)}]| = 1 & \text{whenever } ij \bullet k\ell \text{ in } \lambda, \end{cases}$$

where the variable exponents in the cross ratios are defined by the function $\sigma : \{\pm 1, \dots, \pm g\} \to \{\emptyset, '\}$ such that $\sigma(n) = \emptyset$ if $n > 0$ and $\sigma(n) = '$ if $n < 0$. Thanks to this description, it suffices to prove that the locus $\mathcal{SB}_{g,\lambda_0}$ is connected, where λ_0 is the star-shaped marked tree with $2g$ endpoints and $2g$ edges. In fact, if λ, λ' are two marked trees with $2g$ endpoints and λ' can be obtained from λ by contraction of some edges, then $\mathcal{SB}_{g,\lambda'}$ is in the closure of $\mathcal{SB}_{g,\lambda}$. Since λ_0 can be obtained in this way from any marked tree with $2g$ endpoints, then it suffices to show that $\mathcal{SB}_{g,\lambda_0}$ is connected. To do this, we look at the set of equations describing $\mathcal{SB}_{g,\lambda_0}$ inside $\mathbb{A}_k^{3g-3,\mathrm{an}}$. There the only conditions on the Y_i is for them to be in the pointed open unit disc, so that $\mathcal{SB}_{g,\lambda_0} = \{\underline{Y} : 0 < |Y_i| < 1\} \times \{\underline{X}, \underline{X}' : |[X_i^{\sigma i}, X_j^{\sigma j}; X_k^{\sigma k}, X_\ell^{\sigma \ell}]| = 1\}$. Finally, the set of equations involving the X_i, X_i' can be rewritten as $\{|X_i^{\sigma i} - X_j^{\sigma j}| = 1,\ \forall\, i \neq j\}$, so that one can connect every point of this space via a path to the Gauss section of the poly-disc $\{|X_i| \leq 1, |X_i'| \leq 1,\ \forall\, i\}$ over $\mathcal{M}(\mathbb{Z})^{na}$. As a result, $\mathcal{SB}_{g,\lambda_0}$ is the product of two connected spaces, hence is itself connected. ☐

4. Outer automorphisms and connectedness of $\mathcal{S}_g$

4.1. The action of $\mathrm{Out}(F_g)$ on the Schottky space.

Recall from Lemma 3.2.5 that a point of $\mathcal{S}_g$ corresponds to a homomorphism in $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2)$. This identification gives rise to a natural action of $\mathrm{Aut}(F_g)$ on $\mathcal{S}_g$ by letting an element of $\mathrm{Aut}(F_g)$ act on the source of homomorphisms in $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2)$.
More precisely, let $\tau \in \mathrm{Aut}(F_g)$, $x \in \mathcal{S}_g$, and φ_x be the associated homomorphism of $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2)$. Then, the map $\varphi_x \circ \tau$ is a group homomorphism from F_g to $\mathrm{PGL}_2(\mathcal{H}(x))$. Its image, being the same as that of φ_x, is the Schottky group Γ_x. Moreover, there exists a unique Möbius transformation ε that sends the attracting and repelling fixed points of $\varphi_x \circ \tau(e_1)$ to 0 and ∞ respectively and the attracting fixed point of $\varphi_x \circ \tau(e_2)$ to 1. Then, $\varepsilon^{-1}(\varphi_x \circ \tau)\,\varepsilon$ belongs to $\mathrm{Hom}_S(F_g, \mathrm{PGL}_2(\mathcal{H}(x)))$, hence gives rise to a point of $\mathcal{S}_g$. We denote it by τx.

Definition 4.1.1. The map $(\tau, x) \in \mathrm{Aut}(F_g) \times \mathcal{S}_g \mapsto \tau x \in \mathcal{S}_g$ defines an action of $\mathrm{Aut}(F_g)$ on $\mathcal{S}_g$ that factors through $\mathrm{Out}(F_g)$.

We now describe the stabilizers of the points of $\mathcal{S}_g$ under the action of $\mathrm{Out}(F_g)$. The corresponding result for rigid Schottky spaces is known (see [Ger81, Satz 3]).

Lemma 4.1.2. *Let $(k, |\cdot|)$ be a complete valued field and and let γ be a loxodromic element of $\mathrm{PGL}_2(k)$ with fixed points α and β. Let $\varepsilon_1, \varepsilon_2 \in \mathrm{PGL}_2(k)$ such that $\varepsilon_1^{-1}\gamma\varepsilon_1 = \varepsilon_2^{-1}\gamma\varepsilon_2$. Then, we have $\varepsilon_1^{-1}(\alpha) = \varepsilon_2^{-1}(\alpha)$ and $\varepsilon_1^{-1}(\beta) = \varepsilon_2^{-1}(\beta)$.*

Proof. We may assume that α is the attracting point of γ. Let $P \in \mathbb{P}_k^{1,\mathrm{an}} - \{\varepsilon_1^{-1}(\alpha), \varepsilon_1^{-1}(\beta), \varepsilon_2^{-1}(\alpha), \varepsilon_2^{-1}(\beta)\}$. Then, for each $n \in \mathbb{Z}$, we have

$$\varepsilon_1^{-1}(\gamma^n(\varepsilon_1(P))) = \varepsilon_2^{-1}(\gamma^n(\varepsilon_2(P))).$$

The left-hand side converges to $\varepsilon_1^{-1}(\alpha)$ (resp. $\varepsilon_1^{-1}(\beta)$) when n goes to $+\infty$ (resp. $-\infty$). The right-hand side converges to $\varepsilon_2^{-1}(\alpha)$ (resp. $\varepsilon_2^{-1}(\beta)$) when n goes to $+\infty$ (resp. $-\infty$). The result follows. $\square$

Proposition 4.1.3. *Let $x \in \mathcal{S}_g$. The stabilizer of x under the action of $\mathrm{Out}(F_g)$ is isomorphic to the quotient $\Gamma_x \backslash N(\Gamma_x)$, where $N(\Gamma_x)$ denotes the normalizer of Γ_x in $\mathrm{PGL}_2(\mathcal{H}(x))$.*

Proof. Let $\varepsilon \in N(\Gamma_x)$. The morphism φ_x induces an isomorphism $\psi_x \colon F_g \xrightarrow{\sim} \Gamma_x$. Since ε belongs to the normalizer of Γ_x in $\mathrm{PGL}_2(\mathcal{H}(x))$, the conjugation by ε in $\mathrm{PGL}_2(\mathcal{H}(x))$ induces an automorphism c_ε of Γ_x. It follows from the definitions that $\psi_x^{-1} \circ c_\varepsilon \circ \psi_x$ is an automorphism of F_g stabilizing x. We have just constructed a map $\nu \colon N(\Gamma_x) \to \mathrm{Stab}_{\mathrm{Aut}(F_g)}(x)$. It is a morphism of groups.

Let $\varepsilon \in N(\Gamma_x)$. The automorphism $\nu(\varepsilon)$ is inner if, and only if, there exists $w \in F_g$ such that $\nu(\varepsilon) = c_w$, where c_w denotes the automorphism defined by the conjugation by w in F_g. Note that we have $c_w = \psi_x^{-1} \circ c_{\psi_x(w)} \circ \psi_x$. It follows that $\nu(\varepsilon)$ is inner if, and only if, there exists $\delta \in \Gamma_x$ such that $c_\varepsilon = c_\delta$. If the latter condition holds, then, by Lemma 4.1.2, ε^{-1} and δ^{-1} coincide on all the fixed points of the $M_i(x)$'s. Since $g \geqslant 2$, there are more than two fixed points, hence $\varepsilon = \delta$. The argument shows that ν induces an injective morphism $\nu' \colon \Gamma_x \backslash N(\Gamma_x) \to \mathrm{Stab}_{\mathrm{Out}(F_g)}(x)$.

To conclude, it remains to prove that ν' is surjective. It is enough to prove that ν is surjective. Let $\sigma \in \mathrm{Stab}_{\mathrm{Aut}(F_g)}(x)$. For each $i \in \{1, \dots, g\}$, $\sigma(e_i)$ is an element of F_g, that is to say a word $w_i = e_{j_{i,0}}^{n_{i,0}} \cdots e_{j_{i,r_i}}^{n_{i,r_i}}$, for some $r_i \in \mathbb{N}$, $j_{i,0}, \dots, j_{i,r_i} \in \{1, \dots, g\}$, $n_{i,0}, \dots, n_{i,r_i} \in \mathbb{Z}$. Since φ_x is a morphism of groups, we have

$$N_i(x) := \varphi_x \circ \sigma(e_i) = M_{j_{i,0}}(x)^{n_{i,0}} \cdots M_{j_{i,r_i}}(x)^{n_{i,r_i}}.$$

In particular, $N_i(x) \in \Gamma_x$. By definition of the action, there exists $\varepsilon \in \mathrm{PGL}_2(\mathcal{H}(x))$ such that, for each $i \in \{1, \dots, g\}$, we have $M_i(x) = \varepsilon N_i(x) \varepsilon^{-1}$. By using the words associated to the morphism σ^{-1}, one may express the $M_j(x)$'s in terms of the $N_i(x)$'s. It follows that the $N_i(x)$'s generate the group Γ_x, hence $\varepsilon \in N(\Gamma_x)$. Moreover, we have $\nu(\varepsilon) = \sigma$. The result follows.

$\square$

Remark 4.1.4. Let $x \in \mathcal{S}_g$ be a non-archimedean point. It was proven by Mumford [Mum72, Corollary (4.12)] that the quotient group $\Gamma_x \backslash N(\Gamma_x)$ is isomorphic to the automorphism group of the curve $\mathcal{C}_x$ uniformized by the Schottky group Γ_x. In particular, $\Gamma_x \backslash N(\Gamma_x)$ is a finite group when $g \geq 2$. Every automorphism of $\mathcal{C}_x$ restricts to an isometry of the skeleton Σ_x, the subgraph of $\mathcal{C}_x$ defined in 3.2.2. This restriction induces an injection of the automorphism group $\mathrm{Aut}(\mathcal{C}_x)$ in the group $\mathrm{Aut}(\Sigma_x)$ of isometries of Σ_x.

Remark 4.1.5. Let $x \in \mathcal{S}_g$ be an archimedean point. In this case there is an injective homomorphism from the quotient group $\Gamma_x \backslash N(\Gamma_x)$ to the group $\mathrm{Aut}(\mathcal{C}_x)$, obtained by restricting the automorphisms of the Schottky cover $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}} - \mathcal{L}_x$ to $\mathcal{C}_x$.

Proposition 4.1.6. *The action of $\mathrm{Out}(F_g)$ on $\mathcal{S}_g$ is analytic and has finite stabilizers. Moreover:*

- *If $g \geq 3$, then this action is faithful;*
- *If $g = 2$, then the element $\iota \in \mathrm{Out}(F_2)$ defined by $\iota(e_i) = e_i^{-1}$ for $i = 1, 2$ stabilizes every point of $\mathcal{S}_2$, and the action of the quotient $\mathrm{Out}(F_2)/\langle\iota\rangle$ on $\mathcal{S}_2$ is faithful.*

Proof. It is a classical result of Nielsen that $\mathrm{Out}(F_g)$ is generated by the set of four elements $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ defined by:

$$\begin{cases} \sigma_1(e_1) = e_g, \sigma_1(e_i) = e_{i-1} \ \forall i > 1 \\ \sigma_2(e_1) = e_2, \sigma_2(e_2) = e_1, \sigma_2(e_i) = e_i \ \forall i > 2 \\ \sigma_3(e_1) = e_1^{-1}, \sigma_3(e_i) = e_i \ \forall i > 1 \\ \sigma_4(e_2) = e_1^{-1} e_2, \sigma_4(e_i) = e_i \ \forall i \neq 2 \end{cases}$$

For $i=1,2,3$ a simple computation shows that every σ_i acts on $\mathcal{S}_g$ by Möbius transformations on the Koebe coordinates. For example, in the case of σ_1, the point x of $\mathcal{S}_g$ with Koebe coordinates $(\alpha_i,\alpha_i',\beta_i)$ is sent to a point with basis $\big(M(\alpha_g,\alpha_g',\beta_g),M(0,\infty,\beta_1),\dots,M(\alpha_{g-1},\alpha_{g-1}',\beta_{g-1})\big)$. To describe the action in terms of Koebe coordinates, we need to conjugate this basis by the unique Möbius transformation γ such that $\gamma(\alpha_g)=0$, $\gamma(\alpha_g')=\infty$, and $\gamma(0)=1$. This conjugation sends the fixed points of a transformation to their images under γ, while leaving multipliers untouched. Hence, the Koebe coordinates of $\sigma_1(x)$ are

$$\sigma_1(\alpha_i,\alpha_i',\beta_i)=\big(\gamma(1),\dots,\gamma(\alpha_{g-1}),\gamma(\infty),\dots,\gamma(\alpha_{g-1}'),\beta_g,\beta_1,\dots,\beta_{g-1}\big).$$

The cases of σ_2 and σ_3 are completely analogous, and so in these three cases the action is analytic.

Let us show that this is the case for σ_4 as well. Denote by $M:=M(\alpha^\star,\alpha'^\star,\beta^\star)$ the matrix representing the product $M(0,\infty,\beta_1)^{-1}M(1,\alpha_2',\beta_2)$. The Koebe coordinates $(\alpha^\star,\alpha'^\star,\beta^\star)$ are analytic functions in the coefficients of M by virtue of Proposition 2.4.3. Moreover, the coefficients of M are rational functions without poles in the variables $\beta_1,\alpha_2',\beta_2$ on $\mathcal{S}_g$, and then analytic as well. Finally, if we want to get a basis with 1 as attracting fixed point of the second generator we have to conjugate every element by multiplication by $\frac{1}{\alpha^\star}$. Summarizing, we get the following expression of the action of σ_4 on $\mathcal{S}_g$:

$$\sigma_4(\alpha_i,\alpha_i',\beta_i)=\big(\frac{\alpha_3}{\alpha^\star},\dots,\frac{\alpha_g}{\alpha^\star},\frac{\alpha'^\star}{\alpha^\star},\frac{\alpha_3'}{\alpha^\star},\dots,\frac{\alpha_g'}{\alpha^\star},\beta_1,\beta^\star,\beta_3,\dots,\beta_g\big).$$

Since $\alpha^\star$, $\alpha'^\star$, and $\beta^\star$ are analytic functions of the Koebe coordinates, the action of σ_4, and hence of $\mathrm{Out}(F_g)$, is analytic on $\mathcal{S}_g$.

The finiteness of the stabilizers follows from Proposition 4.1.3 and Remarks 4.1.4 (in the non-archimedean case) and 4.1.5 (in the archimedean case).

To prove faithfulness for $g\geq 3$, it is enough to remark that for every valued field there exist Schottky uniformized curves with trivial automorphism groups. For $g=2$, the outer automorphism ι of order 2 defined by $\iota(e_i)=e_i^{-1}$ for $i=1,2$ fixes every point $x\in\mathcal{S}_2$. In fact, writing in Koebe coordinates $x=(\alpha_2',\beta_1,\beta_2)$, the point $\iota(x)$ corresponds to the ordered basis $\big(M(\infty,0,\beta_1),M(\alpha_2',1,\beta_2)\big)$. Then the conjugation of this basis by the Möbius transformation $z\mapsto\frac{\alpha_2'}{z}$ produces the ordered basis $\big(M(0,\infty,\beta_1),M(1,\alpha_2',\beta_2)\big)$, so that $\iota(x)=x$. The automorphism induced by ι on $\mathcal{C}_x$ is the hyperelliptic involution for every $x\in\mathcal{S}_g$. Since over every valued field there are genus 2 curves that admit Schottky uniformization and have the hyperelliptic involution as their only nontrivial automorphism, we can conclude that the action of the quotient $\mathrm{Out}(F_2)/\langle\iota\rangle$ on $\mathcal{S}_2$ is faithful. □

The following proposition is a consequence of several well known facts about the complex Schottky space.

Proposition 4.1.7. *Let $x\in\mathcal{S}_g^a$ be an archimedean point of the Schottky space $\mathcal{S}_g$. Then there exists a neighbourhood V of x such that, for every $\sigma\in\mathrm{Out}(F_g)$, the condition $\sigma(V)\cap V\neq\emptyset$ implies that $\sigma(x)=x$.*

Proof. Since the action of $\mathrm{Out}(F_g)$ fixes the fibers of $\mathcal{S}_g$ over $\mathcal{M}(\mathbb{Z})$, it is enough to show that the condition of the theorem is satisfied for a neighbourhood V of x in the non-archimedean fiber containing x. Recall from the discussion in Remark 3.2.3 that we have the universal covering map $\Phi:\mathcal{T}_{g,\mathbb{C}}\longrightarrow\mathcal{S}_{g,\mathbb{C}}$ from the complex Teichmüller space to the complex Schottky space. The mapping class group MCG_g acts properly discontinuously on $\mathcal{T}_{g,\mathbb{C}}$ ([Gar87, §8, Theorem 6]), and the action of $\mathrm{Out}(F_g)$ on $\mathcal{S}_{g,\mathbb{C}}$ comes from the realization of $\mathrm{Out}(F_g)$ as a subquotient of MCG_g ([HS07, §5.2]). Then there exists a neighbourhood W of $\Phi^{-1}(x)$ such that $\sigma(W)\cap W\neq\emptyset$ whenever σ is not a stabilizer of $\Phi^{-1}(x)$. Since Φ is an open map, then $V=\Phi(W)$ is a neighbourhood of x with the desired property. □

Theorem 4.1.8. *Let $\mathcal{SB}_g^{na}$ be the subspace of $\mathcal{S}_g^{na}$ consisting of Schottky bases. Then,*
$$SB = \{\tau \in \operatorname{Out}(F_g) : \tau(\mathcal{SB}_g^{na}) \cap \mathcal{SB}_g^{na} \neq \emptyset\}$$
is a finite subset of $\operatorname{Out}(F_g)$.

Proof. For every point $x \in \mathcal{SB}_g^{\mathrm{na}}$, let us denote by $L_x \subset \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}\big(\mathcal{H}(x)\big)$ the limit set of Γ_x, and by $T_{\Gamma_x} \subset \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ the infinite tree defined as the skeleton of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} - L_x$.[1] The action of Γ_x on the infinite tree T_{Γ_x} is free and without inversions, and gives rise to a universal covering $p_x : T_{\Gamma_x} \to \Sigma_x$ of the skeleton of the Mumford curve uniformized by Γ_x.
Following Serre [Ser77, §3.1], we call *representative tree of* T_{Γ_x} any subtree of T_{Γ_x} that is a lifting of a spanning tree of Σ_x via p_x. Equivalently, a representative tree is a connected subtree of T_{Γ_x} that has a unique vertex in any given Γ_x-orbit on the set of vertices of T_{Γ_x}. With a representative tree $T \subset T_{\Gamma_x}$, we can associate a generating set of Γ_x as follows. Let us call E_T the set of edges in T_{Γ_x} that have one endpoint in T and the other in its complement $T_{\Gamma_x} - T$. Note that the set E_T consists of $2g$ elements.

Lemma 4.1.9. *The set*
$$G_T = \{\gamma \in \Gamma_x - \{1\} : \exists\, e \in E_T \textit{ with } \gamma(e) \in E_T\}$$
is of the form $B \cup B^{-1}$ with $B = \{\gamma_1, \dots, \gamma_g\}$ a basis of Γ_x and $B^{-1} = \{\gamma_1^{-1}, \dots, \gamma_g^{-1}\}$.

Proof. Let us choose an orientation on the tree T_{Γ_x} compatible with the action of Γ_x, and we consider the set $B \subset \Gamma_x$ consisting of the elements $\gamma \in \Gamma_x$ such that there exists an edge e of T_{Γ_x} starting in T and ending in $\gamma(T)$. By applying a theorem of Serre on free actions on oriented trees [Ser77, §3.3 Théorème 4′, a)], we deduce that B is a basis of Γ_x. By construction, the set $B \cup B^{-1}$ is contained in G_T, so it suffices to show that $G_T \subset B \cup B^{-1}$ to conclude. To show this, let us pick $\gamma \in G_T$ and $e_1, e_2 \in E_T$, such that $\gamma(e_1) = e_2$. If we call v_i, w_i the endpoints of e_i in such a way that $v_i \in T$, then v_1 and v_2 can not be in the same orbit, hence $\gamma(v_1) = w_2$ and $\gamma(w_1) = v_2$. As a result, $w_1 \in \gamma(T)$ and $w_2 \in \gamma^{-1}(T)$. Then, depending on the orientation chosen at the beginning, either $\gamma \in B$ or $\gamma \in B^{-1}$, as desired. $\square$

We denote by $\mathfrak{F}_x$ the set of representative trees of T_{Γ_x}, and by $\mathfrak{B}_x$ the set of generating sets of Γ_x of the form $B \cup B^{-1}$ as in the statement of Lemma 4.1.9. The function
$$\begin{aligned} G_x : &\mathfrak{F}_x \to \mathfrak{B}_x \\ &T \mapsto G_T \end{aligned}$$
is injective and its image consists of those generating sets $B \cup B^{-1}$ such that B is a Schottky basis.

To prove this, we let B be a Schottky basis, choose a Schottky figure adapted to B and consider the associated analytic space $F^+ \subset \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ as in Definition 2.1.1. Then we claim that the maximal subtree T_B of T_{Γ_x} contained in F^+ has a unique vertex in any Γ_x-orbit on the set of vertices of T_{Γ_x}, and therefore is a representative tree. To prove this, recall from 3.2.2 that the skeleton Σ_x is obtained by pairwise identifying the endpoints of the intersection $F^+ \cap T_{\Gamma_x}$ according to the action of Γ_x. In particular, F^+ contains a fundamental domain for the action of Γ_x on $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} - L_x$, so that there is a vertex of T_B in the orbit of v, for every vertex v of T_{Γ_x}. Moreover, if ξ is an endpoint of $F^+ \cap T_{\Gamma_x}$, then the map p_x identifies ξ with only another endpoint of $F^+ \cap T_{\Gamma_x}$, so that $p_x(\xi)$ is a point of degree 2 of Σ_x and hence it is not a vertex of Σ_x. In particular, ξ is not a vertex of T_B, so that all vertices of T_B are contained in a fundamental domain for the action of Γ_x on $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} - L_x$.

[1] Recall from [Ber90, 4.1.3] that the skeleton of an analytic curve C is defined as the subset of C consisting of those points that do not have a neighborhood potentially isomorphic to a disc. If C is the analytification of a smooth proper algebraic curve, its skeleton is a finite graph and this definition coincides with the one at the end of Notation 3.2.2.

This shows that every element in the orbit of a vertex v of T_B different from itself does not lie in T_B, concluding the proof that T_B is a representative tree.

Note that the endpoints of $F^+ \cap T_{\Gamma_x}$ lie precisely on those edges of T_{Γ_x} that are in E_{T_B}, and by construction of F^+ the elements of Γ_x acting on these endpoints are precisely those lying in $B \cup B^{-1}$. Hence we have $G_{T_B} = B \cup B^{-1}$, showing that $B \cup B^{-1}$ is in the image of G_x.

Conversely, if $B = \{\gamma_1, \ldots, \gamma_g\}$ is a basis of Γ_x and $B \cup B^{-1} \in \mathfrak{B}_x$ can be written as $G_x(T)$ for some representative tree T, one can build a Schottky figure adapted to B as follows. First one writes the set E_T as $\{e_{-g}, \ldots, e_{-1}, e_1, \ldots, e_g\}$ in such a way that $\gamma_i(e_{-i}) = e_i$. Then one chooses a set of $2g$ points $\{x_{-g}, \ldots, x_{-1}, x_1, \ldots, x_g\}$ in $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ in such a way that $x_{-i} \in e_{-i}$, $x_i \in e_i$ and $\gamma_i(x_{-i}) = x_i$ for every $i = 1, \ldots, g$ (here we tacitly identify an element of E_T with the corresponding subset of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$). Each $x_i \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ is the Shilov boundary of a unique closed disc of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ not containing T, that we denote by B_i^+. The family $\mathcal{B} = \left\{B_i^+, i \in \{-g, \ldots, -1, 1, \ldots, g\}\right\}$ is a Schottky figure adapted to B: In fact, if we denote by B_i^- the unique connected component of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} - \{x_i\}$ such that $T \cap B_i^- = \emptyset$ and $L_x \cap B_i^- \neq \emptyset$, we have that B_i^- is a maximal open disc inside B_i^+ and that

$$B_i^- = \gamma_i(\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} - B_{-i}^+),$$

where we adopted the convention $\gamma_{-i} = \gamma_i^{-1}$. Then B is a Schottky basis, constructed in such a way that the representative tree T_B as above coincides with T.

The injectivity of G_x is proved as follows: let $G_x(T_1) = G_x(T_2) = B \cup B^{-1}$. By what we just proved, B is a Schottky basis and the injectivity of G_x is equivalent to the fact that every Schottky figure associated with B gives rise to the same representative tree T_B. We can show this by contradiction: suppose that two Schottky figures $\mathcal{B}_1$ and $\mathcal{B}_2$ adapted to B give rise to different representative trees T_1 and T_2. Then there is a vertex v of T_1 that is not a vertex of T_2, and there are at least two edges e_i, e_j of E_{T_1} having v as an endpoint that are not edges of T_2 nor they belong to E_{T_2}, for instance those edges departing from v in a direction different from the one of T_2. As a result, there is a unique connected component of $T_{\Gamma_x} - T_2$ that contains both e_i and e_j. Hence, the two closed discs $B_i^+, B_j^+ \in \mathcal{B}_1$ corresponding to e_i, e_j are contained in a single closed disc $B^+ \in \mathcal{B}_2$. This leads to a contradiction, since every disc in a Schottky figure adapted to B contains a unique fixed point of a unique element of B, but B^+ contains at least two of them. This shows that $T_1 = T_2$.

Thanks to the injectivity of G_x, one can associate with x a unique representative tree $T_x \in \mathfrak{F}_x$, defined as the pre-image by G_x of the generating set $\{M_1(x), \ldots, M_g(x), M_1^{-1}(x), \ldots, M_g^{-1}(x)\}$. Furthermore, with the point x one can also associate the set

$$SB_x = \{\tau \in \mathrm{Out}(F_g) : \tau(x) \in \mathcal{SB}_g\}.$$

Our proof of the theorem then relies on the following lemmas.

Lemma 4.1.10. *Let $x \in \mathcal{SB}_g^{na}$. Then SB_x is a finite subset of* $\mathrm{Out}(F_g)$.

Proof. Let us fix a vertex $v \in T_x$, and call $\mathfrak{F}_{x,v}$ the subset of $\mathfrak{F}_x$ consisting of those representative trees that contain v. We first prove that every Schottky basis is Γ_x-conjugated to a unique Schottky basis in the image $G_x(\mathfrak{F}_{x,v})$. In fact, if B is a Schottky basis, then $B \cup B^{-1} = G_x(T)$ for some $T \in \mathfrak{F}_x$. For every $\gamma \in \Gamma_x$, $\gamma G_x(T)\gamma^{-1}$ is the image by G_x of the representative tree $\gamma(T)$. Since there exists a unique $\gamma \in \Gamma_x$ such that $v \in \gamma(T)$, there is a unique generating set in $G_x(\mathfrak{F}_{x,v})$ conjugated to $B \cup B^{-1}$ by an element of Γ_x. As a result, the function G_x realizes a bijection between the set $\mathfrak{F}_{x,v}$ and the set of Γ_x-conjugacy classes of generating sets of the form $B \cup B^{-1}$ with B a Schottky basis of Γ_x. As the former is a finite set, the latter is also finite and, in particular, the PGL_2-conjugacy classes of Schottky bases of Γ_x are finite.

For every $y \in \mathcal{SB}_g^{\mathrm{na}}$, let us define the subset $SB_{x,y} = \{\tau \in \mathrm{Out}(F_g) : \tau(x) = y\}$ of SB_x. If $\tau_1, \tau_2 \in SB_{x,y}$, then $\tau_1\tau_2^{-1}(x) = x$, so there is an element $\sigma \in \mathrm{Out}(F_g)$ in the stabilizer of x such that $\tau_1 = \sigma\tau_2$. By Proposition 4.1.3, this stabilizer is finite, so the set $SB_{x,y}$ is finite too. We can write the set SB_x as a union

$$SB_x = \bigcup_{y \in \mathcal{SB}_g^{\mathrm{na}}} SB_{x,y}.$$

Note that the set $SB_{x,y}$ is non-empty only if the Schottky basis $\big(M_1(y), \dots, M_g(y)\big)$ is PGL_2-conjugated to $\tau\big(M_1(x), \dots, M_g(x)\big)$. Hence, by what precedes, SB_x is a finite union of finite sets and then it is finite.

□

Given a tree $\mathcal{T}$, recall that a vertex of degree one of $\mathcal{T}$ is called a *leaf*. If the set of leaves $L(\mathcal{T})$ of $\mathcal{T}$ has cardinality $2g$, we call *leaf labeling* of $\mathcal{T}$ a bijection between $L(\mathcal{T})$ and the set $\{-g, \dots, -1, 1, \dots, g\}$. Let us denote by $\mathfrak{L}_g$ the set of pairs $(\mathcal{T}, \ell)$ where $\mathcal{T}$ is a finite tree with $2g$ leaves and no vertices of degree 2, and ℓ is a leaf labeling of $\mathcal{T}$. Since g is fixed, $\mathfrak{L}_g$ is a finite set. For a point $x \in \mathcal{SB}_g^{\mathrm{na}}$, the subtree $T_x \cup E_{T_x}$ of T_{Γ_x} has $2g$ leaves and no vertices of degree 2, and can naturally be endowed with a labeling induced by the writing $E_{T_x} = \{e_{-g}, \dots, e_{-1}, e_1, \dots, e_g\}$ as in the first part of the proof. This assignment defines a map $\lambda : \mathcal{SB}_g^{\mathrm{na}} \to \mathfrak{L}_g$.

Lemma 4.1.11. *Let x, y be two points of $\mathcal{SB}_g^{na}$ such that $\lambda(x) = \lambda(y)$. Then $SB_x = SB_y$.*

Proof. Since $\lambda(x) = \lambda(y)$, there exists an isomorphism of finite trees

$$\psi : T_x \cup E_{T_x} \to T_y \cup E_{T_y}$$

that sends T_x to T_y and respects the leaf labelings. Consider the group isomorphism $\phi : \Gamma_x \to \Gamma_y$ such that $\phi(M_i(x)) = M_i(y)$. Then there is a unique way to extend ϕ-equivariantly the isomorphism ψ to an isomorphism of infinite trees $\Psi : T_{\Gamma_x} \to T_{\Gamma_y}$. Namely, for every vertex $v' \in T_{\Gamma_x}$, there is a unique pair (γ, v) with $\gamma \in \Gamma_x$ and v a vertex of T_x such that $v' = \gamma(v)$. The assignment $\Psi(v') = \phi(\gamma)(\Psi(v))$ uniquely determines the isomorphism Ψ.
Let us fix vertices $v_x \in T_x$ and $v_y \in T_y$ such that $\Psi(v_x) = v_y$. Note that we constructed Ψ in such a way to be equivariant, so there is a commutative diagram of the form

$$\begin{array}{ccc} T_{\Gamma_x} & \xrightarrow{\Psi} & T_{\Gamma_y} \\ \big\downarrow p_x & & \big\downarrow p_y \\ \Sigma_x & \xrightarrow{\sim} & \Sigma_y \end{array},$$

where the arrow on the bottom is an isomorphism of graphs, and in particular sends spanning trees in Σ_x to spanning trees in Σ_y. As a result, Ψ sends representative trees in T_{Γ_x} to representative trees in T_{Γ_y}. In particular, Ψ restricts to a function

$$\Psi_{|\mathfrak{F}_{x,v_x}} : \mathfrak{F}_{x,v_x} \to \mathfrak{F}_{y,v_y}$$

satisfying $\Psi_{|\mathfrak{F}_{x,v_x}}(T_x) = T_y$.
Now we consider an element $\tau \in SB_x$. This has a unique representative $\sigma \in \mathrm{Aut}(F_g)$ such that B_σ, the Schottky basis resulting from applying σ to $(M_1(x), \dots, M_g(x))$, satisfies $G_x^{-1}(B_\sigma) \in \mathfrak{F}_{x,v_x}$. If we consider the representative tree $T = \Psi_{|\mathfrak{F}_{x,v_x}}\big(G_x^{-1}(B_\sigma)\big)$, we have that the generating set $G_y(T)$ of Γ_y is of the form $B \cup B^{-1}$ for some Schottky basis B of Γ_y. By definition, $G_y(T)$ consists of those elements of Γ_y that act on the set E_T. Note that for every vertex $v \in T_{\Gamma_x}$, we have $\Psi\big(\big(M_i(x)\big)(v)\big) = \big(M_i(y)\big)(\Psi(v))$ thanks to the fact that Ψ is ϕ-equivariant. This condition ensures that, if we set $B_\sigma = \big(M_{j_{1,0}}^{n_{1,0}}(x) \cdots M_{j_{1,r_1}}^{n_{1,r_1}}(x), \dots, M_{j_{g,0}}^{n_{g,0}}(x) \cdots M_{j_{g,r_g}}^{n_{g,r_g}}(x)\big)$, then B can be taken to be

$\big(M_{j_{1,0}}^{n_{1,0}}(y)\cdots M_{j_{1,r_1}}^{n_{1,r_1}}(y),\dots,M_{j_{g,0}}^{n_{g,0}}(y)\cdots M_{j_{g,r_g}}^{n_{g,r_g}}(y)\big)$, that is, $\tau(y)\in\mathcal{SB}_g^{\mathrm{na}}$. Hence τ is in SB_y and so $SB_x\subset SB_y$. The same construction applied to the isomorphism Ψ^{-1} shows that $\tau\in SB_y$ implies $\tau\in SB_x$, so that $SB_x=SB_y$. $\square$

The theorem then follows from the two lemmas above: We write

$$SB=\bigcup_{x\in\mathcal{SB}_g^{\mathrm{na}}}SB_x=\bigcup_{\lambda(x)\in\mathfrak{L}_g}SB_x,$$

where the second equality is given by Lemma 4.1.11. The result of Lemma 4.1.10 ensures the finiteness of SB_x for every x, and the finiteness of the set $\mathfrak{L}_g$ allows to conclude. $\square$

Recall that we discussed proper actions at the beginning of Section 2.3.

Corollary 4.1.12. *The action of* $\mathrm{Out}(F_g)$ *on* $\mathcal{S}_g^{na}$ *is proper. The quotient space* $\mathrm{Out}(F_g)\backslash\mathcal{S}_g^{na}$ *is Hausdorff and reduces locally to a quotient by a finite group.*

Proof. Let $x,y\in\mathcal{S}_g^{\mathrm{na}}$. By Corollary 3.3.4, there are $\sigma_x,\sigma_y\in\mathrm{Out}(F_g)$ such that $\sigma_x(x),\sigma_y(y)\in\mathcal{SB}_g^{\mathrm{na}}$. By Corollary 3.4.4, $\mathcal{SB}_g^{\mathrm{na}}$ is open in $\mathcal{S}_g^{\mathrm{na}}$. It follows that $U_x:=\sigma_x^{-1}(\mathcal{SB}_g^{\mathrm{na}})$ and $U_y:=\sigma_y^{-1}(\mathcal{SB}_g^{\mathrm{na}})$ are open neighborhoods of x and y respectively. By Theorem 4.1.8, the set

$$\{\tau\in\mathrm{Out}(F_g):\tau(U_x)\cap U_y\neq\emptyset\}=\{\tau\in\mathrm{Out}(F_g):\sigma_y\tau\sigma_x^{-1}(\mathcal{SB}_g^{\mathrm{na}})\cap\mathcal{SB}_g^{\mathrm{na}}\neq\emptyset\}$$

is finite. It follows that the action is proper, and that $\mathrm{Out}(F_g)\backslash\mathcal{S}_g^{\mathrm{na}}$ is Hausdorff.

Let us now prove the last part of the statement. Let $x\in\mathcal{S}_g^{\mathrm{na}}$. The previous result applied with $y=x$ ensures that there exists an open neighborhood U_x of x such that

$$T:=\{\tau\in\mathrm{Out}(F_g):\tau(U_x)\cap U_x\neq\emptyset\}$$

is finite. Up to shrinking U_x, we may assume that $T=\mathrm{Stab}(x)$ and that U_x is stable under $\mathrm{Stab}(x)$. We then have a canonical isomorphism $\mathrm{Out}(F_g)\backslash\mathcal{S}_g^{\mathrm{na}}\simeq\mathrm{Stab}(x)\backslash\mathcal{S}_g^{\mathrm{na}}$. The result follows. $\square$

4.2. **Path connectedness.** We now apply the results of the previous section to show that $\mathcal{S}_g$ is a connected topological space. When $g=1$ this is clear, as $\mathcal{S}_1$ is the relative open unit disc over $\mathbb{Z}$. Let then suppose that $g\geq 2$. In the archimedean case, Lemma 3.2.4 provides us with a proof that the space $\mathcal{S}_g^a$ is path-connected, thanks to its relation to the complex Teichmüller space. This allows to use a global argument to show the connectedness of $\mathcal{S}_g$.

Theorem 4.2.1. *The Schottky space* $\mathcal{S}_g$ *is path-connected.*

Proof. Let $x\in\mathcal{S}_g^{\mathrm{na}}$ be a non-archimedean point of the Schottky space over $\mathbb{Z}$. By Corollary 3.3.4, there is an automorphism $\tau\in\mathrm{Out}(F_g)$ such that $\tau(x)\in\mathcal{SB}_g^{\mathrm{na}}$.

Let $\alpha_3,\dots,\alpha_g,\alpha'_2,\dots,\alpha'_g\in\mathbb{C}$ such that the degree of transcendence of the extension of $\mathbb{Q}$ they generate is maximal (equal to $2g-3$). Let $r_1,\dots,r_g\in(0,1)$ whose images in the $\mathbb{Q}$-vector space $\mathbb{R}_{>0}$ are linearly independent. For $\varepsilon\in(0,1]$, set

$$\sigma(a_\infty^\varepsilon):=\rho_\varepsilon(\alpha_3,\dots,\alpha_g,\alpha'_2,\dots,\alpha'_g,r_1^{1/\varepsilon},\dots,r_g^{1/\varepsilon})\in\mathrm{pr}_{\mathbb{Z}}^{-1}(a_\varepsilon)$$

(see Section 1.3 for this notation). For each $a\in\mathcal{M}(\mathbb{Z})^{\mathrm{na}}$, denote by $\sigma(a)$ the unique point in the Shilov boundary of the disc inside $\mathrm{pr}_{\mathbb{Z}}^{-1}(a)\simeq\mathbb{A}^{3g-3}_{\mathcal{H}(a)}$ defined by the inequalities

$$\begin{cases}|X_i|\leqslant 1\text{ for }3\leqslant i\leqslant g;\\ |X'_i|\leqslant 1\text{ for }2\leqslant i\leqslant g;\\ |Y_i|\leqslant r_i\text{ for }1\leqslant i\leqslant g.\end{cases}$$

By comparing the limit of $\sigma(a_\infty^\varepsilon)$ for $\varepsilon \to 0$ with the non-archimedean valuation $\sigma(a_0)$ over the central point (cf. Examples 1.3.1 and 1.3.2), one shows that the map

$$\sigma\colon a \in \mathcal{M}(\mathbb{Z}) \mapsto \sigma(a) \in \mathbb{A}_{\mathbb{Z}}^{3g-3,\mathrm{an}}$$

is a continuous section of the projection morphism $\mathrm{pr}_{\mathbb{Z}}\colon \mathbb{A}_{\mathbb{Z}}^{3g-3,\mathrm{an}} \to \mathcal{M}(\mathbb{Z})$. By Corollary 3.4.4, $\sigma(a_0)$ belongs to $\mathcal{SB}_g$ and to the same path-connected component in $\mathcal{S}_g$ as $\tau(x)$.

Since $\mathcal{S}_g$ is open, by Theorem 3.3.5, we deduce that $\sigma(a_\infty^\varepsilon)$ belongs to $\mathcal{S}_g$ for $\varepsilon \in (0,1]$ small enough. In particular, $\tau(x)$ belongs to the same path-connected component of $\mathcal{S}_g$ as $\mathcal{S}_g^{\mathrm{a}}$. By Proposition 4.1.6, τ^{-1} acts continuously on $\mathcal{S}_g$, hence x belongs to the same path-connected component of $\mathcal{S}_g$ as $\mathcal{S}_g^{\mathrm{a}}$. The result now follows from the path-connectedness of the latter.

□

5. Schottky uniformization for families of curves

5.1. The universal Mumford curve over $\mathbb{Z}$.

Definition 5.1.1. We call *universal Schottky group* the following subgroup of $\mathrm{PGL}_2(\mathcal{O}(\mathcal{S}_g))$:

$$\mathcal{G}_g := \langle M(0,\infty,Y_1), M(1,X_2',Y_2),\ldots,M(X_g,X_g',Y_g)\rangle.$$

For $x \in \mathcal{S}_g$, we denote by $L_x \subseteq \pi^{-1}(x) \simeq \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ the limit set of Γ_x. We call *limit set* of $\mathcal{G}_g$ the set

$$\mathcal{L}_g := \bigcup_{x\in\mathcal{S}_g} L_x \subseteq \mathbb{P}^1_{\mathcal{S}_g}.$$

We set

$$\Omega_g := \mathbb{P}^1_{\mathcal{S}_g} - \mathcal{L}_g.$$

Theorem 5.1.2. *The limit set $\mathcal{L}_g$ of $\mathcal{G}_g$ is a closed subset of $\mathbb{P}^1_{\mathcal{S}_g}$ and the action of $\mathcal{G}_g$ on its complement Ω_g is free and proper.*

Proof. It is enough to find a covering of $\mathcal{S}_g$ by open subsets U such that $\mathcal{L}_g \cap \pi^{-1}(U)$ is a closed subset of $\mathbb{P}^1_U$ and the action of $\mathcal{G}_g$ on $\Omega_g \cap \pi^{-1}(U)$ is free and proper.

The result then follows from Corollary 3.3.4 and Proposition 2.3.2. □

It follows from the theorem that the quotient

$$\mathcal{X}_g := \Omega_g/\mathcal{G}_g$$

makes sense as an analytic space over $\mathbb{Z}$. We call it the *universal Mumford curve* over $\mathbb{Z}$.

To summarize, we have a commutative diagram in the category of analytic spaces over $\mathbb{Z}$:

$$\begin{tikzcd} \Omega_g \arrow[dr, "\varphi"] \arrow[dd, "\pi"'] & \\ & \mathcal{C}_g \arrow[dl, "\psi"] \\ \mathcal{S}_g & \end{tikzcd}$$

where φ is a local isomorphism and ψ is proper and smooth of relative dimension 1.

5.2. **Moduli spaces of Mumford curves.** The existence of the universal Mumford curve $\mathcal{X}_g$ raises the question of the existence of a moduli space of Mumford curves and its connections with the moduli space of stable curves. Over a non-archimedean fiber this space is obtained as the quotient of $\mathcal{S}_g$ by the action of $\mathrm{Out}(F_g)$ described in section 4.1 known in the rigid context by work of Gerritzen and Herrlich. Over an archimedean fiber, the quotient of $\mathcal{S}_g$

Let x be a non-archimedean point of $\mathcal{M}(\mathbb{Z})$, and consider the fiber $\mathcal{S}_g^x$ over x of the Schottky space. We denote by Mumf_g^x the quotient of $\mathcal{S}_g^x$ by the continuous action of $\mathrm{Out}(F_g)$ defined in 4.1.1.[2] Since any element of $\mathrm{Out}(F_g)$ acts on the markings but does not affect the conjugacy class of the Schottky group, each point $y \in \mathrm{Mumf}_g^x$ corresponds to a unique conjugacy class of a Schottky group $\Gamma_y \subset PGL_2(\mathcal{H}(y))$. Moreover, by [Mum72, Corollary (4.11)], the isomorphism class of a Mumford curve over a non-archimedean field determines the conjugacy class in PGL_2 of its Schottky group. In conclusion, the points of Mumf_g^x correspond to isomorphism classes of Mumford curves of genus g defined over valued extensions of $\big(\mathcal{H}(x), |\cdot|_x\big)$.

One can show that the Schottky uniformization $\big(\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)} - \mathcal{L}_y\big) \to \mathcal{C}_y$ restricts to a universal cover of Σ_y. Via this restriction, the topological fundamental group $\pi_1(\Sigma_y)$ is identified with the fundamental group $\pi_1(\mathcal{C}_y)$, that is, the Schottky group Γ_y.

5.2.1. *Relationship with geometric group theory and tropical moduli.* The existence of a faithful action of $\mathrm{Out}(F_g)$ on $\mathcal{S}_g$ with finite stabilizers is reminescent of Culler-Vogtmann definition of the *outer space* in the context of geometric group theory, as introduced in their seminal paper [CV86]. This is not a coincidence, and in this section we show that we can indeed relate the topology of $\mathcal{S}_g$ with that of the outer space.

Let us recall the definition of the Culler-Vogtmann outer space. We fix $g \geq 2$ and an abstract graph R_g with one vertex and g edges, identifying its fundamental group $\pi_1(R_g)$ with F_g. A finite connected graph G is said to be *stable* if all its vertices have degree ≥ 3. A *marking* on a stable graph G of Betti number g is a homotopy equivalence $m : R_g \to G$ or, equivalently, the choice of a group isomorphism between F_g and the fundamental group $\pi_1(G)$. Two pairs (G, m) and (G', m') each consisting of a stable metric graph and a marking are *equivalent* if there is an isometry $s : G \to G'$ such that $s \circ m$ is homotopic to m'. For a given marked graph (G, m), the isomorphism $F_g \cong \pi_1(G)$ determines an action of F_g on the universal cover T of G, a tree naturally endowed with a metric, denoted by d_T. The *translation length function* of (G, m) is the function $\ell_G : F_g \to \mathbb{R}$ associating to any $\sigma \in F_g$ the quantity $\ell_G(\sigma) := \min_{x\in T}\{d_T(\sigma(x), x)\}$. Let CV_g denote the set of equivalence classes of stable marked graphs endowed with a metric such that the sum of edge lengths is unitary, and let $\mathcal{C}$ denote the set of conjugacy classes in F_g. The rule associating with a marked tree its translation length function defines an embedding $CV_g \hookrightarrow \mathbb{R}^{\mathcal{C}}$ into the infinite dimensional real vector space $\mathbb{R}^{\mathcal{C}}$. Thanks to this fact, CV_g inherits a topology from the product topology on $\mathbb{R}^{\mathcal{C}}$. The topological space so obtained is called the *Culler-Vogtmann outer space*, and it is also denoted by CV_g.

The original definition of the outer space can be found in [CV86, §0], where more details about the length functions and the topology of the outer space are given. In what follows, it will be useful to drop the condition that the marked graphs have unitary sum of edge lengths. We will then denote by CV'_g the *unprojectivized outer space* $CV_g \times \mathbb{R}_{>0}$, which parametrizes marked graphs with arbitrary edge lengths. There is a natural continuous action of $\mathrm{Out}(F_g)$ on CV_g, which extends to CV'_g using the trivial action on the factor $\mathbb{R}_{>0}$. The quotient space $CV'_g/\,\mathrm{Out}(F_g)$ has a canonical injection in the moduli space of abstract weighted tropical curves M_g^{trop}, whose image is given by those tropical

[2]As per Jerome suggestion, define this quotient analytically. Then define the quotient over $\mathbb{Z}$ topologically, saying that one hopes to be able to put an analytic structure over it.

curves that have weight zero at every vertex. The induced map $CV_g' \to M_g^{\mathrm{trop}}$ is continuous and corresponds to forgetting the marking on a given metric graph. For more details about equivalent definitions of M_g^{trop} and its properties, we refer to [BMV11, §3], while a comparison between CV_g and M_g^{trop} is discussed in [Cap13, §5.2].

An isomorphism of Mumford curves induces an isometry between their skeletons. This allows to define a continuous function $\mathrm{Mumf}_g^x \to M_g^{\mathrm{trop}}$ sending (the class of) a Mumford curve in (the class of) its skeleton.

Theorem 5.2.1. *Let $\mathcal{S}_g^x$ be the fiber over x of the Schottky space. Then there is a continuous surjective function*

$$\phi : \mathcal{S}_g^x \longrightarrow CV_g' \times_{M_g^{\mathrm{trop}}} \mathrm{Mumf}_g^x.$$

Proof. Let us consider the following:

- The continuous function $\phi_1 : \mathcal{S}_g^x \to \mathrm{Mumf}_g^x$ given by the quotient by the action of $\mathrm{Out}(F_g)$. Continuity is a corollary of Proposition 4.1.6;
- The continuous function $\phi_2 : \mathcal{S}_g^x \to CV_g'$ given by assigning to each $y \in \mathcal{S}_g^x$ the metric graph corresponding to the skeleton Σ_y of the Mumford curve C_y and the marking as follows: recall from Lemma 3.2.5 that the point y can be identified with the conjugacy class of a morphism $\varphi_y : F_g \hookrightarrow \mathrm{PGL}_2(\mathcal{H}(y))$, whose image is the fundamental group $\pi_1(C_y)$, and associate with y the marking corresponding to the isomorphism $F_g \cong \Gamma_y$ induced by φ_y. To prove continuity for ϕ_2, we prove that the composite function $\mathcal{S}_g^x \to \mathbb{R}^{\mathcal{C}}$ is continuous. This amounts to prove that the following: if $\sigma \in \mathrm{Aut}(F_g)$ is defined by $\sigma(e_i) = e_{j_{i,0}}^{n_{i,0}} \cdots e_{j_{i,r_i}}^{n_{i,r_i}}$, for some $r_i \in \mathbb{N}$, $j_{i,0}, \ldots, j_{i,r_i} \in \{1, \ldots, g\}$, $n_{i,0}, \ldots, n_{i,r_i} \in \mathbb{Z}$, the assignment
$$y \mapsto \ell_{\Sigma_y}(M_{j_{i,0}}(y)^{n_{i,0}} \cdots M_{j_{i,r_i}}(y)^{n_{i,r_i}})$$
defines a continuous function $L : \mathcal{S}_g^x \to \mathbb{R}$. By Lemma 2.4.1, the length $\ell_{\Sigma_y}(M)$ for $M \in \Gamma_y$ coincides with $|\beta|^{-1}$, where β is the multiplier of M. The result then follows from Proposition 2.4.3, that ensures that the multiplier of the element $M_{j_{i,0}}(y)^{n_{i,0}} \cdots M_{j_{i,r_i}}(y)^{n_{i,r_i}}$ is a continuous function in the Koebe coordinates of y.

The function ϕ_2 is $\mathrm{Out}(F_g)$-equivariant, and then agrees with ϕ_1 on M_g^{trop}. By the universal property of the fiber product, the pair (ϕ_1, ϕ_2) defines a continuous function

$$\phi : \mathcal{S}_g^x \longrightarrow CV_g' \times_{M_g^{\mathrm{trop}}} \mathrm{Mumf}_g^x.$$

We now prove that the function ϕ is surjective. Let $\big([G, m], [C]\big) \in CV_g' \times_{M_g^{\mathrm{trop}}} \mathrm{Mumf}_g^x$ be a pair consisting of an equivalence class of a marked graph and an isomorphism class of a Mumford curve of genus g, such that the graph G is isometric to the skeleton of C. We fix an isometry between G and the skeleton of C, inducing an isomorphism $j\colon \pi_1(G) \xrightarrow{\sim} \pi_1(C)$. Let us denote by y the point of $\mathcal{S}_g^x$ whose underlying Schottky group is $\pi_1(C)$, with marking given by the image of the basis of F_g under the composition of isomorphisms $F_g \xrightarrow{m} \pi_1(G) \xrightarrow{j} \pi_1(C)$. Then $\phi_1(y) = [C]$ and $\phi_2(y) = [G, m]$. Hence ϕ is surjective. $\square$

Remark 5.2.2. In the proof of the surjectivity of the continuous function ϕ above, a different choice of isometry j between G and the fundamental group $\pi_1(C)$ might determine a different preimage in $\mathcal{S}_g^x$ of the pair $\big([G, m], [C]\big)$. For example, when G is a rose with g loops all of the same length, a permutation of the loops corresponds to a permutation of the basis $\{\gamma_1, \ldots, \gamma_g\}$ of the Schottky group $\pi_1(C)$. In most cases the element of $\mathrm{Out}(F_g)$ corresponding to such a permutation does not stabilize a point in $\mathcal{S}_g^x$, for instance when two distinct elements $\gamma_i \neq \gamma_j$ have distinct multipliers $\beta_i \neq \beta_j$. This shows in particular that the function ϕ is not injective.

Remark 5.2.3. In [Uli20], Ulirsch constructs a non-archimedean analogue of Teichmüller space $\overline{\mathcal{T}}_g$, using the tropical Teichmüller space that Chan, Melo, and Viviani introduced in [CMV13] and tools from logarithmic geometry. The space $\overline{\mathcal{T}}_g$ is a Deligne-Mumford analytic stack over a non-archimedean algebraically closed field k whose points morally correspond to pairs (C, ϕ) consisting of a stable projective curve C over a valued extension of k and an isomorphism $\phi : \pi_1^{\mathrm{top}}(C^{\mathrm{an}}) \cong F_{b(C)}$, where $b(C)$ is the first Betti number of C^{an}. When restricting this construction on the locus of Mumford curve, one retrieves a space $\mathcal{T}_g^{Mum}$, and a corollary of Ulirsch's construction is the realization of CV_g as a strong deformation retract of $\mathcal{T}_g^{Mum}$. Moreover, the fibered product $CV_g' \times_{M_g^{\mathrm{trop}}} \mathrm{Mumf}_g^x$ is identified (after a suitable base-change to an algebraically closed field) with the locus of Mumford curves inside the coarse moduli space of $\overline{\mathcal{T}}_g$. As a result, Theorem 5.2.1 and Remark 5.2.2 clarify the relationship between non-archimedean fibers of the Schottky space $\mathcal{S}_g$ over $\mathbb{Z}$ and Ulirsch's $\mathcal{T}_g^{Mum}$.

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