Galois descent for semi-affinoid analytic spaces

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Plan

1. Arithmetic motivation
2. Discs and Annuli
3. Semi-affinoid spaces
4. Further perspectives

The semi-stable reduction theorem

K (complete) discretely valued field, $\mathcal{O}_K$, $\widetilde{K}$, π_K

C smooth, proper, geometrically irreducible curve over K

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Intuition (Bosch – Lütkebohmert, 1985)

C_L has a semi-stable model if and only if C_L^{an} can be "decomposed" into a union of open discs and a finite number of annuli.

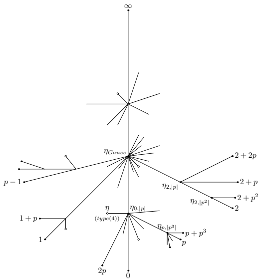
Berkovich analytification

Let k be a field which is complete for a nonarchimedean norm $|\cdot|$ (e.g. $k = \mathbb{Q}_p, \mathbb{C}_p, \mathbb{F}_p((t)), \mathbb{C}((t))$, any trivially valued field, ...).

Definition

Let $X = \mathrm{Spec}(A)$ be an affine k-scheme.
$X^{\mathrm{an}} := \{||\cdot|| : A \longrightarrow \mathbb{R}_{\geq 0}$ *multiplicative semi-norms* $: ||\cdot||_{|k} = |\cdot|\}$.

Ex. $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} := \{||\cdot|| : \mathbb{C}_p[T] \longrightarrow \mathbb{R}_{\geq 0} \dots\} \rightsquigarrow \mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}_p} = \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_p} \cup \{\infty\}$.



Discs and Annuli

A K-analytic space V is said to be:

- an open K-disc if $V \cong \{x \in \mathbb{A}_K^{1,\mathrm{an}} : x(T) < r\}$ for some $r \in |K^\times|$
- an open K-annulus if $V \cong \{x \in \mathbb{A}_K^{1,\mathrm{an}} : r_1 < x(T) < r_2\}$ for some $r_1, r_2 \in |K^\times|$

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Question

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However, if $L|K$ is tamely ramified, V_L disc $\implies$ V fractional disc (Ducros 2013, T. Schmidt 2015).

The main theorem

Theorem (Fantini – T., 2017)

Let $L|K$ be a finite Galois extension of discretely valued fields such that $\mathrm{char}(\widetilde{K}) \nmid [L:K]$, and let X be a K-analytic space.

- *If X_L is a (open, closed, or semi-open) L-annulus, and $\mathrm{Gal}(L|K)$ fixes the branches, then X is a (open, closed, or semi-open) fractional annulus;*
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Elements of proof.

1. Theory of semi-affinoid analytic spaces
2. Representability of Weil restriction and G-fixed locus functors (in the category of semi-affinoid spaces)
3. Linearization theorems for finite order automorphisms

□

Semi-affinoids (after Raynaud, Berthelot, De Jong, ...)

Definition

A topological algebra $\mathcal{A}$ over $\mathcal{O}_K$ is called special if it is a noetherian adic ring, with an ideal of definition J such that $\mathcal{A}/J$ is of finite type over $\widetilde{K}$.

A K-analytic space is called semi-affinoid if it is the "generic fiber" of an affine special formal scheme.

Concretely, $\mathcal{A} \cong \mathcal{O}_K\{X_1, \dots, X_r\}[[Y_1, \dots, Y_s]]/I$.

Example

A K-open disc is the "generic fiber" of $\mathcal{O}_K[[Y]]$

A K-open annulus of modulus e is the "generic fiber" of $\frac{\mathcal{O}_K[[Y_1, Y_2]]}{Y_1 Y_2 - \pi_K^e}$

A non-trivial form of annulus

Let K be such that $\mathrm{char}(\widetilde{K}) \neq 2$

Let $L = K(\sqrt{a})$ for $a \in \mathcal{O}_K^\times$

Let V be the generic fiber of $\mathrm{Spf}\left(\frac{\mathcal{O}_K[[Y_1, Y_2]]}{Y_1^2 - aY_2^2 - \pi_K^e}\right)$

Then, V_L is a L-annulus of modulus $2e$, but V is not a fractional annulus.

Remark

It follows from the main theorem that this is the only possibility outside the case of V fractional annulus.

Further perspectives: Hurwitz trees

Linearization in the wildly ramified case is not an option (e.g. the automorphism of $\mathbb{Z}_3[\zeta_3][[T]]$ defined by $\sigma(T) = \frac{T-3}{T-2}$)

Nevertheless, we have Hurwitz trees, that classify automorphisms of order p of discs and annuli in characteristic $(0, p)$, according to their ramification (studied by Henrio, Brewis–Wewers, T., Temkin, ...)

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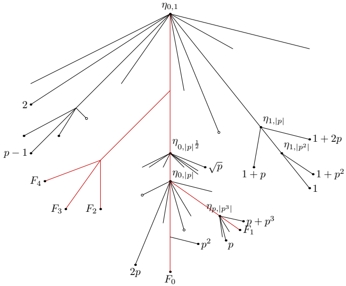

Hope

Use Hurwitz trees to generalize the Main Theorem and give applications to wild monodromy of arithmetic curves.

Thank you!