Moduli spaces of Mumford curves over $\mathbf{Z}$

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July 2, 2020

Outline

1. Schottky uniformization of curves
2. Berkovich spaces over $\mathbf{Z}$
3. Universal Mumford curves over $\mathbf{Z}$
4. Application to modular forms

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Uniformization of complex elliptic curves

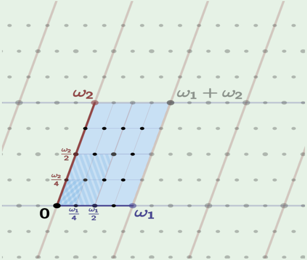
Let

$$E(\mathbf{C}) = \{[x : y : z] \in \mathbf{P}^2_{\mathbf{C}} : zy^2 = x^3 + az^2x + bz^3\}$$

for some $a, b \in \mathbf{C}$ with $4a^3 + 27b^2 \neq 0$.

Uniformization of E

$E(\mathbf{C})$ is a group, isomorphic to $\mathbf{C}/\Lambda$, where $\Lambda = \omega_1\mathbf{Z} \oplus \omega_2\mathbf{Z}$ is a lattice:



Schottky uniformization over $\mathbf{C}$

This isomorphism is of an analytic nature:

$$\mathbf{C}/\Lambda \to E(\mathbf{C})$$
$$w \mapsto \begin{cases} [\wp(w) : \wp'(w) : 1] & \text{if } w \neq 0 \\ [0 : 1 : 0] & \text{if } w = 0 \end{cases}$$

where $\wp$ is the meromorphic Weierstrass $\wp$-function.

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$$E(\mathbf{C}) \simeq \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) \xrightarrow[\sim]{\exp(2\pi i \cdot)} \mathbf{C}^*/q^{\mathbf{Z}}$$

with $\mathrm{Im}(\tau) > 0$ and $q = \exp(2\pi i \tau)$.

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Theorem (Koebe *Rückkehrschnitt* theorem)

Let X^{an} be a compact Riemann surface of genus g. There exist $\Omega \subset \mathbf{C}$ open dense and $\Gamma \subset \mathrm{PGL}_2(\mathbf{C})$ free of rank g, such that $\Omega/\Gamma \cong X^{an}$.

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What if we replace $\mathbf{C}$ with a non-archimedean field $(k, |\cdot|)$?

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Arithmetic analytic geometry

- 1960's John Tate introduces *rigid analytic geometry* and non-archimedean uniformization of elliptic curves
- 1970's Michel Raynaud links rigid spaces and formal geometry
- $\sim$1990 Vladimir Berkovich conceives a new theory using spaces of valuations and spectral theory
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What for?

- Arithmetic geometry: local Langlands program (étale cohomology on Berkovich spaces) and p-adic Hodge theory (Scholze's perfectoid spaces)
- Classical and combinatorial algebraic geometry (via connections to toric and tropical geometries)
- String theory (degeneration of Calabi-Yau, mirror symmetry, SYZ fibration)
- Dynamical systems and potential theory (dynamics on Berkovich spaces)
- p-adic differential equations (radii of convergence on Berkovich curves)
- Diophantine problems (via Arakelov geometry and tropical curves)
- Inverse Galois problem
- ...

The Berkovich analytic space $\mathbf{A}_A^{n,\mathrm{an}}$

Let $(A, \|\cdot\|)$ be a commutative Banach ring with unit. Let $n \in \mathbf{N}$.

The analytic space $\mathbf{A}_A^{n,\mathrm{an}}$ is

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- endowed with the coarsest topology such that that the evaluations

$$\begin{aligned} \mathrm{ev}_f : \mathbf{A}_A^{n,\mathrm{an}} &\longrightarrow \mathbf{R}_+ \\ |\cdot| &\longmapsto |f| \end{aligned}$$

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  are continuous for every $f \in A[T_1 \ldots, T_n]$,
- and with a structure sheaf of rings: $U \to \mathscr{O}(U)$.

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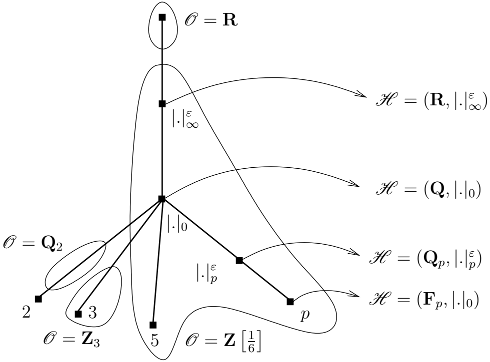
To each $x \in \mathbf{A}_A^{n,\mathrm{an}}$, we associate a complete residue field

$$\mathscr{H}(x) := \text{ completion of the fraction field of } A[T_1, \ldots, T_n]/\mathrm{Ker}(|\cdot|_x)$$

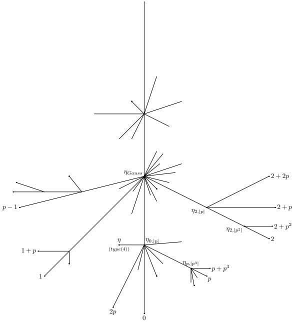
and the resulting evaluation map

$$\chi_x \colon A[T_1, \ldots, T_n] \to \mathscr{H}(x).$$

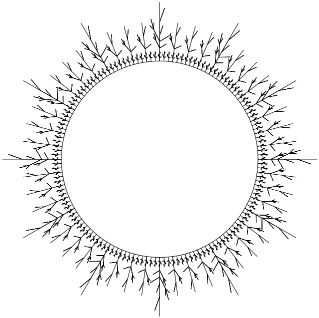
$\mathbf{A}_{\mathbf{Z}}^{0,\mathrm{an}}$



Berkovich curves over $\mathbf{Q}_p$



$\mathbf{P}^{1,an}_{\mathbf{Q}_p}$



$y^2 = x(x-1)(x-p)$

The analytic line $\mathbf{P}^{1,an}_{\mathbf{Z}}$

There is a canonical morphism $\mathrm{pr}\colon \mathbf{P}^{1,an}_{\mathbf{Z}} \to \mathbf{A}^{0,\mathrm{an}}_{\mathbf{Z}}$ and

$$\forall x \in \mathbf{A}^{0,\mathrm{an}}_{\mathbf{Z}},\ \mathrm{pr}^{-1}(x) \simeq \mathbf{P}^{1,an}_{\mathscr{H}(x)}.$$

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Let $\mathbf{D}$ be the open unit disk in $\mathbf{P}^{1,an}_{\mathbf{Z}}$. Then $H^0(\mathbf{D},\mathcal{O})$ is a ring of convergent arithmetic power series (D. Harbater):

$$\begin{aligned} H^0(\mathbf{D},\mathcal{O}) &= \mathbf{Z}[\![T]\!]_{1^-} \\ &= \{f \in \mathbf{Z}[\![T]\!] \text{ with complex radius of convergence } \geqslant 1\}. \end{aligned}$$

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Tate and Mumford's theorems

Let k be a complete non-archimedean field [e.g. $k = \mathbf{Q}_p, \mathbf{C}((t)), \mathbf{F}_p((t))$].

Theorem (Tate)

Let E/k be an elliptic curve with split multiplicative reduction. Then $E^{an} \cong k^\times / q^{\mathbf{Z}}$ for some $q \in k$ with $0 < |q| < 1$.

Theorem (Mumford)

Let X/k a smooth projective curve of genus g whose Jacobian has totally degenerate reduction. Then there exist $\Omega \subset \mathbf{P}_k^{1,\mathrm{an}}$ open dense subset and $\Gamma \subset \mathrm{PGL}_2(k)$ free of rank g such that $\Omega/\Gamma \cong X^{\mathrm{an}}$.

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Aim

Build a "universal" theory of uniformization, that works for every valued field (archimedean and non-archimedean) at once.

Schottky groups

Let $(k, |\cdot|)$ be a complete valued field. Let Γ be a subgroup of $\mathrm{PGL}_2(k)$. It acts on $\mathbf{P}_k^{1,\mathrm{an}}$.

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A Schottky group over k is a subgroup $\Gamma \subset PGL_2(k)$ that satisfies:

- Γ is finitely generated
- Γ is free
- non-trivial elements of Γ are hyperbolic
- the locus of $\mathbf{P}_k^{1,\mathrm{an}}$ where Γ acts discontinuously is non-empty.

Fact

The complement $\mathscr{L}$ of the discontinuity locus, called the limit set, is compact and contains only k-rational points.

Koebe coordinates

To $\gamma \in \mathrm{PGL}_2(k)$ hyperbolic, we associate

- $\alpha \in \mathbf{P}^1(k)$ its attracting fixed point;
- $\alpha' \in \mathbf{P}^1(k)$ its repelling fixed point;
- $\beta \in k$ the quotient of its eigenvalues with absolute value < 1.

For $\alpha, \alpha', \beta \in k$ with $|\beta| \in (0, 1)$, we have

$$\gamma = M(\alpha, \alpha', \beta) = \begin{bmatrix} \alpha - \beta\alpha' & (\beta - 1)\alpha\alpha' \\ 1 - \beta & \beta\alpha - \alpha' \end{bmatrix}.$$

Schottky space

Let $g \geqslant 2$. The Schottky space $\mathscr{S}_g$ is the subset of $\mathbf{A}_{\mathbf{Z}}^{3g-3,\mathrm{an}}$ consisting of the points

$$z = (x_3, \ldots, x_g, x'_2, \ldots, x'_g, y_1, \ldots, y_g)$$

such that the subgroup of $\mathrm{PGL}_2(\mathscr{H}(z))$ defined by

$$\Gamma_z := \langle M(0, \infty, y_1), M(1, x'_2, y_2), M(x_3, x'_3, y_3), \ldots, M(x_g, x'_g, y_g) \rangle$$

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Proposition (Poineau - T.)

For every $(k, |\cdot|)$ and every Schottky group $\Gamma \subset \mathrm{PGL}_2(k)$ of rank g, there is a point $z \in \mathscr{S}_g \times_{\mathbf{Z}} k$ such that $\Gamma_z = h^{-1}\Gamma h$, $h \in \mathrm{PGL}_2(k)$.

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Theorem (Poineau - T.)

The Schottky space $\mathscr{S}_g$ is a connected open subset of $\mathbf{A}_{\mathbf{Z}}^{3g-3,\mathrm{an}}$.

Universal Mumford curve

Denote by $(X_3, \dots, X_g, X'_2, \dots, X'_g, Y_1, \dots, Y_g)$ the coordinates on $\mathbf{A}_{\mathbf{Z}}^{3g-3,\mathrm{an}}$ and consider the subgroup of $\mathrm{PGL}_2(\mathscr{O}(\mathscr{S}_g))$:

$$\Gamma = \langle M(0,\infty,Y_1), M(1,X'_2,Y_2), M(X_3,X'_3,Y_3), \dots, M(X_g,X'_g,Y_g)\rangle.$$

Theorem (Poineau - T.)

There exists a closed subset $\mathscr{L}$ of $\mathbf{P}^{1,\mathrm{an}}_{\mathscr{S}_g} := \mathscr{S}_g \times_{\mathcal{M}(\mathbf{Z})} \mathbf{P}^{1,\mathrm{an}}_{\mathbf{Z}}$ and a relative curve $\mathscr{X}_g \to \mathscr{S}_g$ that is universally uniformized by Γ.

Theorem (Poineau - T.)

The group $Out(F_g)$ acts analytically and properly discontinuously on $\mathscr{S}_g$ with finite stabilizers. The quotient $Mumf_g := Out(F_g)\backslash\mathscr{S}_g$ is a (singular) analytic space over $\mathbf{Z}$ whose non-archimedean locus parametrizes Mumford curves.

What's next?

- Singularities and homotopy type of $Mumf_g$, relationships with tropical moduli (Chan-Galatius-Payne) and outer space (Culler-Vogtmann)
- Hausdorff dimension and capacity of limit sets
- Steinness of $\mathscr{S}_g$
- Periods $(q_{i,j})_{1 \leqslant i,j \leqslant g}$ of Mumford curves (Manin-Drinfeld) over $\mathbf{Z}$
- q-expansions of modular forms
  Schottky problem (= characterizing Jacobians among Abelian varieties)
- Gauß-Manin connections
  Picard-Fuchs equations (Gerritzen):

$$\text{for } 1 \leqslant i \leqslant g, \begin{cases} \nabla\left(\dfrac{du_i}{u_i}\right) = \displaystyle\sum_{j=1}^{g} \beta_j \otimes \dfrac{dq_{i,j}}{q_{i,j}}; \\ \nabla(\beta_i) = 0. \end{cases}$$

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The space $\mathscr{S}_1$

Let $g = 1$.

Schottky group over $k \to \Gamma \sim \left\langle \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$, with $q \in k, 0 < |q| < 1$

Schottky space $\to \mathscr{S}_1 = \mathbf{D}^\circ := \{x \in \mathbf{A}_{\mathbf{Z}}^{1,\mathrm{an}} : 0 < |T(x)| < 1\}$

Universal Tate curve $\to \mathscr{X}_1 = (\mathbf{A}_{\mathscr{S}_1}^{1,\mathrm{an}} - \{0, \infty\})/T^{\mathbf{Z}}$

The sheaf $\Omega^1_{\mathscr{X}_1/\mathscr{S}_1}$ is globally generated by $\frac{dS}{S}$ where S is a parameter for $\mathbf{A}_{\mathscr{S}_1}^{1,\mathrm{an}}$.

q-expansion of modular forms

Let $\omega := \pi_* \Omega^1_{\mathscr{X}_1/\mathscr{S}_1}$ and $f \in H^0(\mathscr{S}_1, \omega^{\otimes k})$.

Then $f = \phi \cdot (\frac{dS}{S})^k$, with $\phi \in H^0(\mathscr{S}_1, \mathscr{O}) = \mathbf{Z}[\![T]\!]_{1^-}[\frac{1}{T}]$.

One can use this to find Fourier expansions of classical modular forms, thanks to the diagram:

$$\begin{array}{ccc} \mathscr{X}_1 & \longrightarrow & \mathscr{E}^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathscr{S}_1 & \longrightarrow & X(N) \end{array}$$

where $\mathscr{E}$ is the universal generalized elliptic curve over the modular curve $X(N)$ (Deligne-Rapoport, Katz-Mazur).

Teichmüller modular forms ($g > 1$)

M_g moduli space of smooth and proper curves of genus g

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The Torelli map τ gives rise to

$$\tau^*\colon S_{g,k}(R) \to T_{g,k}(R),$$

where $S_{g,k}(R)$ denotes the ring of Siegel modular forms over R.

Expansions

T. Ichikawa (1994) defined an expansion map

$$\kappa_R \colon T_{g,k}(R) \to R\left[x_{\pm 1}, \dots, x_{\pm g}, \frac{1}{x_i - x_j}\right][\![y_1, \dots, y_g]\!].$$

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- related to the Fourier expansions of Siegel modular forms (using Yu. Manin - V. Drinfeld "Periods of p-adic Schottky groups", 1972)
- may be helpful for the Schottky problem (characterizing Jacobian varieties among Abelian varieties)

The End (for now)

Thank you for your attention!

Genus 3

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Theorem (S. Tsuyumine, 1991 + T. Ichikawa, 2000)

There exists $\mu_9 \in T_{3,9}(\mathbf{Z})$ *such that*

$$\tau^*(\chi_{18}) = \mu_9^2.$$

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Let $k \subset \mathbf{C}$. Let A/k be a principally polarized indecomposable Abelian threefold that is isomorphic to a Jacobian over $\mathbf{C}$.
Then, A is isomorphic to a Jacobian over k if, and only if,

$$\chi_{18}(A) \in k^2.$$

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$$\text{for } 1 \leqslant i \leqslant g, \begin{cases} \nabla\left(\dfrac{du_i}{u_i}\right) = \displaystyle\sum_{j=1}^{g} \beta_j \otimes \dfrac{dq_{i,j}}{q_{i,j}}; \\ \nabla(\beta_i) = 0. \end{cases}$$

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Definition (M. Culler - K. Vogtmann, 1986)

The Outer Space CV_g is a space of metric graphs X of genus g endowed with a marking (isomorphism $F_g \xrightarrow{\sim} \pi_1(X)$).

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We have a continuous surjective map

$$\mathcal{S}_{g,k} \to CV_g \times_{M_g^{\mathrm{trop}}} \mathrm{Mumf}_{g,k}.$$

See also M. Ulirsch "Non-Archimedean Schottky Space and its Tropicalization", 2020