Lifting Galois covers with non-Archimedean analytic geometry

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Setting

$k = \bar{k}$, $\mathrm{char}(k) = p > 0$,

R complete discrete valuation ring, $\mathrm{char}(R) = 0$, $\mathrm{Frac}(R) = K$, $\widetilde{K} = k$

G finite group

$\mathbb{A}_K^{1,an} :=$ {multiplicative seminorms $\|\cdot\| : K[T] \to \mathbb{R}_{\geq 0} : \|\cdot\|_{|K} = |\cdot|_K$}

$\mathbb{D}(0,1) := \{x \in \mathbb{A}_K^{1,an} : 0 \leq x(T) \leq 1\}$.

Local actions

Lifting Problem

Fix G and $\lambda : G \hookrightarrow Aut_k(k[[t]])$.

- Does there exist $\Lambda : G \hookrightarrow Aut_R(R[[T]])$ whose reduction is λ ?
- Does there exist such Λ for every λ ?

- Equivalent to lift Galois covers of smooth curves $X \to X/G$
- True for $(|G|, p) = 1$ (Grothendieck, 1960-61); G cyclic (Pop, Obus-Wewers, 2013)
- Counterexamples exist (Raynaud, Green-Matignon, Chinburg-Guralnick-Harbater).

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Main tool for local lifting: Hurwitz tree

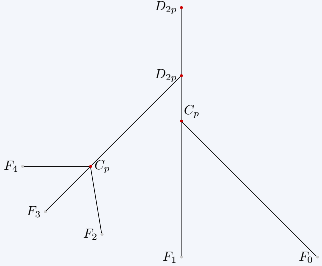
Let $\Lambda : G \hookrightarrow Aut_R(R[[T]])$ be fixed. The Hurwitz tree $\mathcal{H}_\Lambda$ is a finite metric rooted tree that encodes:

A- The relative distance of ramification points in $D(0,1)^- = \{x \in K : |x|_K < 1\}$

B- The ramification theory of the extension $R[[T]]/R[[T]]^G$

C- The deformations of some torsors under R-group schemes and their reduction.

Hurwitz Tree (dihedral case)



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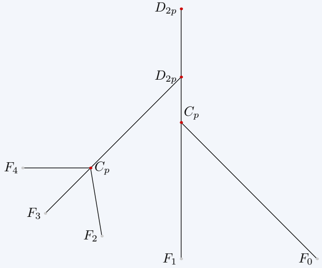

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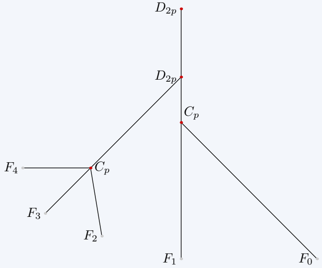

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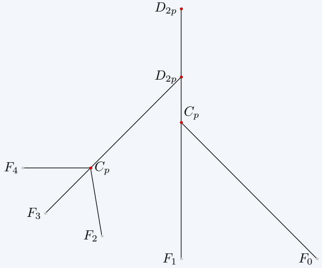

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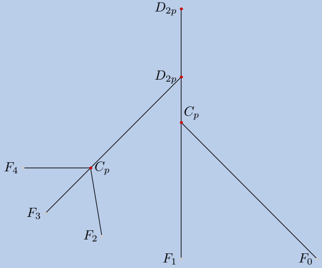

What is it good for:

Strategy to show/disprove lifting to characteristic zero:

1. Start with $\lambda : G \hookrightarrow Aut_k(k[[t]])$;
2. Use Hurwitz tree to classify possible liftings;
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The Berkovich-Hurwitz tree

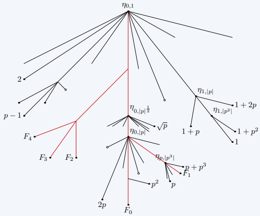
Let $\Lambda : G \hookrightarrow Aut_R(R[[T]])$. We denote by $L_\Lambda \subset \mathbb{D}(0,1)$ the finite set of ramification points.

Theorem (T.)

Let $\mathcal{H}_\Lambda$ be the Hurwitz tree of Λ. There exists a canonical embedding

$$\mathcal{H}_\Lambda \longrightarrow \mathbb{D}(0,1)$$

identifying $\mathcal{H}_\Lambda$ with the skeleton of $\mathbb{D}(0,1)^- \setminus L_\Lambda$.



The Hurwitz data of type B are determined by analytic properties of $\sigma(T) - T \in \mathcal{O}(\mathbb{D}(0,1))$ on $\mathcal{H}_\Lambda$ for every $\sigma \in \Lambda(G)$.

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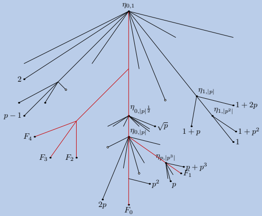

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Temkin reduction and differential forms

Let (X, x) be a K-germ of an analytic space X.

- Temkin: $(X, x) \in \mathit{Germs} \to \widetilde{(X, x)}$, Riemann-Zariski space.

- When $X = \mathbb{D}(0, 1)$ and x is a vertex of Hurwitz tree, $\widetilde{(X, x)}$ is homeomorphic to $\mathbb{P}^1_k$.

Theorem (T.)

There exists a locally free sheaf Ω_Λ over $\mathbb{D}(0, 1)^- \setminus L_\Lambda$ such that Hurwitz data of type C are sections of its Temkin reduction.

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We have a non-Archimedean analytic formalism for Hurwitz trees!

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Perspectives

- Study Ω_Λ for actions of cyclic and dihedral groups. Apply to lifting problem.
- Describe deformations of torsors using Berkovich topology and combinatorics.
- Explore the links between local actions of G and non-Archimedean dynamical systems.

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The end

Thank you!