Resolutions of Monomial Ideals of Projective Dimension 1

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Abstract

We show that a monomial ideal I has projective dimension 1 if and only if the minimal free resolution of I is supported on a graph that is a tree. We do this by constructing specific graphs which support the resolution of the ideal. As a result, we also give a new proof to a result by Herzog, Hibi, and Zheng which characterizes monomial ideals of projective dimension 1 in terms of quasi-trees.

1 Introduction

A free resolution of an ideal is a long exact sequence of free modules that represents the relations between the generators of that ideal. For an ideal generated by monomials it is always possible to find a simplicial complex whose simplicial chain complex completely describes a free resolution of the ideal. It is, however, known that the *minimal* free resolution may not be described by simplicial complexes; see Velasco [14] and Reiner and Welker [12]. In other words, the minimal resolution of a monomial ideal need not be supported on a simplicial complex. A natural question is which monomial ideals do have such minimal simplicial resolutions.

In this paper, we study the smallest case: the case of monomial ideals of projective dimension 1. We show that if a monomial ideal I has $\mathrm{pd}(I) = 1$ then the minimal free resolution of I is supported on a 1-dimensional acyclic simplicial complex, i.e. a (graph) tree. We also provide an alternate proof to a result by Herzog, Hibi, and Zheng ([8], 2.2) which characterizes monomial ideals of projective dimension 1 in terms of quasi-trees. The benefit of our approach is that we construct the specific graph which supports the resolution of the ideal.

Our main results can be summed up in the following statement, in which $\mathcal{N}(I^\vee)$ stands for the Alexander dual of the Stanley-Reisner complex of I.

Theorem 1 (Theorem 18)**.** *Let I be a square-free monomial ideal in a polynomial ring S. Then the following statements are equivalent.*

1. $\mathrm{pd}_S(I) = 1$
2. *I has a minimal free resolution supported on a graph-tree.*
3. *$\mathcal{N}(I^\vee)$ is a quasi-forest*

The equivalence of statements 1 and 2 was already known from the work of Herzog, Hibi, and Zheng [8], but it also follows directly from our construction of a graph tree supporting a resolution of (the polarization of) any ideal of projective dimension 1.

2 Quasi-trees

Let $V = \{v_1, ..., v_n\}$ be a finite set. A (finite) **simplicial complex**, Δ, on V is a collection of non-empty subsets of V such that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$. The elements of Δ are called **faces**. Faces containing one element are called **vertices** and maximal faces are called **facets**. For each face $F \in \Delta$, we define $\dim(F) = |F| - 1$ to be the **dimension of the face** F. We define $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$ to be the **dimension of the simplicial complex** Δ. If Δ is a simplicial complex with only 1 facet and r vertices, we call Δ an **r-simplex**.

If $W \subseteq V$, we define the **induced subcomplex on** W in Δ, denoted Δ_W, to be the set $\Delta_W = \{F \in \Delta | F \subseteq W\}$.

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A **subcollection** of Δ is a simplicial complex whose facets are also facets of Δ.

We say Δ is **connected** if for every $v_i, v_j \in V$ there is a sequence of faces $F_0, ..., F_k$ such that $v_i \in F_0$, $v_j \in F_k$ and $F_i \cap F_{i+1} \neq \emptyset$ for $i = 0, ..., k-1$.

It is easy to see from the definition that a simplicial complex can be described completely by its facets, since every face is a subset of a facet and every subset of every facet is in a simplicial complex. So, if Δ has facets $F_0, ..., F_q$, we use the notation $\langle F_0, ..., F_q \rangle$ to describe Δ.

The f**-vector** of a d-dimensional simplicial complex Δ is the sequence $f(\Delta) = (f_0, ..., f_d)$, where each f_i is the number of i-dimensional faces of Δ.

Definition 2 (Faridi [4])**.** A facet F of a simplicial complex Δ is called a **leaf** if either F is the only facet of Δ or for some facet $G \in \Delta$ with $G \neq F$ we have that $F \cap H \subseteq G$ for all facets $H \neq F$ of Δ. The facet G is said to be the **joint** of F.

A simplicial complex Δ is a **simplicial forest** if every nonempty subcollection of Δ has a leaf. A connected simplicial forest is called a **simplicial tree**.

If a facet F of a simplicial complex is a leaf, then F necessarily has a **free vertex**, which is a vertex of Δ that belongs to exactly one facet.

One of the properties of simplicial trees that we will make particular use of is that whenever Δ is a simplicial tree we can always order the facets $F_1, ..., F_q$ of Δ so that F_i is a leaf of the induced subcollection $\langle F_1, ...F_i \rangle$. Such an ordering on the facets is called a **leaf order** and it is used to make the following definition.

Definition 3. (Zheng [15]) A simplicial complex Δ is a **quasi-forest** if Δ has a leaf order. A connected quasi-forest is called a **quasi-tree**.

Equivalently, we could have defined quasi-trees to be simplicial complexes such that every induced subcomplex has a leaf. This is not clear from the definition and we give a proof below.

Proposition 4 (A characterization of quasi-forests)**.** *A simplicial complex Δ with vertex set V is a quasi-forest if and only if for every subset $W \subset V$, the induced subcomplex Δ_W has a leaf.*

Proof. ($\Rightarrow$) Since Δ has a leaf order, we may label the facets of Δ, $F_0, ..., F_q$, so that F_i is a leaf of $\Delta_i = \langle F_0, ..., F_i \rangle$. For a subset $W \subset V$, choose the smallest i such that W is a subset of the vertex set of Δ_i, which we will denote V_i.

We claim that the complex induced on W in Δ_i is Δ_W. It is clear that $(\Delta_i)_W \subseteq \Delta_W$. To see the converse, let F be a face of Δ_W, then $F \subseteq F_j$ for some facet $F_j \in \Delta$. If $j \leq i$ then $F \in \Delta_i$ and we are done. If $j > i$ then let F_k be the joint of F_j in Δ_j and note that $k < j$. Since $F \subseteq W \subseteq \Delta_i \subseteq \Delta_j \setminus \langle F_j \rangle$ we have that $F \subseteq F_j \cap \big(\Delta_j \setminus \langle F_j \rangle\big) \subset F_k$. If $k \leq i$ then we are done. If not we may iterate this argument as many times as necessary until we get a facet $F_a \in \Delta_i$ for which $F \subseteq F_a$. Hence $(\Delta_i)_W = \Delta_W$.

We will show that $F_i \cap W$ is a leaf of Δ_W. Since $F_i \in \Delta_i$, $F_i \cap W$ is a face of Δ_W. Also, $V_i = V_{i-1} \cup \{\text{free vertices of } F_i \text{ in } \Delta_i\}$ which means that $W \cap \{\text{free vertices of } F_i \text{ in } \Delta_i\} \neq \emptyset$, otherwise W would be contained in the vertex set of Δ_{i-1}. Therefore $F_i \cap W$ is not a subset of any other face in Δ_W, i.e. $F_i \cap W$ is a facet of Δ_W. If F_j is the joint of F_i in Δ_i, then for any face $F \in \Delta$, $F \cap F_i \cap W \subset F_j \cap F_i \cap W$. This means that any facet of Δ_W (except for $F_i \cap W$) that contains $F_j \cap F_i \cap W$ is a joint for $F_i \cap W$ in Δ_W, since the faces of Δ_W are also faces of Δ. If no such facet exist (except for $F_i \cap W$) then $F_i \cap W$ is disjoint from the rest of Δ_W. In either scenario, $F_i \cap W$ is a leaf of Δ_W.

($\Leftarrow$) This is done by induction on the size of the vertex set V of Δ. For $|V|$ = 1 or 2, a quick inspection shows that all simplicial complexes with vertex set V have a leaf order and every induced subcomplex has a leaf. Now assume that every simplicial complex on $\leq n$ vertices for which every induced subcomplex has a leaf is a quasi-forest.

Suppose Δ is a simplicial complex on $n+1$ vertices and that every induced subcomplex of Δ has a leaf. Since Δ is an induced subcomplex of itself, it also has a leaf, call it F, with free vertices $v_1, ..., v_k$. The simplicial complex $\Delta \setminus \langle F \rangle$ is given by the induced subcomplex Δ_W where $W = V \setminus \{v_1, ..., v_k\}$. Every induced subcomplex of Δ_W has a leaf and Δ_W is a simplicial complex on $\leq n$ vertices, hence Δ_W has a leaf order $G_1, ..., G_j$. This gives us a leaf order $G_1, ..., G_j, F$ for Δ. $\square$

It is known that every induced subcomplex of a simplicial forest is also a simplicial forest ([6]), but this property does not characterize simplicial forests.

Let Δ be a simplicial complex on the vertex set $\{x_1, ..., x_r\}$. The **Stanley-Reisner ideal** $\mathcal{N}(\Delta)$ of Δ is is a squarefree monomial ideal generated by the minimal "non-faces" of Δ:

$$\mathcal{N}(\Delta) = (x_{i_1} \cdots x_{i_p} | \{x_{i_1}, ..., x_{i_p}\} \notin \Delta).$$

The **Alexander dual** of Δ is the simplicial complex

$$\Delta^\vee = \{\{x_1, ..., x_r\} \setminus \tau \mid \tau \notin \Delta\}.$$

The following lemma shall be used later in the paper. The proof is a straightforward application of the definitions above (see e.g. Faridi [5])

Lemma 5. *Let $\Delta = \langle F_1, ..., F_q \rangle$ be a simplicial complex on the vertex set $V = \{x_1, ..., x_n\}$. The minimal generating set of $\mathcal{N}(\Delta^\vee)$ is*

$$\{ \prod_{x_i \notin F_1} x_i, ..., \prod_{x_i \notin F_q} x_i \}$$

3 Simplicial resolutions

Let $S = k[x_1, \dots, x_n]$, where k is a field. A **minimal graded free resolution** of a graded S-module M is a chain complex of the form

$$\mathbf{F}: \qquad \cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

such that each F_i is a free S-module, $H_0(\mathbf{F}) \cong M$ a degree zero isomorphism , $H_i(\mathbf{F}) = 0$ for $i \geq 1$, and $\partial_{i+1}(F_{i+1}) \subseteq \mathbf{m}F_i$ for all $i \geq 0$ where $\mathbf{m} = (x_1, ..., x_n)$.

The i^{th} **Betti number** of M over S is defined as $\beta_i^S(M) = \text{rank}(F_i)$. Since $\mathbf{F}$ is graded, each free module F_i is a direct sum of modules of the form $S(-p)$. We define the **graded Betti numbers** of M by

$$\beta_{i,p}^S(M) = \text{number of summands in } F_i \text{ of the form } S(-p)$$

for an integer p. Similarly, If $\mathbf{F}$ is multigraded, we define the **multigraded Betti numbers** of M to be

$$\beta_{i,m}^S(M) = \text{number of summands in } F_i \text{ of the form } S(-m)$$

for a monomial m. The definition tells us that for a fixed i, $\beta_i^S(M) = \sum_p \beta_{i,p}^S(M) = \sum_m \beta_{i,m}^S(M)$. Furthermore, if our minimal resolution admits a multigrading, we will have that for each i,

$$F_i = \bigoplus_{m \in S} S(-m) = \bigoplus_{p \in \mathbb{Z}} \bigoplus_{m \in S_p} S(-m) = \bigoplus_{p \in \mathbb{Z}} S(-p)$$

and we conclude that $\beta_{i,p}^S(M) = \sum_{m \in S_p} \beta_{i,m}^S(M)$.

It is known that every two minimal free resolutions of M are isomorphic and of the same (finite) length, which is called the **projective dimension** of M:

$$\text{pd}_S(M) = \max\{i \mid \beta_i^S(M) \neq 0\}$$

Let I be a monomial ideal in S minimally generated by monomials $m_1, \dots, m_t$. If Δ is a simplicial complex on t vertices, one can label each vertex of Δ with one of the generators $m_1, \dots, m_t$ and each face with the least common multiple of the labels of its vertices. For any monomial m, we denote by Δ_m be the subcomplex of Δ induced on the vertices of Δ whose labels divide m.

Theorem 6 (Resolutions via simplicial trees ([6]))**.** *Let Δ be a simplicial tree labeled by monomials $m_1, \dots, m_t \in S$, and let $I = (m_1, \dots, m_t)$ be the ideal in S generated by the vertex labels. The simplicial chain complex $\mathcal{C}(\Delta) = \mathcal{C}(\Delta; S)$ is a free resolution of S/I if and only if the induced subcomplex Δ_m is connected for every monomial m.*

An example of a simplicial tree Δ "supporting" a free resolution of a monomial ideal (that is, the simplicial chain complex of Δ being a free resolution of the ideal) is the Taylor resolution [13], in which case Δ is a simplex (one facet).

Theorem 6 implies that the Betti vector of I (that is, the vector whose i-th entry is the i-th Betti number of I) is bounded by the f-vector of a simplicial tree Δ that supports a resolution of it:

$$\beta(I) = (\beta_0(I), \ldots, \beta_q(I)) \leq (f_0(\Delta), \ldots, f_q(\Delta)) = \mathbf{f}(\Delta).$$

Equality holds if some extra conditions are satisfied:

Theorem 7 (Bayer, Peeva, Sturmfels ([1]))**.** *With notation as in Theorem 6, $\mathcal{C}(\Delta)$ is a minimal free resolution of S/I if and only if $m_A \neq m_{A'}$ for every proper subface A' of a face A of Δ.*

4 Monomial ideals of projective dimension 1

It is known that not all monomial ideals have simplicial, or even cellular resolutions ([12, 14]). It is also known that if a simplicial complex supports a minimal resolution of a monomial ideal, then it must be acyclic ([11, 10]), and that simplicial trees are acyclic ([6]).

Therefore a natural question in view of Theorem 6 is: Which ideals have minimal resolutions supported on a simplicial tree?

The most basic case is that of a graph-tree, which is a 1-dimensional simplicial tree. In this case, the corresponding ideal will have to have projective dimension 1. Moreover, graph-trees are the only acyclic 1-dimensional simplicial complexes. It turns out that all monomial ideals of projective dimension 1 have simplicial resolutions supported on graph-trees.

Theorem 8. *A monomial ideal I has $\mathrm{pd}(I) = 1$ if and only if I has a minimal resolution supported on a (graph) tree*

Proof. ($\Leftarrow$) Clear.
($\Rightarrow$) If $\mathrm{pd}(I) = 1$ then S/I has a minimal resolution of the form

$$0 \longrightarrow S^t \xrightarrow{\psi} S^r \xrightarrow{\phi} S \longrightarrow 0$$

where $\phi(e_i) = m_i$ for the basis elements e_i of S^r, and $\psi(g_j) = f_j$ where the g_j form a basis of S^t and the f_j form a minimal generating set of $\ker(\phi)$. It is shown (see [3], Corollary 4.13) that $\ker(\phi)$ can be generated (though not necessarily minimally) by the elements

$$\frac{\mathrm{lcm}(m_i, m_j)}{m_i} e_i - \frac{\mathrm{lcm}(m_i, m_j)}{m_j} e_j$$

Let $f_1, \ldots, f_t$ be a minimal generating set of $\ker(\phi)$ which have this form. This gives us a complete description of the map ψ as a matrix with exactly two non-zero monomial entries in each column with coefficients corresponding to those appearing in the f_i (i.e one column entry has coefficient 1 and the other has coefficient -1). Dehomogenizing this resolution (i.e. tensoring the complex by $\dfrac{S}{(x_1 - 1, \ldots, x_n - 1)}$) gives us the sequence of vector spaces

$$0 \longrightarrow k^t \xrightarrow{A} k^r \xrightarrow{(11 \ldots 1)} k \longrightarrow 0 \tag{1}$$

which is exact (Theorem 3.8 of [10]) and where A is a matrix in which every column has exactly one entry which is 1, one entry which is -1, and the rest equal to zero. If we consider each basis element of k^r as a vertex and each basis element e_i of k^t as an edge between the two vertices determined by the basis elements of k^r to which e_i is sent, we may construct a graph G for which $\mathcal{C}(G; k)$ is the chain complex in (1). Since this chain complex is exact the graph G is acyclic, hence a tree (this would also imply that $t = r - 1$). $\square$

In fact, more is true.

Proposition 9. *If I is a monomial ideal which has a resolution supported on a tree T then that resolution is minimal.*

Proof. If $m_1, ..., m_r$ are the minimal generators of I then T would have to have r vertices and $r-1$ edges. When we regard T as a simplicial complex we get the simplicial chain complex

$$\mathcal{C}(T;k) : 0 \longrightarrow k^{r-1} \xrightarrow{\partial_2} k^r \xrightarrow{(11...1)} k \longrightarrow 0$$

where ∂_2 is a matrix in which every column has one entry equal to 1, one entry equal to -1, and the rest equal to zero. Fix a basis $u_{i,j}$ for $\mathcal{C}(T;k)$. The I-homogenization of T ([10]) would then give a resolution of I of the form

$$\mathbf{G}:\ 0 \longrightarrow \bigoplus_{j=1}^{r-1} S(-\alpha_{2,j}) \xrightarrow{d_2} \bigoplus_{j=1}^{r} S(-\alpha_{1,j}) \xrightarrow{d_1} S \longrightarrow 0$$

with multihomogeneous basis $e_{i,j}$ such that $\mathrm{mdeg}(e_{i,j}) = \alpha_{i,j}$. We know that

$$\alpha_{1,j} = \mathrm{mdeg}(e_{1,j}) = \mathrm{mdeg}(m_j)$$

for $j = 1, ..., r$ and the $\alpha_{2,j}$ are given by

$$\alpha_{2,j} = \mathrm{mdeg}\big(\mathrm{lcm}(\mathrm{mdeg}(e_{1,s})|\ a_{s,j} \neq 0)\big)$$

where the $a_{s,j}$ come from the boundary map

$$\partial_2(u_{2,j}) = \sum_{s=1}^{q} a_{s,j} u_{1,s}$$

For each j, exactly 2 of the $a_{s,j} \neq 0$, so the multidegrees of the $e_{2,j}$ are actually of the form $\mathrm{mdeg}(e_{2,j}) = \mathrm{mdeg}(\mathrm{lcm}(m_{i_1}, m_{i_2}))$ where m_{i_1} and m_{i_2} are minimal generators of I. With this in mind we consider the boundary map

$$d_2(e_{2,j}) = \sum_{s=1}^{q} a_{s,j} \frac{\mathrm{mdeg}(e_{2,j})}{\mathrm{mdeg}(e_{1,s})} e_{1,s}$$

which tells us that the matrix representation of d_2 has entries

$$[d_2]_{s,j} = a_{s,j} \frac{\mathrm{mdeg}(e_{2,j})}{\mathrm{mdeg}(e_{1,s})}$$

If $a_{s,j} = 0$ then $[d_2]_{s,j} = 0$. If $a_{s_1,j}, a_{s_2,j} \neq 0$ then we have that $\mathrm{mdeg}(e_{2,j}) = \mathrm{lcm}(m_{s_1}, m_{s_2})$. Since m_{s_1}, m_{s_2} are minimal generators of I we know that m_{s_1} and m_{s_2} strictly divide $\mathrm{mdeg}(e_{2,j}) = \mathrm{lcm}(m_{s_1}, m_{s_2})$, so that $[d_2]_{s,j} \in \mathbf{m}$ for all s, j. By construction, all entries of d_1 are in $\mathbf{m}$ and we can conclude that this resolution is minimal. □

Next we show that all monomial ideals of projective dimension 1 (or their square-free polarizations) can be characterized as $\mathcal{N}(\Delta^\vee)$ where Δ is a quasi-forest. This fact itself is known: Herzog, Hibi, and Zheng [8] proved it by using the Hilbert-Burch Theorem [2], and interpreting aspects of this theorem in the context of the Stanley-Reisner ring of the Alexander Dual of a quasi-tree.

Our proof, on the other hand, gives a specific and simple construction of graph trees that support a resolution of $\mathcal{N}(\Delta^\vee)$. The minimality of the resolution is guaranteed by the previous lemma.

Theorem 10. *If Δ is a quasi-forest, then $\mathcal{N}(\Delta^\vee)$ has a minimal resolution which is supported on a tree.*

Proof. First we shall construct a tree T whose vertices will be labelled by the monomial generators of $\mathcal{N}(\Delta^\vee)$. Then we will show that the forest induced by the lcm of any two of the vertex labels is connected. If these induced forests are connected then so is any forest induced by an element of the lcm-lattice of I and the rest follows from Theorem 3.2 of [6].

To construct the tree we do the following:

1) Order the facets of Δ as $F_0, ..., F_q$, so that F_i is a leaf of $\Delta_i = \langle F_1, ..., F_i \rangle$.

2) Start with the one vertex tree $T_0 = (V_0, E_0)$ where $V_0 = \{v_0\}$ and $E_0 = \emptyset$

3) For $i = 1, ..., q$ do the following:

   - Pick $u < i$ such that F_u is a joint of the leaf F_i in Δ_i
   - Set $V_i = V_{i-1} \cup \{v_i\}$
   - Set $E_i = E_{i-1} \cup \{(v_i, v_u)\}$

What we get is a graph $T = (V_q, E_q)$ which, by construction, is a tree. To complete our construction we determine a labelling of the vertices of T by which to homogenize. To do this we label the vertex v_i with the monomial

$$m_i = \prod_{x_j \in W \setminus F_i} x_j$$

where $W = \{x_1, ..., x_n\}$ is the vertex set of Δ. By Lemma 5, these labels are the monomial generators of $\mathcal{N}(\Delta^\vee)$, so we have constructed a tree and specified a labelling. The I-homogenization of T with respect to this labelling results in the I-complex $\mathbf{F}_T$. We are left with proving that $\mathbf{F}_T$ is a resolution.

Since T is a tree, and hence a simplicial tree, to show that $\mathbf{F}_T$ supports a resolution of $\mathcal{N}(\Delta^\vee)$ it is sufficient to show that T is connected on the subgraphs $T_{i,j}$ which are the induced subgraphs on the vertices m_k such that $m_k \big| \mathrm{lcm}(m_i, m_j)$, for any minimal generators $m_i,\ m_j$ in I. We first observe that

$$\mathrm{lcm}(m_i, m_j) = \prod_{x_l \in W \setminus F_i \cap F_j} x_l$$

so that

$$m_k \big| \mathrm{lcm}(m_i, m_j) \iff F_i \cap F_j \subset F_k$$

Now, to show that every $T_{i,j}$ is connected we first make the set

$$A_{i,j} = \{0 \le k \le n : m_k | \mathrm{lcm}(m_i, m_j)\} = \{0 \le k \le n : F_i \cap F_j \subset F_k\}$$

and let l be the smallest integer in $A_{i,j}$. We will show that for each $k \in A_{i,j}$, there is a path in $T_{i,j}$ connecting v_k and v_l.

If $k \in A_{i,j},\ k \neq l$ then we can consider the facet F_k in Δ_k which is a leaf, so it has a joint F_{k_J} for some $k_J < k$. Since $l < k$, F_l is a facet of Δ_k as well. This means that

$$F_i \cap F_j \subset F_k \cap F_l \subset F_{k_J} \implies F_i \cap F_j \subset F_{k_J} \implies k_J \in A_{i,j}.$$

Since $k_J \in A_{i,j}$ for any joint of $F_k \in \Delta_k$, it is true for the specific joint we used in Step (3) of our construction of T. We may also conclude that $k_J \ge l$, by the minimality of l. Hence it is the case that the edge $\{v_k, v_{k_J}\} \in T$ which in turn implies that $\{v_k, v_{k_J}\} \in T_{i,j}$. Since $l \le k_J < k$, we can iterate this argument for k_J and its joint in Δ_{k_J}, and so on, finitely many times to get a path from v_k to v_l in $T_{i,j}$. $\square$

Remark 11. In the construction of T, we had some choice as to what joint we chose for a facet F_k in the simplicial complex Δ_k, hence the tree that we constructed is not unique. Furthermore, the proof follows through regardless of our choices, so that any tree that we may have constructed would give us a resolution of $\mathcal{N}(\Delta^\vee)$.

Example 12. Let Δ be the simplicial tree

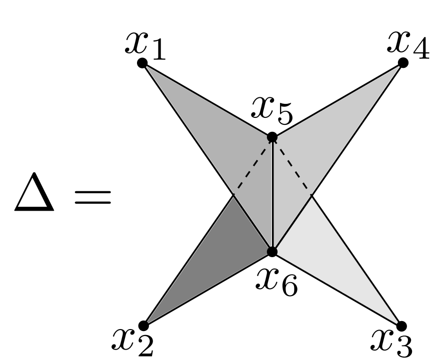

Figure 1: Quasi-tree with many leaf orders

Every order on the facets of Δ is a leaf order, every facet is a leaf, and every facet is the joint of every other facet. This means that if we use the construction given in the proof of Theorem 10, we could produce any tree on four vertices. The monomial generators of $\mathcal{N}(\Delta^\vee)$ are $x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_4,\ x_2x_3x_4$ and the lcm of any two of these generators is $x_1x_2x_3x_4$, so that each $T_{i,j} = T$ for any tree T we choose to consider. Hence, the $T_{i,j}$ are always connected and we get a minimal free resolution of $\mathcal{N}(\Delta^\vee)$.

Remark 13. Fløystad [7] also constructs specific trees supporting minimal resolutions for the class of Cohen-Macaulay monomial ideals of projective dimension 1. Let $I = (m_1, \ldots, m_q)$ be such an ideal and without loss of generality we assume that I is square-free (otherwise replace generators with their polarizations), and that the generators have been arranged so that each m_i corresponds to the complement of a facet F_i of a quasi-tree Δ, and $F_1, \ldots, F_q$ is a leaf ordering of Δ. These extra arrangements are in place so that we can compare the resulting graph with the one in Theorem 10.

Consider the complete graph $\mathcal{K}$ on q vertices, and label its vertices with $m_1, \ldots, m_q$, and label each edge with the lcm of the labels of its vertices. Starting at $i = 1$, let $\mathcal{K}_i$ be the subgraph of $\mathcal{K}$ consisting of all vertices and edges whose monomial labels have total degree $\leq i$, and let $\mathcal{U}_i$ be a spanning forest of $\mathcal{K}_i$, with the condition that $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \ldots$. Let d be the smallest integer for which $\mathcal{U}_d$ is connected and contains all the vertices of $\mathcal{K}$. We use the notation $\mathcal{T}_G$ for the tree $\mathcal{U}_d$. Fløystad shows in [7] that $\mathcal{T}_G$ supports a resolution of I.

We now show that a tree $\mathcal{T}_B$ obtained using the algorithm in Theorem 10 is an instance of a $\mathcal{T}_G$ as described above. Suppose we have such a tree $\mathcal{T}_B$, and consider for every i its subgraph $(\mathcal{T}_B)_i$ consisting of edges and vertices whose monomial labels have total degree $\leq i$. Then $(\mathcal{T}_B)_i$ is a spanning forest of $\mathcal{K}_i$, and we have the chain of inclusions $(\mathcal{T}_B)_1 \subseteq (\mathcal{T}_B)_2 \subseteq \ldots$.

Now suppose that the maximum degree of a vertex or edge label in $\mathcal{T}_B$ is d so that $\mathcal{T}_B = (\mathcal{T}_B)_d$. Then, if you drop the edges and vertices with label of degree d, we have $(\mathcal{T}_B)_{d-1} \subsetneq (\mathcal{T}_B)_d = \mathcal{T}_B$, which shows that $\mathcal{T}_B$ is an example of a $\mathcal{T}_G$.

In order to prove a converse statement to Theorem 10, we are going to need a couple of auxiliary results.

Lemma 14. *Let Δ be a simplicial complex with vertex set $V = \{x_1, ..., x_n\}$, let $W = \{x_1, ..., x_t\} \subseteq V$, and let Δ_W be the subcomplex of Δ induced on W. If $m_1, ..., m_r$ are the minimal generators of $\mathcal{N}(\Delta^\vee)$ then the generators of $\mathcal{N}\big((\Delta_W)^\vee\big)$ are a subset of* $\{\gcd(m_1, x_1 \cdots x_t), ..., \gcd(m_r, x_1 \cdots x_t)\}$

Before we begin it is worth noting that restricting to the first t vertices is notationally convenient, but the statement will hold for any subset of V (just make an appropriate relabelling of the vertices).

Proof. If we present Δ as $\langle F_1, ..., F_r\rangle$ then the generators of $\mathcal{N}(\Delta^\vee)$ have the form $m_i = \prod\limits_{x_j \in V\setminus F_i} x_j$. We also know that the facets of Δ_W are subsets of the facets of Δ, so we can present Δ_W as $\langle \overline{F}_{i_1}, ..., \overline{F}_{i_s}\rangle$, where $\{i_1, ..., i_s\} \subseteq \{1, ..., r\}$ and $\overline{F}_{i_j} \subseteq F_{i_j}$. Since $\overline{F}_{i_j} = F_{i_j} \cap W$ we get that

$$W \setminus \overline{F}_{i_j} = W \setminus (F_{i_j} \cap W) = (V \setminus F_{i_j}) \cap W$$

and the generators of $\mathcal{N}\big((\Delta_W)^\vee\big)$ are

$$\overline{m}_{i_j} = \prod_{\substack{x_s \notin \overline{F}_{i_j} \\ x_s \in W}} x_s = \prod_{\substack{x_s \in V\setminus F_{i_j} \\ x_s \in W}} x_s = \gcd(m_{i_j}, x_1 \cdots x_t)$$

so $\overline{m}_{i_j} \in \{\gcd(m_1, x_1 \cdots x_t), ..., \gcd(m_r, x_1 \cdots x_t)\}$. □

Remark 15. In the above proof we used the fact that there is a correspondence between the facets of Δ_W and a subset of the facets of Δ. If F_q is a facet of Δ where $q \notin \{i_1, ..., i_s\}$ we still have that $F_q \cap W$ is a face of Δ_W. Therefore, $F_q \cap W$ must be a subset of some facet $\overline{F}_{i_j}$ of Δ_W. With this information we can deduce that

$$\gcd(m_q, x_1 \cdots x_t) = \big(\gcd(m_{i_j}, x_1 \cdots x_t)\big) \prod_{\substack{x_s \in F_{i_j}\setminus F_q \\ x_s \in W}} x_s$$

This tells us that $\gcd(m_q, x_1 \cdots x_t) \in \mathcal{N}\big((\Delta_W)^\vee\big)$. What this allows us to do is say that

$$\mathcal{N}\big((\Delta_W)^\vee\big) = \big(\gcd(m_1, x_1 \cdots x_t), ..., \gcd(m_r, x_1 \cdots x_t)\big)$$

With this fact we are able to prove the following corollary of Lemma 14.

Corollary 16. *Let Δ be a simplicial complex with vertex set $V = \{x_1, ..., x_n\}$. Let $W = \{x_1, ..., x_t\}$ for some $t \leq n$ and let $S' = k[x_1, ..., x_t]$. Then*

$$\frac{S'}{\mathcal{N}\big((\Delta_W)^\vee\big)} \cong \frac{S}{\mathcal{N}(\Delta^\vee)} \otimes_S \frac{S}{(x_{t+1} - 1, ..., x_n - 1)}$$

Proof. Let $m_1, ..., m_r$ be the minimal generators for $\mathcal{N}(\Delta^\vee)$. Remark 15 tells us that

$$\mathcal{N}\big((\Delta_W)^\vee\big) = \big(\gcd(m_1, x_1 \cdots x_t), ..., \gcd(m_r, x_1 \cdots x_t)\big)$$

which is the same as saying that we can form the generators of $\mathcal{N}\big((\Delta_W)^\vee\big)$ by taking the the generators of $\mathcal{N}(\Delta^\vee)$ and setting the variables $x_{t+1}, ..., x_n$ equal to 1. When we are using quotient modules we can do this by adding the desired relations to the ideal by which we are taking the quotient. Specifically, what we mean is

$$\frac{S'}{\mathcal{N}\big((\Delta_W)^\vee\big)} \cong \frac{S}{\mathcal{N}(\Delta^\vee) + \big(x_{t+1} - 1, ..., x_n - 1\big)}$$

Moreover, we have that

$$\frac{S}{\mathcal{N}(\Delta^\vee) + (x_{t+1} - 1, ..., x_n - 1)} \cong \frac{S}{\mathcal{N}(\Delta^\vee)} \otimes_S \frac{S}{(x_{t+1} - 1, ..., x_n - 1)}$$

and we have our desired result.

□

With these additional results we are now able to provide a new proof the following theorem.

Theorem 17 (Herzog, Hibi, Zheng [8]). *Let Δ be a simplicial complex, then $\mathrm{pd}(\mathcal{N}(\Delta^\vee)) = 1$ if and only if Δ is a quasi-forest.*

Proof. ($\Leftarrow$) Follows from Theorem 10.

($\Rightarrow$) Without loss of generality let $W = \{x_1, ..., x_k\}$. Recalling Proposition 4, it is enough to show that Δ_W has a leaf to conclude that Δ is a quasi-forest. Let $\mathbf{F}$ be the minimal free resolution

$$0 \longrightarrow S^{r-1} \longrightarrow S^r \longrightarrow S \longrightarrow 0$$

of $S/\mathcal{N}(\Delta^\vee)$. The elements $x_{t+1} - 1, ..., x_n - 1$ form an $S/\mathcal{N}(\Delta^\vee)$-sequence, so we can construct the resolution

$$\mathbf{F} \otimes_S \frac{S}{(x_{t+1} - 1, ..., x_n - 1)}$$

of $S'/\mathcal{N}\big((\Delta_W)^\vee\big)$, where $S' = k[x_1, ..., x_t]$ (See Chapters 20 and 21 of [9] for further details). Since the length of the resulting resolution is no greater than the length of $\mathbf{F}$, we find that $\mathrm{pd}(\mathcal{N}\big((\Delta_W)^\vee\big)) \leq \mathrm{pd}(\mathcal{N}(\Delta^\vee)) = 1$.

If $\mathrm{pd}(\mathcal{N}\big((\Delta_W)^\vee\big)) = 0$, then it must be the case that $\mathcal{N}\big((\Delta_W)^\vee\big) = 0$ which can only happen if Δ_W is a simplex, so it has a leaf.

If $\mathrm{pd}(\mathcal{N}\big((\Delta_W)^\vee\big)) = 1$, then Theorem 8 tells us that $\mathcal{N}\big((\Delta_W)^\vee\big)$ has a minimal resolution supported on a tree T. Choose a labelling of the vertices of T for which the $\mathcal{N}\big((\Delta_W)^\vee\big)$-homogenization yields a resolution, let $\overline{m}_l$ be the label of one of the free vertices of T and let $\overline{m}_j$ be the label of the vertex which shares an edge with m_l. For any other minimal generator $\overline{m}_i$ of $\mathcal{N}\big((\Delta_W)^\vee\big)$ we must have that $\overline{m}_j \big| \mathrm{lcm}(\overline{m}_l, \overline{m}_i)$ to ensure connectivity of the induced forest generated by the lcm of $\overline{m}_l$ and $\overline{m}_i$. In the proof of Theorem 10 we saw that

$$\overline{m}_j \big| \mathrm{lcm}(\overline{m}_l, \overline{m}_i) \Longleftrightarrow \overline{F}_l \cap \overline{F}_i \subset \overline{F}_j$$

which is exactly the condition needed for $\overline{F}_l$ to be a leaf of Δ_W with joint $\overline{F}_j$. Hence, we can conclude that Δ is a quasi-forest. □

To sum up the results of this paper, we can make the following statement.

Theorem 18. *Let I be a square-free monomial ideal in a polynomial ring S. Then the following are equivalent.*

1. $\mathrm{pd}_S(I) = 1$
2. $\mathcal{N}(I^\vee)$ *is a quasi-forest*
3. I *has a minimal free resolution supported on a graph-tree.*

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