# Simplicial Trees: Properties and Applications

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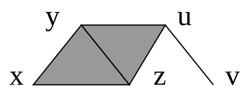
This talk focuses on properties of simplicial trees. The basic idea is to study a square-free monomial ideal by considering it as the facet ideal of a simplicial complex. We review this construction, and compare it with that in Stanley-Reisner Theory. We introduce a special class of simplicial complexes called "simplicial trees"; this definition generalizes the concept of a tree from graph theory. We then focus on simplicial trees, and discuss their Cohen-Macaulay properties. In particular, we show that the facet ideal of a simplicial tree is sequentially Cohen-Macaulay (following a definition of Richard Stanley), and if this ideal is equidimensional, it will have a Cohen-Macaulay quotient. We also discuss further problems in facet ideal theory.

## 1 Facet ideals and relations to Stanley-Reisner theory

**Definition 1.1 ([F1]).** Let $\Delta$ be a simplicial complex with vertex set $\{x_1, \ldots, x_n\}$, and let $k$ be a field. We define the *facet ideal of* $\Delta$ to be the square-free monomial ideal $\mathcal{F}(\Delta) = k[x_1, \ldots, x_n]$, where each generator is the product of the vertices of a facet (a *facet* is a maximal face of a simplicial complex).

**Example 1.2.** If $\Delta$ is the simplicial complex $\langle xyz, yzu, uv \rangle$ drawn below, then $\mathcal{F}(\Delta) = (xyz, yzu, uv)$ is its facet ideal.




For simplicity, we do not distinguish in the notation used for a facet $F$ and the monomial that is the product of the vertices of $F$. It is easy to see that there is a one-to-one correspondence between square-free monomial ideals and simplicial complexes via this construction.

**Definition 1.3.** A subset $A$ of the vertex set of $\Delta$ is called a *vertex covering of* $\Delta$ if every facet of $\Delta$ intersects $A$. If $A$ is a minimal element of the set of vertex covers of $\Delta$ it is called a *minimal vertex cover.*

Note that $A$ is a (minimal) vertex cover of $\Delta \Leftrightarrow$ the ideal generated by $A$ is a (minimal) prime of $\mathcal{F}(\Delta)$.

**Example 1.4.** If $\Delta$ is the simplicial complex in Example 1.2, then the vertex covers of $\Delta$ (or the generators of the primes of $\mathcal{F}(\Delta)$) are listed below. The first five vertex covers (highlighted in bold) are the minimal vertex covers of $\Delta$.

$$\{\mathbf{x}, \mathbf{u}\}, \{\mathbf{y}, \mathbf{u}\}, \{\mathbf{y}, \mathbf{v}\}, \{\mathbf{z}, \mathbf{u}\}, \{\mathbf{z}, \mathbf{v}\}, \{x, y, u\}, \{x, z, u\}, \{x, y, v\}, \ldots.$$

We now construct a new simplicial complex using the minimal vertex covers of a given simplicial complex $\Delta$.

**Definition 1.5 ([F3]).** Given a simplicial complex $\Delta$, the simplicial complex $\Delta_M$ called the *cover complex of* $\Delta$, is the simplicial complex whose facets are the minimal vertex covers of $\Delta$. We say that $\Delta$ is *unmixed* if all of its minimal vertex covers have the same cardinality (i.e., if $\Delta_M$ is *pure*).

**Example 1.6.** The simplicial complex $\Delta = \langle xyz, yzu, uv \rangle$ in Example 1.2 is unmixed, as $\Delta_M = \langle xu, yu, yv, zu, zv \rangle$ is pure.

**Proposition 1.7 ([F3]).** *The simplicial complex $\Delta_M$ is a dual of $\Delta$; i.e. $\Delta_{MM} = \Delta$.*

We now focus on some basic definitions from Stanley-Reisner theory.

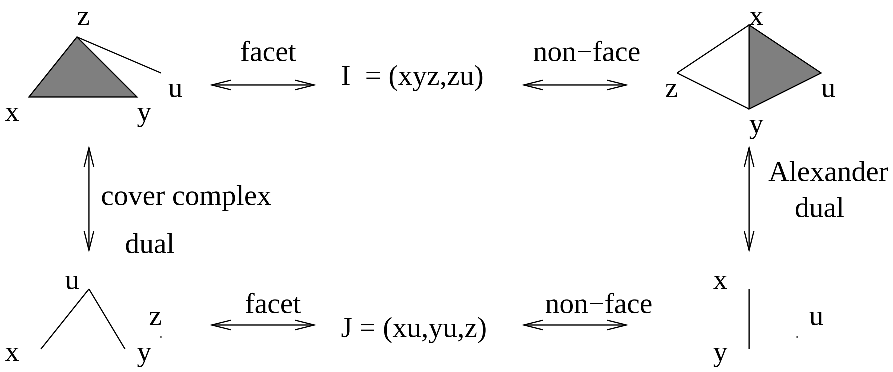
**Definition 1.8 ([S]).** If $\Delta$ is a simplicial complex, the *Stanley-Reisner ideal* or *non-face ideal* $\mathcal{N}(\Delta)$ of $\Delta$ is a square-free monomial ideal such that $F \in \Delta$ if and only if the product of the vertices of $F$ is not in $\mathcal{N}(\Delta)$.

**Definition 1.9.** The simplicial complex $\Delta^\vee = \{F \subset V(\Delta) \mid V(\Delta) \setminus F \notin \Delta\}$ is the *Alexander dual* of $\Delta$.

Note that $\Delta^{\vee\vee} = \Delta$.

**Proposition 1.10 ([F3]).** *If for two simplicial complexes $\Delta$ and $\Gamma$ we have $\mathcal{N}(\Gamma) = \mathcal{F}(\Delta)$, then $\mathcal{N}(\Gamma^\vee) = \mathcal{F}(\Delta_M)$.*

The following diagram clarifies Proposition 1.10.

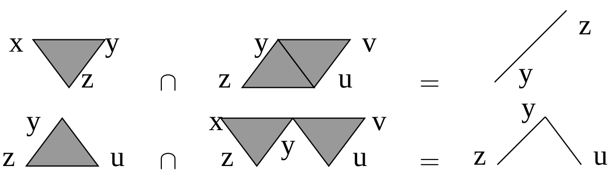



## 2 Simplicial Trees

Considering simplicial complexes as higher dimensional graphs, one can define the notion of a *tree* which generalizes the same notion from graph theory.

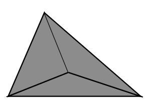
**Definition 2.1 ([F1]).** A facet $F$ of a simplicial complex is called a **leaf** if either $F$ is the only facet of $\Delta$, or for some facet $G \in \Delta \setminus \langle F \rangle$ we have $F \cap \Delta \setminus \langle F \rangle \subseteq G$.

**Example 2.2.** Let $\Delta = \langle xyz, yzu, zuv \rangle$. Then $F = xyz$ is a leaf, but $H = yzu$ is not, as one can see in the picture below.




**Definition 2.3 ([F1]).** A connected simplicial complex $\Delta$ is a **tree** if every nonempty *subcollection* of $\Delta$ (that is a subcomplex of $\Delta$ whose facets are also facets of $\Delta$) has a leaf.

**Example 2.4.** The simplicial complexes in examples 1.2 and 2.2 are both trees, but the one below is not.



**Properties of trees:** If $\mathcal{F}(\Delta) \subseteq R = k[x_1, \ldots, x_n]$ is the facet ideal of a simplicial tree $\Delta$ , then

1. $\mathcal{F}(\Delta)$ satisfies Sliding Depth condition ([F1]).

   This property puts bounds on the depths of the Koszul homology modules of $\mathcal{F}(\Delta)$. Sliding Depth results in:

2. $\mathcal{F}(\Delta)$ has a normal and Cohen-Macaulay Rees ring.

3. Trees are Strongly Cohen-Macaulay; i.e., if $R/\mathcal{F}(\Delta)$ is Cohen-Macaulay, then all Koszul homology modules of $\mathcal{F}(\Delta)$ are Cohen-Macaulay.

   These results were proved in the case of graphs in [SVV] and were generalized to simplicial complexes in [F1]. In fact, simplicial trees were defined this way so that they satisfied Sliding Depth condition. Later, it turned out that they also satisfy the following ([F2]):

4. (König's Theorem generalized to simplicial trees): if the minimum of the cardinalities of the minimal vertex covers of $\Delta$ is $r$, then $\Delta$ has a collection of $r$ pairwise disjoint facets.

   The last fact tells us much about the combinatorial structure of trees, and allows us to extend a result of Villarreal from graphs to simplicial complexes ([F2]):

5. $R/\mathcal{F}(\Delta)$ is Cohen-Macaulay $\iff$ $\Delta$ is unmixed.

   This last statement is stated in a much stronger language in [F2]: For a tree to be unmixed it has to have a specific combinatorial structure, namely it has to be *grafted* (see [F2] for a detailed description of grafting), and one can show that any grafted simplicial complex (not necessarily a tree) is Cohen-Macaulay ([F2]).

   But what about a simplicial tree that is not unmixed? We can show that in general ([F3]):

6. $\mathcal{F}(\Delta)$ is a sequentially Cohen-Macaulay ideal.

   A square-free monomial ideal $I$ is *Sequentially Cohen-Macaulay* if for every $i$, the pure $i$-dimensional subcomplex of the non-face complex $\Delta_N$ of $I$ is Cohen-Macaulay ([S], [D]).

   From the dualities discussed in Section 1 it easily follows that $\mathcal{F}(\Delta)$ being Sequentially Cohen-Macaulay is equivalent to $\mathcal{F}(\Delta_M)$ being *componentwise linear*: this means that every homogeneous component of $\mathcal{F}(\Delta_M)$ (that is every ideal that is generated by all square-free monomials in $\mathcal{F}(\Delta_M)$ of the same degree) has a linear resolution. Componentwise linear ideals were introduced in [HH] and the equivalence mentioned above was proved there in terms of Alexander Duality.

   More is true: if $\Delta$ is a tree, then every homogeneous component of $\mathcal{F}(\Delta_M)$ has linear quotients. This property (due to [HT]), not only implies that the component has a linear resolution, but also implies the following ([F3]):

7. If $R/\mathcal{F}(\Delta)$ is Cohen-Macaulay, and $\Gamma$ is a simplicial complex such that $\mathcal{N}(\Gamma) = \mathcal{F}(\Delta)$, then $\Gamma$ is shellable.

# References

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