The notion of a sequentially Cohen-Macaulay module was introduced by Stanley [7], following the introduction of a nonpure shellable simplicial complex by Björner and Wachs [BW]. It was known that the Stanley-Reisner ideal of a shellable simplicial complex is Cohen-Macaulay (see [BH]). A shellable simplicial complex is by definition pure (all facets have the same dimension), which is equivalent to its Stanley-Reisner ideal being unmixed. A nonpure shellable simplicial complex, on the other hand, may not be pure, so its Stanley-Reisner ideal may not be unmixed, and hence not Cohen-Macaulay. As it turns out, however, the Stanley-Reisner ideal of a nonpure simplicial complex is “sequentially Cohen-Macaulay” (Definition 1 below).

If the Stanley-Reisner ideal of a simplicial complex is sequentially Cohen-Macaulay, the complex has Cohen-Macaulay pure subcomplexes (see Duval [D] Theorem 3.3, or Stanley [7] Chapter III, Proposition 2.10). In the language of commutative algebra, this is equivalent to all equidimensional components appearing in the primary decomposition of a square-free monomial ideal being Cohen-Macaulay (see [F] for more details).

The purpose of this note is to establish that, more generally, this is what being sequentially Cohen-Macaulay means for any module. Below we use basic facts about primary decomposition of modules to study the structure of the submodules appearing in the (unique) filtration of a sequentially Cohen-Macaulay module. The main result (Theorem 5) states that each submodule appearing in the filtration of a sequentially Cohen-Macaulay module $M$ is the intersection of all primary submodules whose associated primes have a certain height and appear in an irredundant primary decomposition of the 0-submodule of $M$. Similar results, stated in a different language, appear in [Sc]; the author thanks Jürgen Herzog for pointing this out.

**Definition 1 ([St] Chapter III, Definition 2.9).** Let $M$ be a finitely generated $\mathbb{Z}$-graded module over a finitely generated $\mathbb{N}$-graded $k$-algebra, with $R_0 = k$. We say that $M$ is sequentially Cohen-Macaulay if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

of $M$ by graded submodules $M_i$ satisfying the following two conditions.

(a) Each quotient $M_i/M_{i-1}$ is Cohen-Macaulay;
(b) \( \dim (M_1/M_0) < \dim (M_2/M_1) < \ldots < \dim (M_r/M_{r-1}) \), where “\( \dim \)” denotes Krull dimension.

Before we begin our study of sequentially Cohen-Macaulay modules, we record two basic lemmas that we shall use later. Throughout the discussions below, we assume that \( R \) is a finitely generated algebra over a field, and \( M \) is a finite module over \( R \).

**Lemma 2.** Let \( Q_1, \ldots, Q_t, P \) all be primary submodules of an \( R \)-module \( M \), such that \( \text{Ass}(M/Q_i) = \{q_i\} \) and \( \text{Ass}(M/P) = \{\wp\} \). If \( Q_1 \cap \ldots \cap Q_i \subseteq P \) and \( Q_i \not\subseteq P \) for some \( i \), then there is a \( j \neq i \) such that \( q_j \subseteq \wp \).

**Proof.** Let \( x \in Q_i \setminus P \). For each \( j \neq i \), pick the positive integer \( m_j \) such that \( q_j^{m_j} x \subseteq Q_j \). So we have that

\[
q_1^{m_1} \ldots q_i^{m_i-1} q_{i+1}^{m_{i+1}} \ldots q_t^{m_t} x \subseteq Q_1 \cap \ldots \cap Q_i \subseteq P
\]

which implies that, since \( x \not\in P \),

\[
q_1^{m_1} \ldots q_i^{m_i-1} q_{i+1}^{m_{i+1}} \ldots q_t^{m_t} \subseteq \wp
\]

and hence for some \( j \neq i \), \( q_j \subseteq \wp \). \( \Box \)

**Lemma 3.** Let \( M \) be an \( R \)-module and \( N \) be a submodule of \( M \). Then for every \( \wp \in \text{Ass}(M/N) \), if \( \wp \not\supseteq \text{Ann}(N) \), then \( \wp \in \text{Ass}(M) \).

**Proof.** Since \( \wp \in \text{Ass}(M/N) \), there exists \( x \in M \setminus N \) such that \( \wp = \text{Ann}(x) \); in other words

\[
\wp x \subseteq N.
\]

Suppose \( \text{Ann}(N) \not\subseteq \wp \), and let \( y \in \text{Ann}(N) \setminus \wp \). Now \( y\wp x = 0 \), and so \( \wp \subseteq \text{Ann}(yx) \) in \( M \).

On the other hand, if \( z \in \text{Ann}(yx) \), then \( zyx = 0 \subseteq N \) and so \( zy \in \wp \). But \( y \not\in \wp \), so \( z \in \wp \). Therefore \( \wp \in \text{Ass}(M) \). \( \Box \)

Suppose \( M \) is a sequentially Cohen-Macaulay module with filtration as in Definition 1. We adopt the following notation. For a given integer \( j \), we let

\[
\text{Ass}(M)_j = \{\wp \in \text{Ass}(M) \mid \text{height } \wp = j\}.
\]

Suppose that all the \( j \) where \( \text{Ass}(M)_j \neq \emptyset \) form the sequence of integers

\[
0 \leq h_1 < \ldots < h_c \leq \dim R
\]

so that

\[
\text{Ass}(M) = \bigcup_{1 \leq j \leq c} \text{Ass}(M)_{h_j}.
\]

We can now make the following observations.

**Proposition 4.** For all \( i = 0, \ldots, r-1 \), we have

1. \( \text{Ass}(M_{i+1}/M_i) \cap \text{Ass}(M) \neq \emptyset \);
Proof. 1. We use induction on the length $r$ of the filtration of $M$. The case $r = 1$ is clear, as we have a filtration $0 \subset M$, and the assertion follows. Now suppose the statement holds for sequentially Cohen-Macaulay modules with filtrations of length less than $r$. Notice that $M_{r-1}$ that appears in the filtration of $M$ in Definition 1 is also sequentially Cohen-Macaulay, and so by the induction hypothesis, we have

$$\text{Ass}(M_{i+1}/M_i) \cap \text{Ass}(M_{r-1}) \neq \emptyset$$

and since $\text{Ass}(M_{r-1}) \subseteq \text{Ass}(M)$ it follows that

$$\text{Ass}(M_{i+1}/M_i) \cap \text{Ass}(M) \neq \emptyset$$

for $i = 0, \ldots, r - 2$. It remains to show that $\text{Ass}(M/M_{r-1}) \cap \text{Ass}(M) \neq \emptyset$.

For each $i$, $M_{i-1} \subset M_i$, so we have ([B] Chapter IV)

$$\text{Ass}(M_1) \subseteq \text{Ass}(M_2) \subseteq \text{Ass}(M_1) \cup \text{Ass}(M_2/M_1) \tag{1}$$

The inclusion $M_2 \subseteq M_3$ along with the inclusions in (1) imply that

$$\text{Ass}(M_2) \subseteq \text{Ass}(M_3) \subseteq \text{Ass}(M_2) \cup \text{Ass}(M_3/M_2) \subseteq \text{Ass}(M_1) \cup \text{Ass}(M_2/M_1) \cup \text{Ass}(M_3/M_2).$$

If we continue this process inductively, at the $i$-th stage we have

$$\text{Ass}(M_i) \subseteq \text{Ass}(M_{i-1}) \cup \text{Ass}(M_i/M_{i-1}) \subseteq \text{Ass}(M_1) \cup \text{Ass}(M_2/M_1) \cup \text{Ass}(M_3/M_2) \cup \ldots \cup \text{Ass}(M_i/M_{i-1})$$

and finally, when $i = r$ it gives

$$\text{Ass}(M) \subseteq \text{Ass}(M_1) \cup \text{Ass}(M_2/M_1) \cup \text{Ass}(M_3/M_2) \cup \ldots \cup \text{Ass}(M/M_{r-1}). \tag{2}$$

Because of Condition (b) in Definition 1, and the fact that each $M_{i+1}/M_i$ is Cohen-Macaulay (and hence all its associated primes have the same height; see [BH] Chapter 2), if for every $i$ we pick $\wp_i \in \text{Ass}(M_{i+1}/M_i)$, then

$$h_c \geq \text{height } \wp_0 > \text{height } \wp_1 > \ldots > \text{height } \wp_{r-1}.$$

where the left-hand-side inequality comes from the fact that $\text{Ass}(M_1) \subseteq \text{Ass}(M)$. By our induction hypothesis, $\text{Ass}(M)$ intersects $\text{Ass}(M_{i+1}/M_i)$ for all $i \leq r - 2$, and so because of (2) we conclude that

$$\text{height } \wp_i = h_{c-i}, \text{ and } \text{Ass}(M) h_{c-i} \subseteq \text{Ass}(M_{i+1}/M_i) \text{ for } 0 \leq i \leq r - 2.$$

And now $\text{Ass}(M) h_0$ has no choice but to be included in $\text{Ass}(M/M_{r-1})$, which settles our claim. It also follows that $c = r$. 

3. If $\wp \in \text{Ass}(M_{i+1})$, then $\text{height } \wp \geq h_{r-1}$.

4. If $\wp \in \text{Ass}(M_{i+1}/M_i)$, then $\text{Ann}(M_i) \not\subseteq \wp$.

5. $\text{Ass}(M_{i+1}/M_i) \subseteq \text{Ass}(M)$.

6. $\text{Ass}(M_{i+1}/M_i) = \text{Ass}(M) h_{r-1}$.

7. $\text{Ass}(M/M_i) = \text{Ass}(M) h_{r-1}$.

8. $\text{Ass}(M_{i+1}) = \text{Ass}(M) \geq h_{r-1}$. 

2. See the proof for part 1.

3. We use induction. The case \( i = 0 \) is clear, since for every \( \wp \in \text{Ass}(M_1) = \text{Ass}(M_1/M_0) \) we know from part 2 that height \( \wp = h_r \). Suppose the statement holds for all indices up to \( i - 1 \). Consider the inclusion
\[
\text{Ass}(M_i) \subseteq \text{Ass}(M_{i+1}) \subseteq \text{Ass}(M_i) \cup \text{Ass}(M_{i+1}/M_i).
\]
From part 2 and the induction hypothesis it follows that if \( \wp \in \text{Ass}(M_{i+1}) \) then height \( \wp \geq h_{r-i} \).

4. Suppose \( \text{Ann}(M_i) \subseteq \wp \). Since \( \sqrt{\text{Ann}(M_i)} = \bigcap_{\wp' \in \text{Ass}(M_i)} \wp' \), we have
\[
\bigcap_{\wp' \in \text{Ass}(M_i)} \wp' \subseteq \wp
\]
so there is a \( \wp' \in \text{Ass}(M_i) \) such that \( \wp' \subseteq \wp \). But by part 2 and part 3 above height \( \wp' \geq h_{r-i+1} \) and height \( \wp = h_{r-i} \), which is a contradiction.

5. From part 4 and Lemma 3, it follows that
\[
\text{Ass}(M_{i+1}/M_i) \subseteq \text{Ass}(M_{i+1}) \subseteq \text{Ass}(M).
\]

6. This follows from parts 2 and 5, and the fact that \( M_{i+1}/M_i \) is Cohen-Macaulay, and hence all associated primes have the same height.

7. We show this by induction on \( e = r - i \). The case \( e = 1 \) (or \( i = r - 1 \)) is clear, because by part 6
\[
\text{Ass}(M/M_{r-1}) = \text{Ass}(M)_{h_1} = \text{Ass}(M)_{\leq h_1}.
\]
Now suppose the equation holds for all integers up to \( e - 1 \) (namely \( i = r - e + 1 \)), and we would like to prove the statement for \( e \) (or \( i = r - e \)). Since \( M_{i+1}/M_i \subseteq M/M_i \), we have
\[
\text{Ass}(M_{i+1}/M_i) \subseteq \text{Ass}(M/M_i) \subseteq \text{Ass}(M_{i+1}/M_i) \cup \text{Ass}(M/M_{i+1}) \quad (3)
\]
By the induction hypothesis and part 6 we know that
\[
\text{Ass}(M_{i+1}) = \text{Ass}(M)_{\leq h_{r-i-1}} \quad \text{and} \quad \text{Ass}(M_{i+1}/M_i) = \text{Ass}(M)_{h_{r-i}},
\]
which put together with (3) implies that
\[
\text{Ass}(M)_{h_{r-i}} \subseteq \text{Ass}(M/M_i) \subseteq \text{Ass}(M)_{\leq h_{r-i}}.
\]
We still have to show that \( \text{Ass}(M/M_i) \supseteq \text{Ass}(M)_{\leq h_{r-i-1}} \).

Let \( \wp \in \text{Ass}(M)_{\leq h_{r-i-1}} = \text{Ass}(M/M_{i+1}) = \text{Ass}((M/M_i)/(M_{i+1}/M_i)) \).
If \( \wp \supseteq \text{Ann}(M_{i+1}/M_i) \), then (by part 6)
\[
\wp \supseteq \bigcap_{q \in \text{Ass}(M)_{h_{r-i}}} q \implies \wp \supseteq q \text{ for some } q \in \text{Ass}(M)_{h_{r-i}}
\]
which is a contradiction, as height \( \wp \leq h_{r-i-1} < \text{height } q \).
It follows from Lemma 3 that \( \wp \in \text{Ass}(M/M_i) \).
8. The argument is based on induction, and exactly the same as the one in part 4, using more information: from

\[ \text{Ass}(M_i) \subseteq \text{Ass}(M_{i+1}) \subseteq \text{Ass}(M_i) \cup \text{Ass}(M_{i+1}/M_i), \]

the induction hypothesis, and part 6 we deduce that

\[ \text{Ass}(M) \supseteq_{h_{r-i+1}} \text{Ass}(M_{i+1}) \subseteq \text{Ass}(M) \cup \text{Ass}(M)_{h_{r-i}}, \]

which put together with part 4, along with Lemma 3 produces the equality.

Now suppose that as a submodule of \( M \), \( M_0 = 0 \) has an irredundant primary decomposition of the form:

\[ M_0 = 0 = \bigcap_{1 \leq j \leq r} Q_{h_j}^j \cap \ldots \cap Q_{s_j}^j, \tag{4} \]

where for a fixed \( j \leq r \) and \( e \leq s_j \), \( Q_{h_j}^e \) is a primary submodule of \( M \) with

\[ \text{Ass}(M/Q_{h_j}^e) = \{ \wp_{h_j}^e \} \text{ and } \text{Ass}(M)_{h_j} = \{ \wp_{1}^{h_j}, \ldots, \wp_{s_j}^{h_j} \}. \]

**Theorem 5.** Let \( M \) be a sequentially Cohen-Macaulay module with filtration as in Definition 1, and suppose that \( M_0 = 0 \) has a primary decomposition as in \((4)\). Then for each \( i = 0, \ldots, r - 1 \), \( M_i \) has the following primary decomposition

\[ M_i = \bigcap_{1 \leq j \leq r-i} Q_{h_j}^j \cap \ldots \cap Q_{s_j}^j. \tag{5} \]

**Proof.** We prove this by induction on \( r \) (length of the filtration). The case \( r = 1 \) is clear, as the filtration is of the form \( 0 = M_0 \subset M \). Now consider \( M \) with filtration

\[ 0 = M_0 \subset M_1 \subset \ldots \subset M_r = M. \]

Since \( M_{r-1} \) is a sequentially Cohen-Macaulay module of length \( r - 1 \), it satisfies the statement of the theorem. We first show that \( M_{r-1} \) has a primary decomposition as described in \((5)\). From part 7 of Proposition 4 it follows that

\[ \text{Ass}(M/M_{r-1}) = \text{Ass}(M)_{h_1} \]

and so for some \( \wp_{e}^{h_1} \)-primary submodules \( P_{e}^{h_1} \) of \( M \) \( (1 \leq e \leq s_j) \), we have

\[ M_{r-1} = P_{1}^{h_1} \cap \ldots \cap P_{s_1}^{h_1}. \tag{6} \]

We would like to show that \( Q_{e}^{h_1} = P_{e}^{h_1} \) for \( e = 1, \ldots, s_1 \).

Fix \( e = 1 \) and assume \( Q_{1}^{h_1} \not\subset P_{1}^{h_1} \). From the inclusion \( M_0 \subset P_1^{h_1} \) and Lemma 2 it follows that for some \( e \) and \( j \) (with \( e \neq 1 \) if \( j = 1 \)), we have \( \wp_{e}^{h_j} \subset \wp_{1}^{h_1} \). Because of the difference in heights of these ideals the only conclusion is \( \wp_{e}^{h_j} = \wp_{1}^{h_1} \), which is not possible. With a similar argument we deduce that \( Q_{e}^{h_1} \subset P_{e}^{h_1} \), for \( e = 1, \ldots, s_1 \).

Now fix \( j \in \{1, \ldots, r\} \) and \( e \in \{1, \ldots, s_j\} \). If \( M_{r-1} = Q_{e}^{h_j} \) we are done. Otherwise, note that for every \( j \) and \( \wp_{e}^{h_j} \)-primary submodule \( Q_{e}^{h_j} \) of \( M \),

\[ Q_{e}^{h_j} \cap M_{r-1} \]
is a $\wp_{e}^{h_{j}}$-primary submodule of $M_{r-1}$ (as $\emptyset \neq \Ass(M_{r-1}/(Q_{e}^{h_{j}} \cap M_{r-1})) = \Ass((M_{r-1} + Q_{e}^{h_{j}})/Q_{e}^{h_{j}}) \subseteq \Ass(M/Q_{e}^{h_{j}}) = \{\wp_{e}^{h_{j}}\}$). So $M_{0} = 0$ as a submodule of $M_{r-1}$ has a primary decomposition

$$M_{0} \cap M_{r-1} = 0 = \bigcap_{1 \leq j \leq r} (Q_{1}^{h_{j}} \cap M_{r-1}) \cap \ldots \cap (Q_{s_{j}}^{h_{j}} \cap M_{r-1}).$$

From Proposition 4 part 8 it follows that

$$\Ass(M_{r-1}) = \Ass(M)_{\geq h_{2}}$$

so the components $Q_{t}^{h_{1}} \cap M_{r-1}$ are redundant for $t = 1, \ldots, s_{1}$, so for each such $t$ we have

$$\bigcap_{Q_{e}^{h_{j}} \neq Q_{t}^{h_{1}}} (Q_{1}^{h_{j}} \cap M_{r-1}) \subseteq Q_{t}^{h_{1}} \cap M_{r-1}.$$ 

If $Q_{e}^{h_{j}} \cap M_{r-1} \not\subseteq Q_{t}^{h_{1}} \cap M_{r-1}$ for some $e$ and $j$ (with $Q_{e}^{h_{j}} \neq Q_{t}^{h_{1}}$), then by Lemma 2 for some such $e$ and $j$ we have $\wp_{e}^{h_{j}} \subseteq \wp_{t}^{h_{1}}$, which is a contradiction (because of the difference of heights).

Therefore, for each $t$ ($1 \leq t \leq s_{1}$), there exists indices $e$ and $j$ (with $Q_{e}^{h_{j}} \neq Q_{t}^{h_{1}}$) such that

$$Q_{e}^{h_{j}} \cap M_{r-1} \subseteq Q_{t}^{h_{1}} \cap M_{r-1}.$$ 

It follows now, from the primary decomposition of $M_{r-1}$ in (6) that for a fixed $t$

$$P_{1}^{h_{1}} \cap \ldots \cap P_{t}^{h_{1}} \cap Q_{s_{1}}^{h_{1}} \subseteq Q_{t}^{h_{1}}.$$ 

Assume $P_{t}^{h_{1}} \not\subseteq Q_{t}^{h_{1}}$. Applying Lemma 2 again, we deduce that

$$\wp_{e}^{h_{j}} \subseteq \wp_{t}^{h_{1}}$$

or there is $t' \neq t$ such that $\wp_{t'}^{h_{1}} \subseteq \wp_{t}^{h_{1}}$.

Neither of these is possible, so $P_{t}^{h_{1}} \subseteq Q_{t}^{h_{1}}$ for all $t$. We have therefore proved that

$$M_{r-1} = Q_{1}^{h_{1}} \cap \ldots \cap Q_{s_{1}}^{h_{1}}.$$ 

By induction hypothesis, for each $i \leq r-2$, $M_{i}$ has the following primary decomposition

$$M_{i} = \bigcap_{2 \leq j \leq r-i} (Q_{1}^{h_{j}} \cap M_{r-1}) \cap \ldots \cap (Q_{s_{j}}^{h_{j}} \cap M_{r-1}) = \bigcap_{1 \leq j \leq r-i} Q_{1}^{h_{j}} \cap \ldots \cap Q_{s_{j}}^{h_{j}}$$

which proves the theorem.

\[
\square
\]

References


