The blowup closure of a set of ideals with applications to TI closure

Sara Faridi *

October 1, 2001

Abstract

We introduce a new closure operation on sets of ideals in a commutative Noetherian ring of characteristic p, called the blowup closure. We develop the theory of this operation and prove that given a set of ideals in a Noetherian ring of characteristic p, under mild conditions, the tight integral closure of this set agrees with the blowup closure of certain extensions of those ideals in an extension of the original ring. We then use properties of blowup closure to settle open questions on tight integral closure posed by Hochster in [Ho2]. In particular, we show that under mild conditions on the ring, tight integral closure persists under ring maps, and that it commutes with localization if and only if tight closure does.

In [Ho2] Hochster defined an operation on a set of ideals in a commutative Noetherian ring of positive characteristic : *tight integral closure* (or *TI* closure). This operation mixes the ideas of tight and integral closure in the sense that one can describe the tight closure or the integral closure of any given ideal as the TI closure of a certain set of ideals. Using this operation, Hochster was able to prove a Briançon-Skoda type theorem for sets of ideals in a regular ring which significantly improved the original tight closure version in [HH1]. However, it turned out to be very difficult to verify basic properties of *TI* closure, even though from the definition one would expect most of the properties of tight closure to generalize to TI closure.

In this paper we introduce the *blowup closure* of a set of ideals which has the advantage that one can reduce its study to the case where all the ideals are principal. In practice, this is the simplest case in which one can work with several ideals at the same time. Moreover, we show that under mild conditions, TI closure can be described as a blowup closure in a larger ring (Corollary 3.5).

Consequently, we show that the TI closure of a set of ideals can be expressed as the tight closure of one ideal in an extension ring of the original one. We apply this fact to address the questions on TI closure that were stated in [Ho2]. We settle the question of persistence of TI closure under ring maps (Theorem 3.4), and show (Theorem 3.6) that TI closure commutes with localization if and only if tight closure does.

We also develop a theory of test elements for TI closure in Section 3.3. Test elements are the key ingredients for tight closure arguments, and the existence of test elements is one of the most important results in tight closure theory (see [HH2] or [HH3]). However, this notion does not exist for integral closure, and so we alter the traditional definition of test elements in tight closure theory to make sense of test elements for TI closure. TI closure test elements, unlike those in tight closure, depend on the ideals that one works with. In Theorem 3.12, we describe specific test elements for the TI closure of a set of ideals in an affine algebra, via a similar result of [HH2] for tight closure.

*Mathematics Department, George Washington University, Washington, DC 20052. E-mail: faridi@gwu.edu

This result produces an easy method for calculating TI closure test elements when the defining set of equations for the algebra is known.

In Section 3.4 we define test exponents for TI closure (see [HH4] for this notion in tight closure theory) and show that the existence of test exponents in TI closure is related to the problem of TI closure commuting with localization. The final section of the paper deals with TI closure in characteristic zero.

Acknowledgments. The contents of this paper are part of my PhD thesis at The University of Michigan. I am grateful to Karen Smith and Mel Hochster for their guidance during the preparation of the thesis.

1 Basic Definitions

In all the discussions, we assume that all rings are commutative with identity. When we refer to a ring of characteristic p, we mean p is a positive prime integer.

Notation. If R is a ring, then $x \in R^o$ means that x is an element of R that is not in any minimal prime of R, and $\mathcal{M}(R)$ is the set of all minimal primes of R. By R' we mean the normalization of R (see Definition 1.1). When R has prime characteristic p, q is a power of p, and I is an ideal of R, $I^{[q]}$ denotes the ideal generated by the qth powers of the elements of I. In particular, if I is generated by $x_1, \ldots, x_n$, then $I^{[q]}$ is generated by ${x_1}^q, \ldots, {x_n}^q$. For a graded ring S and f a homogeneous element of S, by $S_{(f)}$ we mean the zeroth graded piece of the localized ring S_f, i.e., $S_{(f)} = (S_f)_0$.

1.1 Integral Closure

Definition 1.1. Let R be a Noetherian ring, and let $p_1, \ldots, p_m$ be the minimal primes of R. Then we define the *normalization* of R, denoted by R', to be

$$(R/p_1)' \times \ldots \times (R/p_m)',$$

where for $i = 1, \ldots, m$, $(R/p_i)'$ is the integral closure of the domain R/p_i in the field of fractions of R/p_i.

We refer the reader to [M] or [BH] for basic properties of integral closure of rings and ideals. We state some well-known facts on integral closure that we will use later in this paper; for written proofs we refer the reader to [F].

Proposition 1.2. *Let R be a reduced Noetherian ring, and let $p_1, \ldots, p_m$ be the minimal primes of R. Then:*

(a) The minimal primes of R' are ${p_1}^\sharp, \ldots, {p_m}^\sharp$, where for $1 \le i \le m$,

$${p_i}^\sharp = (R/p_1)' \times \ldots \times (R/p_{i-1})' \times (0) \times (R/p_{i+1})' \times \ldots \times (R/p_m)';$$

(b) $R'/{p_i}^\sharp \simeq (R/p_i)'$, for all i.

We next focus on the notion of integral closure of an ideal.

Definition 1.3. For a ring S and an ideal J of S, the *integral closure* of J in S, denoted by $\overline{J}$, is defined as the set of all all $x \in S$ that satisfy an equation of the form $x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ where $a_i \in J^i$, $i = 1, \ldots, n$; or equivalently, the set of all $x \in S$ for which there exists a $c \in S$ that does not belong to any minimal prime of S, such that $cx^n \in J^n$ for all positive integers n ([Ho2] 1.2).

Below we record a useful feature of integral closure that we use often in the later sections.

Proposition 1.4. *Let R be a domain, and let $I = (g_1, \ldots, g_s)$ be an ideal in R. For every i, let S_i be the normalization of $R[g_1/g_i, \ldots, g_s/g_i]$. Then $x \in \overline{I}$ if and only if $x \in IS_i$ for all $i = 1, \ldots, s$.*

Proof. Suppose that $x \in \overline{I}$. Then it follows that $x \in \overline{IS_i} = IS_i$ for all i, since $IS_i = (g_i)S_i$ is a principal ideal, and S_i is a normal ring, and so IS_i is integrally closed.

Now suppose that $x \in IS_i$ for all i. Let V be a valuation domain containing R. Since I is a finitely generated ideal, the image of I in V will be generated by one of the g_i. It follows that for some i, $S_i \subseteq V$. Then $x \in IS_i$ implies that $x \in IV$, and so $x \in \overline{I}$ (see [ZS2] page 353). $\square$

The integral closure of an ideal I in a ring R is closely related to the normalization of the Rees ring of I, where the Rees ring of I is the graded subring of the polynomial ring $R[t]$ of the form:

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \ldots .$$

A well known description for the integral closure of Rees rings, which follows from Proposition 20 in Chapter 5 of [B], is the following:

Theorem 1.5. *Let R be a normal domain, and I an be ideal of R, and t be an indeterminate over R. Then the normalization of $R[It]$ in its field of fractions is*

$$R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \ldots ,$$

where $\overline{J}$ denotes the integral closure of the ideal J in R.

1.2 Tight Closure

Tight closure is an operation on ideals in Noetherian rings of positive prime characteristic p and Noetherian rings that contain a field. The main idea for characteristic p arguments is that when R is a ring of prime characteristic p, the Frobenius map (the map from R to R that takes an element r to its pth power r^p) will be a ring homomorphism. The theory is extended to rings of characteristic zero via the method of reduction to characteristic p. For details and references on tight closure, and for proofs of the facts stated below, see [HH1], [Hu] or [BH].

Definition 1.6. Let R be a Noetherian ring of characteristic $p > 0$. Let I be an ideal of R. Then the *tight closure* of I , denoted by I^*, is the collection of all $z \in R$ for which there exists $c \in R^o$ such that $cz^q \in I^{[q]}$ for all large $q = p^e$. We say that an ideal I is *tightly closed* if $I = I^*$.

It is easy to check that the tight closure of an ideal is itself an ideal. In general the integral closure of an ideal is much larger than its tight closure.

Example 1.7. Let $R = k[x, y]$ be a polynomial ring over a field of characteristic p. Let $I = (x^2, y^2)$. Then $\overline{I} = (x^2, xy, y^2)$, while $I^* = I = (x^2, y^2)$ since R is a regular ring.

A key element in tight closure theory is the existence of *test elements*.

Definition 1.8. Let R be a Noetherian ring of characteristic p. An element $c \in R^o$ is called a *test element* for R, if for every ideal I of R and every $x \in I^*$, $cx^q \in I^{[q]}$ for all $q = p^e$, $e \geq 0$. The ideal generated by the test elements for R is called the *test ideal* of R, and is denoted by $\tau(R)$.

Under certain mild conditions on the ring, Hochster and Huneke proved that test elements exist. Before stating this result, we introduce some terminology.

Note 1.9. If R is a ring of characteristic p, R is *essentially of finite type* over a ring S if it is a localization of a finitely generated algebra over S. R is *F-finite* if the Frobenius map is a finite map. Below, as well as in the next sections, we frequently assume that our rings are F-finite or essentially of finite type over an excellent local ring. It is worth pointing out that these are not very restrictive conditions on a ring. For example, any reduced finitely generated algebra over a perfect field or any complete local ring with perfect residue field will satisfy these properties (see Chapter 10 of [BH] or [Hu]). It is straightforward to check that both these properties are preserved after localizing at a multiplicative set in R, and after taking finitely generated algebra extensions.

Theorem 1.10 (existence of test elements). *Let R be a Noetherian reduced ring of positive characteristic p, and suppose that R is essentially of finite type over an excellent local ring, or that R is F-finite. Let $c \in R^o$ be such that R_c is regular. Then c has a power that is a test element for R, and remains so after localizing or completing R.*

A very useful property of tight closure follows from this theorem.

Theorem 1.11 (persistence of tight closure). *Let $R \longrightarrow S$ be homomorphism of Noetherian rings of positive characteristic p. Suppose that R is essentially of finite type over an excellent local ring, or that R_{red} (i.e. R modulo the nilradical of R) is F-finite. Then*

$$I^*S \subseteq (IS)^*.$$

These theorems are [Hu] 2.1 and 2.3, respectively.

Remark 1.12. Persistence improves several features of tight closure. One property that we will use is the following: If tight closure persists for the map $f : R \longrightarrow S$ of Noetherian rings, then the contraction of a tightly closed ideal of S will be a tightly closed ideal of R. To see this, let $J \subseteq S$ be tightly closed in S, and take $u \in (f^{-1}(J))^*$. Then $f(u) \in (f^{-1}(J)S)^* \subseteq J^* = J$ (see Proposition 10.1.2 of [BH]). Hence $f(u) \in J$, and so $u \in f^{-1}(J)$.

1.2.1 Tight Closure in Equal Characteristic Zero

The method of reduction to characteristic p allows one to extend tight closure theory to rings of equal characteristic zero, that is, rings of characteristic zero containing a field. Consequently, tight closure results in characteristic p find analogous statements in characteristic zero. For a brief but more detailed treatment of this topic see [Ho1] or [HH1]. The complete and general source on tight closure theory in equal characteristic zero is [HH2].

The definition of tight closure in equal characteristic zero is based on the existence of *descent data*, which provide us with a finitely generated subalgebra R_D of the ring over a finitely generated algebra D over the integers. We can then look at fibers of R_D over maximal ideals of D, which are rings of positive characteristic over finite (and therefore perfect) fields. We define tight closure for R via the (usual positive characteristic) definition of tight closure for these fibers.

Definition 1.13. Let R be a finitely generated algebra over a field K of characteristic zero, let I be an ideal of R and let $x \in R$. By *descent data* for R, I, and x we mean a triple (D, R_D, I_D), where D is a finitely generated $\mathbf{Z}$-subalgebra of K, R_D is a finitely generated D-subalgebra of R, and I_D is an ideal of R_D such that:

(a) I_D and R_D/I_D are D-free.

(b) The canonical map $K \otimes_D R_D \to R$ induced by the inclusions of K and R_D in R is a K-algebra isomorphism.

(c) $I = I_D R$.

(d) $x \in R_D$.

Descent data always exists: see [HH2] Section 2.1.

Definition 1.14. Let R be a finitely generated algebra over a field K of characteristic zero and let I be an ideal of R. We say that an element x of R is in the *tight closure* of I, denoted by I^*, if there exist descent data (D, R_D, I_D) such that for every maximal ideal m of D, if $k = D/m$, then $x_k \in {I_k}^*$ in R_k, where the subscript k denotes images after applying $k \otimes_D$. If $I = I^*$, then we say that I is *tightly closed*.

In fact, by replacing D by a localization at a single element, one can see that it suffices to check that for *almost all* $m \in \text{MaxSpec}D$, i.e. for all m in a Zariski dense open subset of $\text{MaxSpec}D$, if $k = D/m$, then $x_k \in {I_k}^*$.

The following theorem follows from [HH2] 2.5.2 and 2.5.3:

Theorem 1.15 (independence of choice of descent and uniform multipliers). *Let K be a field of characteristic zero, let R be a finitely generated K-algebra, let I be an ideal of R and let $u \in R$. Let (D, R_D, I_D) be descent data for R, I, and u.*

(a) If $u \in I^*$, then for every maximal ideal m of D, if $k = D/m$, then $x_k \in {I_k}^*$ in R_k.

(b) There is an element c_D of R_D^o such that $u \in I^*$ iff for almost all maximal ideals m of D and $k = D/m$, $c_k {u_k}^q \in {I_k}^{[q]}$ for all positive powers q of p.

Part (b) involves the existence of *universal test elements*; these are the characteristic zero analogues of test elements ([HH2] 2.4.2). When R is reduced and equidimensional, $I \subseteq R$, $u \in I^*$ and $c_D \in R_D$ is a universal test element, then for almost all $m \in \text{MaxSpec}D$, if $k = D/m$, we have $c_k {u_k}^q \in {I_k}^{[q]}$ for all positive powers q of p. We will state a theorem due to Hochster and Huneke which enables us to explicitly calculate universal test elements in Section 3.5 (Theorem 3.23).

1.3 Tight Integral Closure

Tight integral closure (or TI closure) is an operation on a *set* of ideals in a Noetherian ring that generalizes the ideas of tight and integral closure. This notion was introduced by Hochster in 1998 ([Ho2]). Below, we give a brief description of TI closure and refer the reader to [Ho2] for properties of this operation.

Definition 1.16. Let R be a Noetherian commutative ring of prime characteristic p, and let $I_1, \ldots, I_n$ be ideals in R. We define $x \in R$ to be in $(I_1, \ldots, I_n)^{\underline{*}}$, called the *tight integral closure* (or *TI closure*) of $I_1, \ldots, I_n$, if and only if there exists an element $c \in R^o$ such that $cx^q \in I_1^q + \ldots + I_n^q$, for all large powers q of p.

Note that if $n = 1$, $(I)^{\underline{*}} = \overline{I}$, and if $I_1, \ldots, I_n$ are all principal ideals, then $(I_1, \ldots, I_n)^{\underline{*}} = (I_1 + \ldots + I_n)^*$.

It turns out that TI closure has several of the basic properties of both tight and integral closure: its study reduces to the case of domains and it respects inclusions of ideals. In the case of a set of monomial ideals in a polynomial ring, their TI closure can be computed as the sum of their integral closures (see [Ho2] for details of these results).

Using TI closure, Hochster was able to prove an improved version of the Briançon-Skoda theorem ([Ho2] Theorem 2.3). Below we demonstrate by an example this result in a case where TI closure can be computed.

Example 1.17. Let R be the polynomial ring $k[x, y, u, v]$ over a field k of characteristic p, and let $I = (u^2, v^2)$ and $J = (x^2, y^2)$ be ideals of R. Then it follows from the TI closure version of the Briançon-Skoda theorem ([Ho2] Theorem 2.3), that $\overline{(I+J)^2} \subseteq (I, J)^{\underline{*}}$. So

$$(u, v, x, y)^4 = \overline{(u^2, v^2, x^2, y^2)^2} \subseteq \big((u^2, v^2), (x^2, y^2)\big)^{\underline{*}} = \overline{(u^2, v^2)} + \overline{(x^2, y^2)} = (u^2, v^2, uv, x^2, y^2, xy).$$

This is an improvement of the tight closure Briançon-Skoda theorem ([HH1] Theorem 5.4), which would yield $\overline{(I+J)^4} \subseteq (I+J)^*$, and hence

$$(u,v,x,y)^8 = \overline{(u^2,v^2,x^2,y^2)^4} \subseteq (u^2,v^2,x^2,y^2)^* = (u^2,v^2,x^2,y^2).$$

TI closure can be defined for affine algebras over fields of characteristic zero by the method of reduction to characteristic p, described in Section 1.2. In [Ho2], characteristic zero analogues of several of the results on TI closure are stated.

Definition 1.18. Let R be a finitely generated algebra over a field K of characteristic zero, and let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a set of ideals of R. We say that an element x of R is in the *tight integral closure of* $\mathcal{I}$, denoted by $\mathcal{I}^{\underline{*}}$, if there exist descent data $(D, R_D, I_{1,D}, \ldots, I_{n,D})$, such that for every maximal ideal m of D, if $k = D/m$, then $x_k \in \mathcal{I}_k{}^{\underline{*}}$, where $\mathcal{I}_k$ denotes the set of ideals $\{I_{1,k}, \ldots, I_{n,k}\}$, with $I_{t,k} = k \otimes_k I_{t,D}$.

1.4 Questions on TI closure

As can be seen above, TI closure is able to generalize several statements, and to tie up tight closure and integral closure into one definition. However, there are many useful properties that both tight and integral closure satisfy, but have been turned out to be difficult to verify for TI closure using Definition 1.16. In [Ho2], Hochster stated the following questions:

Question 1.19. Does TI closure persist? That is, if $h: R \longrightarrow S$ is a homomorphism of Noetherian rings of characteristic p, and $I_1, \ldots, I_n$ are ideals of R, then is it true that $(I_1, \ldots, I_n)^{\underline{*}}\, S \subseteq (I_1 S, \ldots, I_n S)^{\underline{*}}$?

This property holds for both tight closure (with mild conditions on the ring, see Theorem 1.11) and for integral closure. It is easy to show that it also holds for for TI closure if $h(R^o) \subseteq S^o$; this happens for example when h is a flat map, or if h is any injective map of domains. In general, the lack of a test element theory for TI closure makes the problem obscure in its original setting. This question will be answered in Section 3.2.

Question 1.20. If R is a Noetherian ring of characteristic p, $I_1, \ldots, I_n$ are ideals of R and $c \in R^o$ and $x \in R$ are such that

$$cx^{p^e} \in {I_1}^{p^e} + \ldots + {I_n}^{p^e},$$

for infinitely many e (rather than *all large* e), then can one conclude that $x \in (I_1, \ldots, I_n)^{\underline{*}}$?

This is again a property that holds for both tight and integral closure, and it is reasonable to expect it for TI closure. In Section 3.2 we give an affirmative answer to this question.

Question 1.21. Can one develop a theory of test elements for TI closure (see Definition 1.8)?

Such a theory exists for tight closure (see Section 1.2), but it is not possible to define such a notion for integral closure that does not depend on the ideal:

Example 1.22. Let R be the polynomial ring $k[x,y]$, where k is a field. If $c \in R - \{0\}$ were an integral closure test element for R, then one would have $cz^m \in I^m$ for all ideals I of R, all $z \in \overline{I}$ and all positive integers m.

Now take the family of ideals $\{I_n\}_{n \in \mathbf{N}}$, where for each n, $I_n = (x^{2n}, y^{2n})$. Then $x^n y^n \in \overline{I_n}$ for all n, since $(x^n y^n)^2 = x^{2n} y^{2n} \in {I_n}^2$. As c is a test element, this implies that $cx^n y^n \in (x^{2n}, y^{2n})$ for all n. Since this holds in the polynomial ring $k[x,y]$, one can reduce to the case where c is a monomial in $R = k[x,y]$, which is not possible, since the degree of c will have to grow larger when n gets large.

Still, one could hope to find elements c that work for Definition 1.16, but depend on the ideals $I_1, \ldots, I_n$. Such specific elements of R are introduced in Section 3.3.

Question 1.23. Does TI closure commute with localization?

It is known that integral closure commutes with localization, and the same is conjectured for tight closure. In Section 3.2 we show that the question of TI closure commuting with localization is equivalent to the question of tight closure commuting with localization.

Question 1.24. Let R be an affine algebra over a field K of characteristic zero, and $\mathcal{I}$ be a set of ideals in R and $x \in \mathcal{I}^{\underline{*}}$. Let $(D, R_D, \mathcal{I}_D)$ be descent data. Then can one find a $c \in R_D^o$ such that for all $m \in \mathrm{MaxSpec}D$ and $k = D/m$, $c_k {x_k}^q \in {\mathcal{I}_k}^{[q]}$ for all positive powers q of p?

The last question will be addressed in Section 3.5.

2 Blowup Closure

We now explore a new notion: the blowup closure of a set of ideals. Here, we are motivated by the fact that the extension of an ideal I to its blowup scheme $\mathrm{Proj}R[It]$ is locally principal. For a set of ideals, we consider the blowup scheme of their product, and each one of the original ideals will be locally principal there. This is the simplest situation to handle several ideals at the same time.

Although the definition of blowup closure makes sense for any commutative ring of positive characteristic, we immediately restrict ourselves to rings R such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite. The reason is that tight closure persists for maps from such rings (Theorem 1.11), and this property simplifies most blowup closure arguments. These conditions are not very restrictive, since the class of such rings includes most rings that one would normally encounter in commutative algebra and in algebraic geometry (see Note 1.9).

Definition 2.1. Let R be a Noetherian commutative ring of prime characteristic p, and let $I_1, \ldots, I_n$ be ideals in R. We define $x \in R$ to be in $(I_1, \ldots, I_n)^{\sim}$, called the *blowup closure* of $I_1, \ldots, I_n$, if and only if for every affine open set of the blowup of the product ideal $I = I_1 \ldots I_n$, if S is the coordinate ring of that affine set, then $x \in (JS)^*$, where $J = I_1 + \ldots + I_n$.

The following discussion shows how for certain rings, one can reduce the process of checking if an element x is in $(I_1, \ldots, I_n)^{\sim}$ to checking if it is in $(JS_i)^*$, $1 \leq i \leq m$, for any fixed open affine cover $\mathrm{Spec}S_1, \ldots, \mathrm{Spec}S_m$ of the blowup of I.

Lemma 2.2. *Let R be a ring of positive characteristic p, and let J be an ideal of R. Then for $x \in R$, $x \in J^*$ if and only if there are $g_1, \ldots, g_s \in R$ that generate the unit ideal in R, and for each j, $1 \leq j \leq s$, $x \in (J_{g_j})^*$.*

Proof. If $x \in J^*$, it is immediate that $x \in (JR_f)^*$ for all $f \in R$, therefore one direction is clear. Suppose we are given a sequence of elements $g_1, \ldots, g_s \in R$ such that $(g_1, \ldots, g_s) = R$, and for each j, $1 \leq j \leq s$, $x \in (J_{g_j})^*$.

For each j, let c_j be such that $c_j x^{p^e} \in {J_{g_j}}^{[p^e]}$ for all large enough e. One can replace each c_j by its product with a large enough power of g_j, so that $c_j \in R$. Let c be the product of all the c_j's for $j = 1, \ldots, s$. Then $c \in R$ and $cx^{p^e} \in {J_{g_j}}^{[p^e]}$ for all $j = 1, \ldots, s$, and all large enough e.

Fix p^e. Then for each j, there is some power N_j of g_j such that $g_j^{N_j} cx^{p^e} \in J^{[p^e]}$. Let q be a power of p that is larger than all the N_j for $j = 1, \ldots, s$. Then for all j, $g_j^q cx^{p^e} \in J^{[p^e]}$. On the other hand, since $(g_1, \ldots, g_s)$ is the unit ideal, so is $(g_1^q, \ldots, g_s^q)$, and so it follows that $cx^{p^e} \in J^{[p^e]}$. Therefore $x \in J^*$.

$\square$

Now let X be a scheme, and let $\mathcal{J}$ be a sheaf of ideals on X. Suppose that there is an open affine cover $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of X, such that for each i, $\mathcal{U}_i = \mathrm{Spec} B_i$, where B_i is essentially of finite type over an excellent local ring, or $(B_i)_{red}$ is F-finite. Suppose $x \in \mathcal{J}(\mathcal{U}_i)^*$ in $\mathcal{O}_X(\mathcal{U}_i)$, for $i = 1, \ldots, n$. Let $\mathcal{U}$ be any open affine set of X. We will show that $x \in \mathcal{J}(\mathcal{U})^*$.

One can write $\mathcal{U}$ as $(\mathcal{U} \cap \mathcal{U}_1) \cup \ldots \cup (\mathcal{U} \cap \mathcal{U}_n)$. If $\mathcal{U} = \mathrm{Spec} A$, then one can refine this cover of $\mathcal{U}$ into

$$\mathrm{Spec} A_{f_1} \cup \ldots \cup \mathrm{Spec} A_{f_s},$$

where the elements $f_1, \ldots, f_s$ of A generate the unit ideal of A. So for every j, we have an inclusion $\mathrm{Spec} A_{f_j} \subseteq \mathcal{U}_i$, for some i, $1 \leq i \leq n$. This corresponds to a homomorphism of rings $B_i \longrightarrow A_{f_j}$. Since $x \in \mathcal{J}(\mathrm{Spec} B_i)^*$, from the persistence of tight closure it follows that $x \in \mathcal{J}(\mathrm{Spec} A_{f_j})^*$. Since the $f_1, \ldots, f_s$ generate the unit ideal, Lemma 2.2 implies that $x \in \mathcal{J}(\mathcal{U})^*$.

We have thus proved that:

Theorem 2.3. *Let R be a Noetherian commutative ring of prime characteristic p, such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals in R. If x is an element of R, then $x \in (I_1, \ldots, I_n)^\sim$ if and only if for every non-nilpotent element f of a fixed set of generators for the product ideal $I = I_1 \ldots I_n$,*

$$x \in (JR[It]_{(ft)})^*,$$

where $J = I_1 + \ldots + I_n$, and $R[It]_{(ft)}$ is the zeroth graded piece of the localized Rees ring $R[It]_{ft}$.

Note 2.4. Let R be a ring as above, and $I_1, \ldots, I_n$ be ideals of R with $I = I_1 \ldots I_n$ and $J = I_1 + \ldots + I_n$. Let $\mathcal{G}$ be a fixed set of generators for the ideal I such that every f in $\mathcal{G}$ is of the form $f = f_1 \ldots f_n$, where each f_i is an element of a fixed set of generators $f_1^i, \ldots, f_{s_i}^i$ for I_i. For future reference we show what $(JR[It]_{(ft)})^*$ looks like in practice when f is not nilpotent.

$$\begin{aligned}
(JR[It]_{(ft)})^* &= ((I_1 + \ldots + I_n)R[I_1 \ldots I_n t]_{(f_1 \ldots f_n t)})^* \\
&\simeq ((I_1 + \ldots + I_n)R[\tfrac{I_1}{f_1}\tfrac{I_2}{f_2} \ldots \tfrac{I_n}{f_n}])^* \\
&= ((f_1, \ldots, f_n)R[\tfrac{I_1}{f_1}\tfrac{I_2}{f_2} \ldots \tfrac{I_n}{f_n}])^* \\
&= ((f_1, \ldots, f_n)R[\tfrac{f_1^1}{f_1}, \ldots, \tfrac{f_{s_1}^1}{f_1}, \ldots, \tfrac{f_1^n}{f_n}, \ldots, \tfrac{f_{s_n}^n}{f_n}])^* \\
&= ((f_1, \ldots, f_n)R[\tfrac{I_1}{f_1}, \tfrac{I_2}{f_2}, \ldots, \tfrac{I_n}{f_n}])^*.
\end{aligned}$$

2.1 Blowup Closure Can Be Tested Modulo Minimal Primes

Theorem 2.5. *Let R be a Noetherian ring of positive characteristic such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite, and let $I_1, \ldots, I_n$ be ideals in R. Then $x \in (I_1, \ldots, I_n)^\sim$ if and only if $\overline{x} \in (I_1 R/p, \ldots, I_n R/p)^\sim$, for all minimal primes p of R, where $\overline{x}$ is the image of x in R/p.*

We first specify the structure of the minimal primes of Rees rings.

Proposition 2.6 (minimal primes of Rees rings). *Let R be a Noetherian ring, and $p_1, \ldots, p_m$ be the minimal primes of R. If I is any ideal of R, then:*

(a) The minimal primes of $R[It]$ are $\hat{p_1}, \ldots, \hat{p_m}$, where for $1 \leq i \leq m$,

$$\hat{p_i} = p_i R[t] \cap R[It] = p_i \oplus (I \cap p_i)t \oplus (I^2 \cap p_i)t^2 \oplus \ldots ;$$

(b) For all primes p of R, $R[It]/\hat{p} \simeq (R/p)[I(R/p)t]$.

Proof. (*a*) See [Va], Proposition 1.1 part (*iii*).

(*b*) For $p \in \operatorname{Spec} R$, construct the map

$$\phi : R[It] \longrightarrow R/p[I(R/p)t]$$

with $\phi(xt^n) = \overline{x}t^n$, where $\overline{x}$ is the image of x in R/p.

This is a surjective homomorphism of graded rings. To find the kernel, we observe that $\phi(xt^n) = 0$ if and only if $\overline{x}t^n = 0$. So xt^n is in the kernel of ϕ if and only if $x \in I^n \cap p$. So the kernel of ϕ is equal to $\hat{p}$, and hence we have an isomorphism $R[It]/\hat{p} \simeq (R/p)[I(R/p)t]$. □

Corollary 2.7. *Let R be a Noetherian ring, and let I be an ideal of R. Suppose ft^n is a homogeneous element of $R[It]$ for some positive integer n.*

(a) There is a one to one correspondence between the minimal primes of $R[It]$ that do not contain ft^n, and the minimal primes of $R[It]_{(ft^n)}$.

(b) Suppose $f \in I$ and p is a minimal prime of R not containing f. Let $\hat{p}$ be the minimal prime of $R[It]$ corresponding to p, and let p' be the minimal prime of $R[It]_{(ft)}$ corresponding to $\hat{p}$. Then

$$\frac{R[It]_{(ft)}}{p'} \simeq \left(\frac{R[It]}{\hat{p}}\right)_{(ft)} \simeq (R/p[(IR/p)t])_{(ft)}.$$

Proof. The statement of part (a) is equivalent to saying that there is a one to one correspondence between $\mathcal{M}(R[It]_{ft^n})$ and $\mathcal{M}(R[It]_{(ft^n)})$.

When $n = 1$, we have $R[It]_{ft} \simeq R[It]_{(ft)}[u, u^{-1}]$. This is because $R[It]_{(ft)} \simeq R[g_1/f, \ldots, g_s/f]$, where $g_1, \ldots, g_s$ is a fixed set of generators for I. We can then define the map

$$R[g_1/f, \ldots, g_s/f][u, u^{-1}] \longrightarrow R[It]_{ft}$$

by sending u^m to $(ft)^m$ for all nonzero integers m. It is easy to check that this map is an isomorphism, and it follows that all members of $\mathcal{M}(R[It]_{ft})$ are extensions of those in $\mathcal{M}(R[It]_{(ft)})$.

If $n > 1$, then $R[It]_{(ft^n)} \simeq R[\frac{I^n}{f}]$, where by $\frac{I^n}{f}$ we mean all elements of the form $\frac{x}{f}$, where $x \in I^n$. This is isomorphic to $(R[It]^{(n)})_{(ft)}$, where $R[It]^{(n)} = R[I^n t]$ is the nth Veronese subring of $R[It]$. By the previous paragraph, we know that that there is a one to one correspondence between $\mathcal{M}((R[It]^{(n)})_{ft})$ and $\mathcal{M}((R[It]^{(n)})_{(ft)})$. On the other hand, the homogeneous primes of $R[It]^{(n)}$ are contractions of the homogeneous primes of $R[It]$ (see [E]). Since all the minimal primes of $R[It]$ are homogeneous by Proposition 2.6, it follows again that $\mathcal{M}(R[It]_{ft^n})$ and $\mathcal{M}(R[It]_{(ft^n)})$ correspond. This settles part (a).

To prove part (b), from part (a) we notice that since $R[It]_{ft} \simeq R[It]_{(ft)}[u, u^{-1}]$, we have

$$\frac{R[It]_{(ft)}}{p'}[u, u^{-1}] \simeq \frac{R[It]_{ft}}{\hat{p}R[It]_{ft}} \simeq \left(\frac{R[It]}{\hat{p}}\right)_{ft} \simeq \left(\frac{R[It]}{\hat{p}}\right)_{(ft)}[u, u^{-1}],$$

where the second isomorphism is because localization is flat, and the third follows again from part (a) of this theorem along with part (b) of Proposition 2.6. Therefore

$$\frac{R[It]_{(ft)}}{p'} \simeq \left(\frac{R[It]}{\hat{p}}\right)_{(ft)},$$

and combining this with Proposition 2.6 part (b), we obtain the desired result. □

Proof of Theorem 2.5. Let $I = I_1 \dots I_n$, $J = I_1 + \dots + I_n$, and $\mathcal{G}$ be a fixed set of generators for I.

Take $x \in (I_1, \dots, I_n)^{\sim}$. Then for any $f \in \mathcal{G}$,

$$x \in (JR[It]_{(ft)})^*. \tag{1}$$

Since tight closure can be tested modulo minimal primes (see Section 1.2), we see that Equation 1 is equivalent to

$$\overline{x} \in \left(J\frac{R[It]_{(ft)}}{p'}\right)^*,$$

for every minimal prime p' of $R[It]_{(ft)}$, which by part (a) of Corollary 2.7 and Proposition 2.6 corresponds to a minimal prime p of R that does not contain f. From Corollary 2.7 part (b) we see that

$$\overline{x} \in \left(J\frac{R[It]_{(ft)}}{p'}\right)^* \simeq \left(J\left(\frac{R[It]}{\hat{p}}\right)_{(ft)}\right)^* \simeq \left(JR/p[I(R/p)t]_{(ft)}\right)^*.$$

This holds for all minimal primes p of R that do not contain f. Hence we equivalently have

$$\overline{x} \in (I_1R/p, \dots, I_nR/p)^{\sim}$$

for all minimal primes p of R. ☐

2.2 Basic Properties of Blowup Closure

The following fact, which follows from the contraction property of tight closure, makes the computation of blowup closure simpler in several cases.

Proposition 2.8. *Let R be a Noetherian commutative ring of prime characteristic p, such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \dots, I_n$ be ideals in R, $I = I_1 \dots I_n$, $J = I_1 + \dots + I_n$, $f \in I$ and $x \in R$. Then*

$$x \in (JR[It]_{(ft)})^* \quad \textit{iff} \quad x \in (JR[It]'_{(ft)})^*,$$

where $R[It]'$ is the normalization of $R[It]$.

Proof. Let $p_1, \dots, p_m$ be the set of minimal primes of $R[It]$ that do not contain ft. These correspond to minimal primes $p'_1, \dots, p'_m$ of $R[It]_{(ft)}$ (Corollary 2.7) and minimal primes ${p_1}^\sharp, \dots, {p_m}^\sharp$ of $R[It]'_{(ft)}$ (Proposition 1.2 and [E] proposition 4.13).

Suppose $x \in (JR[It]'_{(ft)})^*$. Since tight closure can be tested modulo minimal primes (see [BH] Proposition 10.1.2(e)), we see that for $i = 1, \dots, m$,

$$\overline{x} \in \left(J\frac{R[It]'_{(ft)}}{{p_i}^\sharp}\right)^*,$$

where $\overline{x}$ is the image of x in the ring R/p_i. But Proposition 1.2 implies that

$$\frac{R[It]'_{(ft)}}{{p_i}^\sharp} \simeq \left(\frac{R[It]_{(ft)}}{p'_i}\right)',$$

so $\overline{x}$ (or rather the image of $\overline{x}$ under this isomorphism) belongs to

$$\left(J\left(\frac{R[It]_{(ft)}}{{p'}_i}\right)'\right)^*.$$

Therefore, by the contraction property of tight closure ([Hu] Theorem 1.7), for all i,

$$\overline{x} \in \left(J \left(\frac{R[It]_{(ft)}}{p_i'} \right)' \right)^* \bigcap \frac{R[It]_{(ft)}}{p_i'} \subseteq \left(J \left(\frac{R[It]_{(ft)}}{p_i'} \right) \right)^*.$$

Applying Proposition 10.1.2(e) of [BH] again, we see that $x \in (JR[It]_{(ft)})^*$.

The reverse inclusion follows because of the inclusion of $R[It]_{(ft)}$ in $R[It]'_{(ft)}$, and the persistence of tight closure (Theorem 1.11). $\square$

Theorem 2.9. *Let R be a Noetherian ring of prime characteristic p, such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite, and let $I_1, \ldots, I_n$ be ideals of R. Then:*

(a) The ideal $(I_1, \ldots, I_n)^\sim$ is tightly closed, and

$$(I_1, \ldots, I_n)^{\underline{*}} \subseteq (I_1, \ldots, I_n)^\sim.$$

(b) If all $I_1, \ldots, I_n$ are principal, then $(I_1, \ldots, I_n)^\sim = (I_1 + \ldots + I_n)^*$.

(c) If $n = 1$, then $(I_1)^\sim = \overline{I_1}$.

(d) $(\overline{I_1}, \ldots, \overline{I_n})^\sim = (I_1, \ldots, I_n)^\sim$.

Proof. Throughout the proof, we let $\mathcal{G}$ be a fixed set of generators for the ideal $I = I_1 \ldots I_n$, such that every f in $\mathcal{G}$ is of the form $f = f_1 \ldots f_n$, where each f_i is an element of a fixed set of generators for I_i. We let $J = I_1 + \ldots + I_n$.

(a) We can assume that R is a domain (Theorem 2.5 and [Ho2] Proposition 1.4).

For every non-nilpotent element f in $\mathcal{G}$, tight closure persists under the map $R \longrightarrow R[It]_{(ft)}$, and so $(JR[It]_{(ft)})^* \cap R$ is tightly closed in R (see Remark 1.12). On the other hand, the intersection of tightly closed ideals is tightly closed, and since $(I_1, \ldots, I_n)^\sim$ is the intersection of finitely many ideals of the form $(JR[It]_{(ft)})^* \cap R$, it follows that $(I_1, \ldots, I_n)^\sim$ is tightly closed in R.

Now let $z \in (I_1, \ldots, I_n)^{\underline{*}}$. Then $cz^q \in I_1^q + \ldots + I_n^q$, for all large $q = p^e$ and some nonzero $c \in R$. If $f \in \mathcal{G}$, and $f = f_1 \ldots f_n$ where f_i is a generator of I_i as above, then this equation can be extended to

$$\begin{aligned} cz^q &\in I_1^q R[It]_{(ft)} + \ldots + I_n^q R[It]_{(ft)} \\ &= (f_1)^q R[It]_{(ft)} + \ldots + (f_n)^q R[It]_{(ft)} \quad \text{(see Note 2.4)} \\ &= (f_1, \ldots, f_n)^{[q]} R[It]_{(ft)} \\ &= J^{[q]} R[It]_{(ft)} \\ &= (JR[It]_{(ft)})^{[q]} \end{aligned}$$

for all large powers q of p. It follows that $z \in (JR[It]_{(ft)})^*$. Since this holds for all $f \in \mathcal{G}$, by 2.3 we conclude that $z \in (I_1, \ldots, I_n)^\sim$.

(b) We can again assume that R is a domain. If $I_1, \ldots, I_n$ are principal, then so is their product I. So for each $f \in \mathcal{G}$, $R[It]_{(ft)}$ is the same as R. We therefore have

$$(I_1, \ldots, I_n)^\sim = (I_1 + \ldots + I_n)^*.$$

(c) Assume that R is a domain. In this case, $I = J = I_1$. We use the "normalized" definition of blowup closure, following Proposition 2.8, and we obtain

$$(I)^\sim = \bigcap_{f \in \mathcal{G}} IR[It]'_{(ft)} \cap R = \overline{I},$$

since the extension of I to the normal ring $R[It]'_{(ft)}$ is principal , and therefore tightly closed (see Proposition 1.4 and [HH1] Corollary 5.8).

(d) We use the "normalized" definition of blowup closure, following Proposition 2.8.

Let $J' = \overline{I_1} + \ldots + \overline{I_n}$ and $I' = \overline{I_1} \ldots \overline{I_n}$. Then $R[It]' = R[I't]'$, since $\overline{(I')^n} = \overline{I^n}$, and so $\mathrm{Proj} R[I't]'$ like $\mathrm{Proj} R[It]'$ is covered by affines of the form $\mathrm{Spec} R[It]'_{(ft)}$ for $f \in \mathcal{G}$. On the other hand, suppose $f = f_1 \ldots f_n \in \mathcal{G}$ where $f_i \in I_i$, $i = 1, \ldots, n$.

Then

$$(J'R[It]'_{(ft)})^* = (J'R[I't]'_{(ft)})^* = ((f_1, \ldots, f_n)R[I't]'_{(ft)})^* = (JR[It]'_{(ft)})^*,$$

as pointed in Note 2.4. It follows that $(\overline{I_1}, \ldots, \overline{I_n})^{\sim} = (I_1, \ldots, I_n)^{\sim}$. □

Theorem 2.10 (blowup closure from contractions). *Suppose that R and S are Noetherian domains of prime characteristic p, such that R is either essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals of R. Suppose S is a module finite extension of R. Then*

$$(I_1 S, \ldots, I_n S)^{\sim} \cap R \subseteq (I_1, \ldots, I_n)^{\sim}$$

.

Proof. Let I and J be the product and the sum of $I_1, \ldots, I_n$, respectively, and let $\mathcal{G}$ be a fixed set of generators for I in R.

Suppose S is generated as an R-module by elements $u_1, \ldots, u_m$. Then $u_1, \ldots, u_m$ also generate $S[(IS)t]$ as a module over $R[It]$. To see this, take a homogeneous element $zt^n \in S[(IS)t]$. Then $z \in I^n S$, and so if $I^n = (v_1, \ldots, v_r)$ in R, we can write $z = a_1 v_1 + \ldots + a_r v_r$ for $a_1, \ldots, a_r \in S$. On the other hand for $i = 1, \ldots, r$, $a_i = a_{i1} u_1 + \ldots + a_{im} u_m$, where $a_{ij} \in R$ for $j = 1, \ldots, m$. Rewriting the equation describing z above, we have $z = b_1 u_1 + \ldots + b_m u_m$, where $b_j \in I^n$ for $j = 1, \ldots, m$. It follows that $zt^n = (b_1 t^n) u_1 + \ldots + (b_m t^n) u_m$. So $S[(IS)t]$ is module finite over $R[It]$. A similar argument shows that $S[(IS)t]_{(ft)}$ is module finite over $R[It]_{(ft)}$.

Now let $z \in (I_1 S, \ldots, I_n S)^{\sim} \cap R$. Then by definition, z belongs to

$$\begin{aligned} & \left(JS[(IS)t]_{(ft)}\right)^* \cap R \\ & = \left(JS[(IS)t]_{(ft)}\right)^* \cap R[It]_{(ft)} \cap R \\ & \subseteq \left(JR[It]_{(ft)}\right)^* \cap R \qquad \text{by contraction of tight closure ([Hu] 1.7)} \end{aligned}$$

This holds for all $f \in \mathcal{G}$, which implies that $z \in (I_1, \ldots, I_n)^{\sim}$. □

Theorem 2.11 (persistence of blowup closure). *Let $\phi : R \longrightarrow S$ be a homomorphism of Noetherian rings of prime characteristic p, and let $I_1, \ldots, I_n$ be ideals of R. Suppose that either R is essentially of finite type over an excellent local ring, or that R_{red} is F-finite. Then:*

$$(I_1, \ldots, I_n)^{\sim} S \subseteq (I_1 S, \ldots, I_n S)^{\sim}.$$

Proof. Let $I = I_1 \ldots I_n$ and $J = I_1 + \ldots + I_n$. Fix a finite set of generators for I, and let $f = f_1 \ldots f_n$, $f_i \in I_i$, be in that set. If R is essentially of finite type over an excellent local ring or R_{red} is F-finite, then $R[It]_{(ft)}$ will have the same property since it is an algebra of finite type over R.

On the other hand, if $\phi(f)$ is not nilpotent, then ϕ induces a map

$$R[It]_{(ft)} \to S[(IS)t]_{(\phi(f)t)} ,$$

under which tight closure persists (see Theorem 1.11). It follows that when $\phi(f)$ is not nilpotent,

$$\left(R \cap (JR[It]_{(ft)})^*\right) S \subseteq (JR[It]_{(ft)})^* S[(IS)t]_{(\phi(f)t)} \subseteq \left(JS[(IS)t]_{(\phi(f)t)}\right)^*.$$

If $g_1, \ldots, g_r$ is a set of generators for I in R, then IS is generated by the $\phi(g_i)$, $i = 1, \ldots, r$, that are nonzero. From the discussion above and Theorem 2.3, it follows that

$$(I_1, \ldots, I_n)^{\sim} S \subseteq (I_1 S, \ldots, I_n S)^{\sim}.$$

$\square$

Blowup closure satisfies most properties that TI closure does. However, in many cases these two operations do not produce the same ideal. Here is an example of ideals in a polynomial ring for which these two operations are not the same:

Example 2.12. Let $R = k[x, y]$ be a polynomial ring over a field k of characteristic p. Consider the ideals $I = (y^3)$ and $J = (x^3, x^2y)$. Notice that I and J are both integrally closed ideals. We know from [Ho2] Proposition 1.9 that

$$xy^2 \notin (I, J)^{\underline{*}} = \overline{I} + \overline{J} = (x^3, y^3, x^2y).$$

We show that, however, $xy^2 \in (I, J)^{\sim}$. To show this, we check the ideal $I + J$ against two localized Rees rings. With notation as in Note 2.4, if we take $f_1 = y^3$ and $f_2 = x^2y$, we have

$$xy^2 \in (y^3, x^2y)k[x, y, \frac{x}{y}],$$

since $xy^2 = y^3(x/y)$.

If we take $f_1 = y^3$ and $f_2 = x^3$, we have

$$xy^2 \in (y^3, x^3)k[x, y, \frac{y}{x}],$$

since $xy^2 = x^3(y/x)^2$.

Therefore $xy^2 \in (I, J)^{\sim}$, but $xy^2 \notin (I, J)^{\underline{*}}$.

Moreover, blowup closure fails to respect inclusions: If $J_1, \ldots, J_n$ is a set of ideals such that $I_i \subseteq J_i$, for $i = 1, \ldots, n$, then the inclusion $(I_1, \ldots, I_n)^{\sim} \subseteq (J_1, \ldots, J_n)^{\sim}$ does not necessarily hold. Here is an example:

Example 2.13. Let $R = k[x, y, u, v]$. Let $I_1 = (x^3, x^2y)$, $I_2 = (y^3)$, $J_1 = (x^3, x^2y, u)$ and $J_2 = (y^3, v)$. Then $xy^2 \in (I_1, I_2)^{\sim}$ as was shown in the previous example. But $xy^2 \notin (J_1, J_2)^{\sim}$, because looking at the affine patch corresponding to the generators u and v of J_1 and J_2, respectively, we can see that

$$xy^2 \notin (u, v)k[x, y, u, v, \frac{x^3}{u}, \frac{x^2y}{u}, \frac{y^3}{v}].$$

We show in the next section that TI closure can be described as the contraction of the blowup closure of some ideals in a larger ring. Before discussing this, we demonstrate a case where blowup closure can be directly calculated.

2.3 The Case of Monomial Ideals in a Polynomial Ring

Theorem 2.14. *Let $R = k[x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^n, \ldots, x_{m_n}^n]$ be a polynomial ring in distinct variables x_j^i, $1 \leq i \leq n$ and $1 \leq j \leq m_i$, over an algebraically closed field k of prime characteristic p. Let $I_1, \ldots, I_n$ be monomial ideals in R, where the generators of I_i are monomials in the variables $x_1^i, \ldots, x_{m_i}^i$ for $1 \leq i \leq n$. Then $(I_1, \ldots, I_n)^{\sim} = \overline{I_1} + \ldots + \overline{I_n}$.*

To prove this theorem, we will show that $(I_1, \ldots, I_n)^\sim$ is a monomial ideal. Then, for a given monomial $M \in (I_1, \ldots, I_n)^\sim$, we will show that for some $1 \leq \alpha \leq n$,

$$M \in R \cap I_\alpha R[It]' = \overline{I_\alpha},$$

and it will follow that $(I_1, \ldots, I_n)^\sim = \overline{I_1} + \ldots + \overline{I_n}$.

We begin by fixing the notation. For each i, let $I_i = (f_1^i, \ldots, f_{s_i}^i)$, where f_j^i is a monomial in the polynomial ring $S_i = k[x_1^i, \ldots, x_{m_i}^i]$. Let $J = I_1 + \ldots + I_n = (f_1^1, \ldots, f_{s_1}^1, \ldots, f_1^n, \ldots, f_{s_n}^n)$, and $I = I_1 \ldots I_n = \left(f_{i_1}^1 \ldots f_{i_n}^n : 1 \leq i_j \leq s_j, 1 \leq j \leq n\right)$.

Our first goal is to show that

$$(I_1, \ldots, I_n)^\sim = \bigcap_{\substack{1 \leq i_j \leq s_j \\ 1 \leq j \leq n}} (f_{i_1}^1, \ldots, f_{i_n}^n) R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)} \cap R.$$

By 2.4, for a fixed index set $i_1, \ldots, i_n$, we are interested in the tight closure of the ideal $(f_{i_1}^1, \ldots, f_{i_n}^n)$ in the ring:

$$k[x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^n, \ldots, x_{m_n}^n, \frac{f_1^1}{f_{i_1}^1}, \ldots, \frac{f_{s_1}^1}{f_{i_1}^1}, \ldots, \frac{f_1^n}{f_{i_n}^n}, \ldots, \frac{f_{s_n}^n}{f_{i_n}^n}]. \tag{2}$$

Equivalently, by Theorem 2.8, we can study the tight closure of this ideal in the normalization of the ring described in 2. We claim that the tight closure of $(f_{i_1}^1, \ldots, f_{i_n}^n)$ in the normalization of the ring in 2 is equal to $(f_{i_1}^1, \ldots, f_{i_n}^n)$ itself.

To see this, let $f = f_{i_1}^1 \ldots f_{i_n}^n$. Since $R[It]$ is a monomial subring of $R[t]$, $R[It]'$ is also a monomial ring, and it is weakly F-regular (i.e., all ideals are tightly closed; see [Sm2]). Also, tight closure commutes with localization for $R[It]$ ([Sm2]). Since normalization also commutes with localization ([E] Proposition 4.13) it follows that

$$(JR[It]'_{ft})^* = JR[It]'_{ft}.$$

On the other hand,

$$\left(JR[It]'_{(ft)}\right)^* \subseteq \left((JR[It]'_{ft})^*\right)_0 = \left((JR[It]'_{ft})\right)_0 = JR[It]'_{(ft)},$$

and so $(JR[It]'_{(ft)})^* = JR[It]'_{(ft)}$.

We have therefore shown that if $I_1, \ldots, I_n$ are monomial ideals in a polynomial ring R, then

$$(I_1, \ldots, I_n)^\sim = \bigcap_{\substack{1 \leq i_j \leq s_j \\ 1 \leq j \leq n}} (f_{i_1}^1, \ldots, f_{i_n}^n) R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)} \cap R. \tag{3}$$

Notice that this argument does not require the distinction of the sets of variables generating $I_1, \ldots, I_n$.

Proposition 2.15. *Let $I_1, \ldots, I_n$ be monomial ideals (not necessarily generated by distinct variables) in a polynomial ring $R = k[u_1, \ldots, u_m]$ where k is an infinite field. Then $(I_1, \ldots, I_n)^\sim$ is also a monomial ideal in R.*

Proof. We use the fact that an ideal I in a polynomial ring $R = k[u_1, \ldots, u_m]$ over an infinite field k is generated by monomials, if and only if I is invariant under the action of the torus $(k^*)^m$ (see [Ho3] page 319), where k^* denotes $k - \{0\}$, and the torus action on R is defined as follows. If $\lambda = (\lambda_1, \ldots, \lambda_m) \in (k^*)^m$, and x is a monomial ${u_1}^{c_1} \ldots {u_m}^{c_m}$ of R, then

$$\lambda x = (\lambda_1 u_1)^{c_1} \ldots (\lambda_m u_m)^{c_m} = {\lambda_1}^{c_1} \ldots {\lambda_m}^{c_m} x,$$

and if x is a sum of monomials $M_1 + \ldots + M_r$, then $\lambda x = \lambda M_1 + \ldots + \lambda M_r$.

The action of the torus on monomials with negative powers is defined in a similar way. In the ring $R[{u_1}^{-1}, \ldots, {u_m}^{-1}]$, where R is as above, if $\lambda = (\lambda_1, \ldots, \lambda_m) \in (k^*)^m$ and $u = {u_1}^{c_1} \ldots {u_m}^{c_m}$, where the c_i are integers, then

$$\lambda u = (\lambda_1 u_1)^{c_1} \ldots (\lambda_m u_m)^{c_m} = {\lambda_1}^{c_1} \ldots {\lambda_m}^{c_m} u,$$

and if $u = M_1 + \ldots + M_r$, where M_i are monomials in the u_i with integer powers, then $\lambda u = \lambda M_1 + \ldots + \lambda M_r$.

We show that $(I_1, \ldots, I_n)^\sim$ is invariant under the action of the torus $(k^*)^m$. Suppose each I_i is generated by $f_1^i, \ldots, f_{s_i}^i$. By the discussion preceding the theorem we only need to prove that for any given index set $i_1, \ldots, i_n$, if

$$x \in (f_{i_1}^1, \ldots, f_{i_n}^n) R[It]'_{(f_{i_1}^1 \ldots f_{i_n}^n t)}$$

and $\lambda \in (k^*)^m$, then

$$\lambda x \in (f_{i_1}^1, \ldots, f_{i_n}^n) R[It]'_{(f_{i_1}^1 \ldots f_{i_n}^n t)}.$$

Take $\lambda = (\lambda_1, \ldots, \lambda_m)$ and x as above. Then x can be written as

$$x = A_1 f_{i_1}^1 + \ldots + A_n f_{i_n}^n,$$

with $A_1, \ldots, A_n \in R[It]'_{(f_{i_1}^1 \ldots f_{i_n}^n t)}$.

So

$$\begin{aligned} \lambda x &= \lambda(A_1 f_{i_1}^1) + \ldots + \lambda(A_n f_{i_n}^n) \\ &= (\lambda A_1)(\lambda f_{i_1}^1) + \ldots + (\lambda A_n)(\lambda f_{i_n}^n) \end{aligned}$$

Since each $f_{i_j}^j$ is a monomial in the u_i, $\lambda f_{i_j}^j$ will be just some scalar times $f_{i_j}^j$, and will therefore still belong to $(f_{i_1}^1, \ldots, f_{i_n}^n)$.

As for the A_j, we claim that λA_j still remains in $R[It]'_{(f_{i_1}^1 \ldots f_{i_n}^n t)}$ for $j = 1, \ldots, n$. To see this, fix some j. One can write A_j as

$$A_j = \frac{B_j}{(f_{i_1}^1 \ldots f_{i_n}^n)^r t^r},$$

where $B_j \in R[It]'$. We can then write B_j as

$$B_j = M_1 t^r + \ldots + M_{s_j} t^r,$$

where $M_1, \ldots, M_{s_j}$ are monomials of R that belong to $\overline{I^r}$ (see Theorem 1.5). So if we set $f = f_{i_1}^1 \ldots f_{i_n}^n$, then $A_j = M_1 f^{-r} + \ldots + M_{s_j} f^{-r}$, and so

$$\lambda A_j = \alpha_1 M_1 f^{-r} + \ldots + \alpha_{s_j} M_{s_j} f^{-r},$$

where $\alpha_1, \ldots, \alpha_{s_j} \in k$ are scalars. It follows that

$$\lambda A_j = \frac{\alpha_1 M_1 + \ldots + \alpha_{s_j} M_{s_j}}{f^r} \in R[It]'_{(ft)}.$$

Therefore $\lambda x \in (f_{i_1}^1, \ldots, f_{i_n}^n) R[It]'_{(ft)}$, and so we are done.

$\square$

Lemma 2.16. *Let $R = k[u_1, \ldots, u_m, \frac{M_1}{N_1}, \ldots, \frac{M_r}{N_r}]$, where $u_1, \ldots, u_m$ are distinct variables, and $M_1, \ldots, M_r$ and $N_1, \ldots, N_r$ are nonzero monomials in the polynomial ring $k[u_1, \ldots, u_m]$ over a field k. Suppose $M, g_1, \ldots, g_s$ are nonzero monomials in $k[u_1, \ldots, u_m]$ such that $M \in (g_1, \ldots, g_s)R$. Then for some i, $1 \leq i \leq s$, $M \in (g_i)R$.*

Proof. Since $M \in (g_1, \ldots, g_s)R$, there are $A_1, \ldots, A_s$ in R such that $M = A_1 g_1 + \ldots + A_s g_s$. After taking the common denominator A of the right hand side of the equation, and multiplying both sides of the equation by A, we end up with an equation of the form $AM = p_1 g_1 + \ldots + p_s g_s$, where $p_1, \ldots, p_s$ are polynomials in $k[u_1, \ldots, u_m]$, and for each i, $A_i = \frac{p_i}{A}$. Now notice that AM is a monomial, and so when you add the polynomials $p_1 g_1, \ldots, p_s g_s$ all terms that are not equal to a scalar multiple of AM cancel out with each other. So we can without loss of generality for each i, replace p_i with $\alpha_i Q_i$, where $\alpha_i \in k$ is nonzero if the monomial AM appears as a term of $p_i g_i$, and Q_i is a monomial in $k[u_1, \ldots, u_m]$ for which $Q_i g_i = AM$. So $AM = \alpha_1 Q_1 g_1 + \ldots + \alpha_s Q_s g_s$, and $\alpha_1 + \ldots + \alpha_s = 1$. Now it is clear that least one of the α_i must be nonzero; say α_1 is nonzero. Then $\alpha_1 Q_1$ is a term of p_1, and since $A_1 = \frac{p_1}{A} \in R$, $\frac{Q_1}{A} \in R$, and therefore $M = \frac{Q_1}{A} g_1 \in (g_1)R$. $\square$

A Noetherian ring R satisfies Serre's condition (R_n) if R_p is a regular local ring for $p \in \operatorname{Spec} R$ with $\dim R_p \leq n$. We say that R satisfies Serre's condition (S_n) if $\operatorname{depth} R_p \geq \min(n, \dim R_p)$ for all $p \in \operatorname{Spec} R$. For more on these conditions, see the first three chapters of [BH].

Proposition 2.17. *Let R and S be two domains, which are finitely generated k-algebras, where k is an algebraically closed field. Then:*

(a) For a positive integer n, if R and S satisfy (S_n), then so does $R \otimes_k S$.

(b) For a positive integer n, if R and S satisfy (R_n), then so does $R \otimes_k S$.

Proof. (a) This is Theorem 5.5.5 of [V].

(b) One can express R and S as

$$R = \frac{k[u_1, \ldots, u_m]}{(g_1, \ldots, g_s)} \quad \text{and} \quad S = \frac{k[v_1, \ldots, v_r]}{(h_1, \ldots, h_t)},$$

where $u_1, \ldots, u_m$ and $v_1, \ldots, v_r$ are distinct variables. Then

$$R \otimes_k S = \frac{k[u_1, \ldots, u_m, v_1, \ldots, v_r]}{(g_1, \ldots, g_s, h_1, \ldots, h_t)};$$

see [ZS1] for the tensor product of two rings.

Suppose R and S satisfy (R_n), that means the defining ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ of the singular loci R and S, respectively, have heights larger than n. The defining ideal $\mathcal{J}$ of the singular locus of $R \otimes_k S$, is the ideal generated by the $d \times d$ minors of the Jacobian matrix

$$\begin{pmatrix} \frac{\delta g_1}{\delta u_1} & \dots & \frac{\delta g_1}{\delta u_m} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\delta g_s}{\delta u_1} & \dots & \frac{\delta g_s}{\delta u_m} & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{\delta h_1}{\delta v_1} & \dots & \frac{\delta h_1}{\delta v_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\delta h_t}{\delta v_1} & \dots & \frac{\delta h_t}{\delta v_r} \end{pmatrix},$$

where d is the height of the ideal $(g_1, \dots, g_s, h_1, \dots, h_t)$ in the polynomial ring $k[u_1, \dots, u_m, v_1, \dots, v_r]$ (see Corollary 16.20 of [E]). On the other hand, $d = d_1 + d_2$, where d_1 is the height of $(g_1, \dots, g_s)$ in $k[u_1, \dots, u_m]$, and d_2 is the height of $(h_1, \dots, h_t)$ in $k[v_1, \dots, v_r]$ (see Chapter II of [Har]).

It is an easy exercise to see that the only time that the determinant of a square matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is nonzero is when A and B are both square matrices. So $\mathcal{J}$ is generated by determinants of the form

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = (\det A)(\det B),$$

where A and B are square matrices. In particular, it contains products of $d_1 \times d_1$ minors of the Jacobian of R with $d_2 \times d_2$ minors of the Jacobian of S. Therefore $\mathcal{J} \supseteq \mathcal{J}_1\mathcal{J}_2$, and hence $\operatorname{ht} \mathcal{J} \geq \min(\operatorname{ht} \mathcal{J}_1, \operatorname{ht} \mathcal{J}_2) > n$, since R and S are (R_n). Therefore $R \otimes_k S$ is smooth in codimension n.

$\square$

Corollary 2.18. *Let R and S be two normal domains, which are finitely generated k-algebras, where k is an algebraically closed field. Then $R \otimes_k S$ is also a normal domain.*

Proof. Serre's normality criterion says that a ring being normal is equivalent to it satisfying (S_2) and (R_1), and so by Proposition 2.17 $R \otimes_k S$ is normal. If K and K' are the quotient fields of R and S respectively, Theorem III.15.40 of [ZS1] proves that $K \otimes_k K'$ is a domain when k is algebraically closed, and since $R \otimes_k S$ is a subring of $K \otimes_k K'$, it follows that $R \otimes_k S$ is a domain. $\square$

Below, we adopt the following notation. If $R \otimes_k S$ is the tensor product of two finitely generated k-algebras R and S over the field k, and if $x \in R$ and $y \in S$, by $x \otimes_k y$ we mean the product of $x \otimes_k 1$ and $1 \otimes_k y$, where $x \otimes_k 1$ and $1 \otimes_k y$ are the images of x and y in $R \otimes_k S$ under the inclusions $R \longrightarrow R \otimes_k S$ and $S \longrightarrow R \otimes_k S$, respectively (see [ZS1]).

Lemma 2.19. *Let $R = k[x_1^1, \dots, x_{m_1}^1, \dots, x_1^n, \dots, x_{m_n}^n]$, I and J be as in Theorem 2.14. Let M be a monomial of R that belongs to $(I_1, \dots, I_n)^\sim$. Fix an index set $i_1, \dots, i_n$, such that $1 \leq i_v \leq s_v$, for $1 \leq v \leq n$. Suppose that for some fixed β, $1 \leq \beta \leq n$, $M \in (f_{i_\beta}^\beta)R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$. Then for any index set $j_1, \dots, j_n$, $1 \leq j_v \leq s_v$, $1 \leq v \leq n$, such that $j_\beta = i_\beta$ (so that $f_{j_\beta}^\beta = f_{i_\beta}^\beta$), we have $M \in (f_{j_\beta}^\beta)R[It]'_{(f_{j_1}^1 \cdots f_{j_n}^n t)}$.*

Proof. We know that $M \in (I_1, \ldots, I_n)^{\sim}$, so for all index sets $j_1, \ldots, j_n$ one has

$$M \in \left(JR[It]'_{(f^1_{j_1}\cdots f^n_{j_n}t)}\right)^* = (f^1_{j_1}, \ldots, f^n_{j_n})R[It]'_{(f^1_{j_1}\cdots f^n_{j_n}t)},$$

as we proved earlier in this section.

Since the $f^1_{j_1}, \ldots, f^n_{j_n}$ are monomials in distinct sets of variables, Corollary 2.18 implies that

$$R[It]'_{(f^1_{j_1}\cdots f^n_{j_n}t)} = S_1[I_1t]'_{(f^1_{j_1}t)} \otimes_k \ldots \otimes_k S_n[I_nt]'_{(f^n_{j_n}t)}, \tag{4}$$

where for each v, $1 \leq v \leq n$, $S_v = k[x^v_1, \ldots, x^v_{m_v}]$.

So, for the index set $i_1, \ldots, i_n$, following the structure in 4, we can write $M = M_1 \otimes \ldots \otimes M_n$, where each M_v is a monomial in S_v for $1 \leq v \leq n$, and $M_\beta \in (f^\beta_{i_\beta})S_\beta[I_\beta t]'_{(f^\beta_{i_\beta}t)}$.

Let $j_1, \ldots, j_n$, $1 \leq j_v \leq s_v$ be any set of indices such that $j_\beta = i_\beta$, that is, $f^\beta_{j_\beta} = f^\beta_{i_\beta}$. Then since $M_\beta \in (f^\beta_{i_\beta})S_\beta[I_\beta t]'_{(f^\beta_{i_\beta}t)}$, we still have $M \in (f^\beta_{j_\beta})R[It]'_{(f^1_{j_1}\cdots f^n_{j_n}t)}$. $\square$

Proof of Theorem 2.14. We want to show that $(I_1, \ldots, I_n)^{\sim} = \overline{I_1} + \ldots + \overline{I_n}$. Clearly $\overline{I_1} + \ldots + \overline{I_n} \subset (I_1, \ldots, I_n)^{\sim}$. We need to show that the other inclusion holds. From Proposition 2.15 we know that since $I_1, \ldots, I_n$ are monomial ideals, $(I_1, \ldots, I_n)^{\sim}$ is a monomial ideal.

So we pick a monomial M of $R = k[x^1_1, \ldots, x^1_{m_1}, \ldots, x^n_1, \ldots, x^n_{m_n}]$ such that $M \in (I_1, \ldots, I_n)^{\sim}$. Our goal is to show that for some α, $1 \leq \alpha \leq n$, $M \in I_\alpha R[It]'$.

Fix an index set $i_1, \ldots, i_n$, and let $f = f^1_{i_1} \ldots f^n_{i_n}$. Then from Equation 3 we see that

$$M \in \left(JR[It]'_{(ft)}\right)^* = (f^1_{i_1}, \ldots, f^n_{i_n})R[It]'_{(ft)}.$$

From [EGA] Lemma 2.1.6, it follows that $\mathrm{Proj}R[It]' = \mathrm{Proj}R[\overline{I^h}t]$ for some $h \geq 1$, and we obtain

$$R[It]'_{(ft)} \simeq R[\overline{I^h}t]_{(f^ht)}.$$

Since I is a monomial ideal, $\overline{I^h}$ is a monomial ideal (see [E] Chapter 4), and we can write $\overline{I^h} = (H_1, \ldots, H_r)$, where $H_1, \ldots, H_r$ are monomials. So we have

$$M \in (f^1_{i_1}, \ldots, f^n_{i_n})k[x^1_1, \ldots, x^1_{m_1}, \ldots .x^n_1, \ldots, x^n_{m_n}, H_1t, \ldots, H_rt]_{(f^ht)},$$

which is isomorphic to

$$k[x^1_1, \ldots, x^1_{m_1}, \ldots, x^n_1, \ldots, x^n_{m_n}, \frac{H_1}{f^h}, \ldots, \frac{H_r}{f^h}],$$

and hence from Lemma 2.16 it follows that for some β, $1 \leq \beta \leq n$, $M \in (f^\beta_{i_\beta})R[\overline{I^h}t]_{(f^ht)}$ which implies that $M \in (f^\beta_{i_\beta})R[It]'_{(ft)}$.

We would like to prove that this choice is consistent for all the affine sets, that is, there is some β such that $M \in (f^\beta_{i_\beta})R[It]'_{(f^1_{i_1}\cdots f^n_{i_n}t)}$ for all choices of index sets $i_1, \ldots, i_n$.

Suppose that for each $\alpha = 1, ..., n-1$, there is some index γ_α, $1 \leq \gamma_\alpha \leq s_\alpha$, and some index set $i_1, \ldots, i_n$ with $i_\alpha = \gamma_\alpha$, for which $M \notin (f^\alpha_{i_\alpha})R[It]'_{(f^1_{i_1}\cdots f^n_{i_n}t)}$. Then by Lemma 2.19, for all j, $1 \leq j \leq s_n$, if one picks the index set $\gamma_1, \ldots, \gamma_{n-1}, j$,

$$M \notin (f^\alpha_{\gamma_\alpha})R[It]'_{(f^1_{\gamma_1}\cdots f^{n-1}_{\gamma_{n-1}}f^n_jt)}$$

for $1 \leq \alpha \leq n-1$. Therefore for all possible j,

$$M \in (f_j^n)R[It]'_{(f_{\gamma_1}^1 \cdots f_{\gamma_{n-1}}^{n-1} f_j^n t)}.$$

Applying Lemma 2.19 again, one gets that $M \in (f_{i_n}^n)R[It]'_{(f_{i_1}^1 \cdots f_{i_n}^n t)}$, for all index sets $i_1, \ldots, i_n$.

We have now proved that $M \in (I_1 + \ldots + I_n)R[It]'$, implies that for some β, $1 \leq \beta \leq n$, $M \in I_\beta R[It]'$. Therefore

$$M \in R \cap I_\beta R[It]' = \overline{I_\beta}$$

by Proposition 2.20. □

Proposition 2.20. *Let R be a Noetherian domain, and let X be a normal scheme, with a proper birational map $\pi : X \longrightarrow \mathrm{Spec}R$. Suppose that I is an ideal of R such that $I\mathcal{O}_X$ is an invertible sheaf of ideals on X. Then $I\mathcal{O}_X \cap R = \overline{I}$.*

Proof. See Proposition 6.2 of [L1] and the remark following it for the proof. □

3 Applications to TI closure

In this section we apply the notion of blowup closure to study properties of TI closure and to answer the questions mentioned in Section 1.4. It turns out (Corollary 3.5) that the TI closure of a set of ideals is the contraction of the blowup closure of certain extensions of those ideals in an extension of the original ring. In fact, the TI closure of a set of ideals can be described as the contraction of *only one* of the affine patches used to calculate the blowup closure mentioned above. To facilitate the arguments, we call this particular patch the *multiple closure*.

3.1 Definition and Basic Facts

Definition 3.1. Let R be a Noetherian ring of positive characteristic p, and let $I_1, \ldots, I_n$ be ideals in R. We define x in R to be in the *multiple closure* of $I_1, \ldots, I_n$, denoted by $(I_1, \ldots, I_n)^{\overset{*}{\sim}}$, if and only if the image of x is in:

$$\left((w_1, \ldots, w_n)R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}]\right)^*,$$

where $w_1, \ldots, w_n$ are indeterminates.

Recall that the ring

$$S = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}]$$

above is obtained as follows: We take a ring extension $R[w_1, \ldots, w_n]$ of R, and we consider the Rees ring $R[w_1, \ldots, w_n][I't]$ ring of the product I' of the ideals $I_1 + (w_1), \ldots, I_n + (w_n)$. We then localize this Rees ring at the element $w_1 \ldots w_n t$, and take the zeroth graded piece of the localized ring to obtain S. The ideal $(w_1, \ldots, w_n)$ in S is just the sum of the ideals $I_1 + (w_1), \ldots, I_n + (w_n)$ extended to S. So the multiple closure of $I_1, \ldots, I_n$ in the ring R is one of the affine patches to be considered to compute the blowup closure of $I_1 + (w_1), \ldots, I_n + (w_n)$ in the ring $R[w_1, \ldots, w_n]$ (see Theorem 2.3). It turns out that the multiple closure of $I_1, \ldots, I_n$ *is* in fact the blowup closure of $I_1 + (w_1), \ldots, I_n + (w_n)$ in $R[w_1, \ldots, w_n]$, contracted back to R (Corollary 3.5). Multiple closure therefore enjoys all the basic properties of blowup closure.

We verify that multiple closure can be tested modulo minimal primes, and hence one can reduce most arguments to the case of domains.

Proposition 3.2 (multiple closure can be tested modulo minimal primes). *Suppose that R is a Noetherian ring of positive characteristic p, and let $I_1, \ldots, I_n$ be ideals in R. An element x of R is in $(I_1, \ldots, I_n)^{\overset{*}{\sim}}$ if and only if for all minimal primes p of R, the image $\overline{x}$ of x in R/p is in $(I_1R/p, \ldots, I_nR/p)^{\overset{*}{\sim}}$.*

Proof. The proof follows from the description of minimal primes of localized Rees rings, as in Corollary 2.7.

Let $I = (I_1 + (w_1)) \ldots (I_n + (w_n))$. Then by Proposition 2.6 and Corollary 2.7, a minimal prime p' of

$$S = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}] \simeq R[w_1, \ldots, w_n, It]_{(w_1 \ldots w_n t)}$$

corresponds to a minimal prime $p^\dagger$ of $R[w_1, \ldots, w_n]$ that does not contain $w_1 \ldots w_n$, which in turn corresponds to a minimal prime p of R. By the same results, we have the following isomorphisms

$$S/p' \simeq \Big((R[w_1, \ldots, w_n]/p^\dagger)[I(R[w_1, \ldots, w_n]/p^\dagger)t]\Big)_{(w_1 \ldots w_n t)}$$

which is isomorphic to

$$((R/p)[w_1, \ldots, w_n]\,[(I(R/p)[w_1, \ldots, w_n])t])_{(w_1 \ldots w_n t)}$$

which is isomorphic to

$$(R/p)\left[w_1, \ldots, w_n, \frac{I_1(R/p)}{w_1}, \ldots, \frac{I_n(R/p)}{w_n}\right].$$

Now take $x \in (I_1, \ldots, I_n)^{\overset{*}{\sim}}$. By Definition 3.1, this is equivalent to

$$x \in ((w_1, \ldots, w_n)S)^* \cap R.$$

From the isomorphisms above, equivalently for all $p \in \mathcal{M}(R)$

$$\overline{x} \in \Big((w_1, \ldots, w_n)S/p^\dagger\Big)^* \cap R/p,$$

or, equivalently

$$\overline{x} \in \left((w_1, \ldots, w_n)(R/p)[w_1, \ldots, w_n, \frac{I_1(R/p)}{w_1}, \ldots, \frac{I_n(R/p)}{w_n}]\right)^* \cap R/p$$

By Definition 3.1 this is equivalent to $\overline{x} \in (I_1R/p, \ldots, I_nR/p)^{\overset{*}{\sim}}$. $\square$

Theorem 3.3. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals of R. Then*

$$(I_1, \ldots, I_n)^{\overset{*}{\sim}} = (I_1, \ldots, I_n)^{\underline{*}}\,.$$

Proof. Since TI closure and multiple closure can both be tested modulo minimal primes ([ho2] 1.4 and 3.2 above), we can assume that R is a domain. Suppose that $S = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}]$, where $w_1, \ldots, w_n$ are indeterminates. Pick $z \in (I_1, \ldots, I_n)^{\underline{*}}$. Then there is a nonzero c in R such that $cz^q \in I_1{}^q + \ldots + I_n{}^q$ for all large $q = p^e$. Since we have the inclusion $R \subset S$, it follows that for all large q

$$cz^q \in (I_1{}^q + \ldots + I_n{}^q)S \subseteq (w_1{}^q, \ldots, w_n{}^q)S = (w_1, \ldots, w_n)^{[q]}S\,.$$

Hence

$$z \in ((w_1, \ldots, w_n)S)^* \cap R = (I_1, \ldots, I_n)^{\overset{*}{\sim}}.$$

To show the other inclusion, we choose an element c in R such that R_c is regular. Then

$$S_{cw_1 \ldots w_n} = R_c[w_1, \ldots, w_n, \frac{1}{w_1}, \ldots, \frac{1}{w_n}]$$

is regular, and therefore $d = cw_1 \ldots w_n$ has a power c' that is a test element for the ring S (Theorem 1.10). By multiplying c' with appropriate powers of elements in $\frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}$, we may assume that $c' \in R$.

Now take $z \in (I_1, \ldots, I_n)^{\overset{*}{\sim}}$. Then

$$c'z^q \in (w_1{}^q, \ldots, w_n{}^q)S$$

for all $q = p^e$.

So for a given q, we can find $C_1, \ldots, C_n$ in S, such that

$$c'z^q = C_1 w_1{}^q + \ldots + C_n w_n{}^q.$$

By taking common denominators, we can find a positive integer N, which we can take to be larger than q, such that $C_i = \frac{A_i}{(w_1 \ldots w_n)^N}$ for every $i = 1, \ldots, n$, where A_i is a polynomial in $R[w_1, \ldots, w_n]$. So we get

$$c'z^q(w_1 \ldots w_n)^N = A_1 w_1{}^q + \ldots + A_n w_n{}^q.$$

Since $R[w_1, \ldots, w_n]$ is a free module over R generated by the monomials in $w_1, \ldots, w_n$, and $c'z^q \in R$, we can without loss of generality take each A_i to be is a monomial of the form $B_i w_1{}^N \ldots w_{i-1}{}^N w_i{}^{N-q} w_{i+1}{}^N \ldots w_n{}^N$, where $B_i \in R$, for all $i = 1, \ldots, n$. So we can write $c'z^q$ as

$$\frac{A_1}{(w_1 \ldots w_n)^N} w_1{}^q + \ldots + \frac{A_n}{(w_1 \ldots w_n)^N} w_n{}^q = \frac{B_1}{w_1{}^q} w_1{}^q + \ldots + \frac{B_n}{w_n{}^q} w_n{}^q$$

which implies that $B_i \in I_i{}^q$ for all $i = 1, \ldots, n$. So

$$c'z^q \in I_1{}^q + \ldots + I_n{}^q.$$

This holds for all q, hence $z \in (I_1, \ldots, I_n)^{\underline{*}}$. □

This equality translates the TI closure of a set of ideals in R into the tight closure of an ideal in an extension ring of R. In particular, most properties of tight closure can now be extended to TI closure.

3.2 Basic Properties of TI Closure Via Multiple Closure

Theorem 3.4 (persistence of TI closure). *Let R be a Noetherian ring of prime characteristic p that is either essentially of finite type over an excellent local ring or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals of R. Suppose $R \longrightarrow S$ is a homomorphism of rings. Then TI closure persists under this map:*

$$(I_1, \ldots, I_n)^{\underline{*}} S \subseteq (I_1 S, \ldots, I_n S)^{\underline{*}}.$$

Proof. The properties mentioned above for R are preserved when we pass to the ring $R^\dagger = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}]$, since this is just a finitely generated algebra over R. Moreover, we have the obvious induced map

$$R^\dagger \longrightarrow S^\dagger = S[w_1, \ldots, w_n, \frac{I_1 S}{w_1}, \ldots, \frac{I_n S}{w_n}]$$

under which tight closure persists. Therefore, by Theorem 1.11,

$$(w_1, \ldots, w_n)^* S^\dagger \subseteq \left((w_1, \ldots, w_n) S^\dagger\right)^*$$

which implies that $(I_1, \ldots, I_n)^{\underline{*}}\, S \subseteq (I_1 S, \ldots, I_n S)^{\underline{*}}$. □

An interesting corollary is that multiple closure (or TI closure) is indeed a blowup closure in a larger ring.

Corollary 3.5. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals of R. If $S = R[w_1, \ldots, w_n]$, and for $i = 1, \ldots, n$, $I'_i = I_i + (w_i)$ is an ideal of S, then*

$$(I_1, \ldots, I_n)^{\underline{*}} = (I_1, \ldots, I_n)^{\overset{*}{\sim}} = \left((I'_1, \ldots, I'_n)S\right)^{\sim} \cap R.$$

Proof. From Theorem 3.4, [Ho2] Proposition 1.4, and Theorem 2.9 part (a) we have

$$(I_1, \ldots, I_n)^{\underline{*}} S \subseteq (I_1 S, \ldots, I_n S)^{\underline{*}} \subseteq (I'_1, \ldots, I'_n)^{\underline{*}} \subseteq (I'_1, \ldots, I'_n)^{\sim},$$

and so

$$(I_1, \ldots, I_n)^{\underline{*}} \subseteq (I'_1, \ldots, I'_n)^{\sim} \cap R.$$

On the other hand, let I' and J' denote the product and sum of $I'_1, \ldots, I'_n$, respectively, and let $\mathcal{G}'$ be a fixed set of generators for I' such that $w_1 \ldots w_n \in \mathcal{G}'$. By Theorem 2.3,

$$(I'_1, \ldots, I'_n)^{\sim} = \bigcap_{f \in \mathcal{G}'} \left(J' S[I't]_{(ft)}\right)^* \cap S.$$

From the previous paragraph, for every $f \in \mathcal{G}'$, $(I_1, \ldots, I_n)^{\underline{*}} \subseteq \left(J' S[I't]_{(ft)}\right)^* \cap R$. On the other hand, by definition of multiple closure, we know that if $f = w_1 \ldots w_n$, then $(I_1, \ldots, I_n)^{\underline{*}} = \left(J' S[I't]_{(ft)}\right)^* \cap R$. So we have

$$(I_1, \ldots, I_n)^{\underline{*}} = \left((I'_1, \ldots, I'_n)S\right)^{\sim} \cap R.$$

□

One can see that tight closure commuting with localization and TI closure commuting with localization are equivalent properties.

Theorem 3.6. *Let $\mathcal{R}$ denote the class of all Noetherian rings R, such that R is either essentially of finite type over an excellent local ring or R_{red} is F-Finite. Then TI closure commutes with localization for all rings in $\mathcal{R}$ if and only if tight closure commutes with localization for all rings in $\mathcal{R}$.*

Proof. It is clear that if TI closure commutes with localization for any ring, then so will tight closure, since the tight closure of an ideal is equal to the TI closure of a set of principal ideals ([Ho2] proposition 1.4).

Now suppose that tight closure commutes with localization, and take a set of ideals $I_1, \ldots, I_n$ in a ring R described above. Let U be a multiplicative set in R. Then

$$\begin{aligned} U^{-1}(I_1, \ldots, I_n)^{\underline{*}} &= U^{-1}\left[\left((w_1, \ldots, w_n)R[w_1, \ldots, w_n, \tfrac{I_1}{w_1}, \ldots, \tfrac{I_n}{w_n}]\right)^* \cap R\right] \\ &= \left((w_1, \ldots, w_n)U^{-1}R[w_1, \ldots, w_n, \tfrac{I_1}{w_1}, \ldots, \tfrac{I_n}{w_n}]\right)^* \cap U^{-1}R \\ &= (U^{-1}I_1, \ldots, U^{-1}I_n)^{\underline{*}}. \end{aligned}$$

□

In Section 3.4 we strengthen the statement of Theorem 3.6 using the notion of test exponents.

Theorem 3.7. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-Finite. Let $I_1, \ldots, I_n$ be ideals of R. Suppose that $c \in R^o$ and $z \in R$ are such that $cz^q \in {I_1}^q + \ldots + {I_n}^q$ for infinitely many powers q of p. Then $z \in (I_1, \ldots, I_n)^{\underline{*}}$.*

Proof. Let

$$S = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}].$$

Since $c \in R^o$, it immediately follows that $c \in S^o$; see Corollary 2.7. Also, $cz^q \in {I_1}^q + \ldots + {I_n}^q$ implies that $cz^q \in (w_1, \ldots, w_n)^{[q]}$ in S for infinitely many q. Therefore

$$z \in \left((w_1, \ldots, w_n)R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}]\right)^* \cap R = (I_1, \ldots, I_n)^{\underline{*}}.$$

□

3.3 Test Elements for TI Closure

Theorem 3.3 allows us to develop a theory of test elements for TI closure (see Section 1.2). Test elements are useful since they help us decide whether a given element of a ring is in the TI closure of a given set of ideals in that ring. Test elements do not exist for integral closure (see Example 1.22), and since the TI closure of one ideal is the integral closure of that ideal (see Section 1.3), we expect the test elements for TI closure to depend on the ideals. We therefore first specify what we mean by test elements for TI closure.

Definition 3.8. Let R be a Noetherian ring of prime characteristic p. Let $I_1, \ldots, I_n$ be ideals in R. We say that $c \in R^o$ is a *TI closure test element for* $I_1, \ldots, I_n$, or in short, a *test element for* $I_1, \ldots, I_n$, if for every $z \in (I_1, \ldots, I_n)^{\underline{*}}$, $cz^q \in I_1^q + \ldots + I_n^q$, for all nonnegative powers q of p. We call the ideal generated by the test elements for $I_1, \ldots, I_n$ the *test ideal for* $I_1, \ldots, I_n$, and denote it by $\tau(I_1, \ldots, I_n)$.

From the proof of Theorem 3.3 it immediately follows that:

Corollary 3.9. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals of R, and let $S = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}]$. Then*

$$\tau(S) \cap R \subseteq \tau(I_1, \ldots, I_n).$$

Here, $\tau(S)$ is the usual tight closure test ideal for S. We also obtain *locally stable TI* closure test elements for a set of ideals; these are TI closure test elements of a set of ideals $I_1, \ldots, I_n$ that remain test elements for these ideals after we localize the ring at any multiplicative set.

Theorem 3.10. *Let R be a Noetherian ring of prime characteristic p such that either R is essentially of finite type over an excellent local ring, or R_{red} is F-finite. Let $I_1, \ldots, I_n$ be ideals of R. Let c be an element of R^o such that R_c is regular. Then for some positive integer N and all choices $a_1 \in I_1, \ldots, a_n \in I_n$, $(ca_1 \ldots a_n)^N$ is a locally stable TI closure test element for the ideals $I_1, \ldots, I_n$.*

Proof. Let U be a multiplicative set in R. Suppose $z \in (U^{-1}I_1, \ldots, U^{-1}I_n)^{\underline{*}}$. Therefore

$$z \in \left((w_1, \ldots, w_n)U^{-1}R[w_1, \ldots, w_n, \frac{U^{-1}I_1}{w_1}, \ldots, \frac{U^{-1}I_n}{w_n}]\right)^* = \left((w_1, \ldots, w_n)U^{-1}S\right)^*,$$

where

$$S = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}],$$

and U denotes the multiplicative set U in R, as well as the image of U in S.

Since R_c is regular, so is $S_{cw_1 \ldots w_n}$, and so $cw_1 \ldots w_n$ to some positive power N is a locally stable test element for S (see Theorem 1.10). This means that $(cw_1 \ldots w_n)^N$ is a test element for $U^{-1}S$, and therefore $(ca_1 \ldots a_n)^N = (cw_1 \ldots w_n)^N (\frac{a_1}{w_1} \ldots \frac{a_n}{w_n})^N$ is a test element for $U^{-1}S$ for any $a_i \in I_i$.

For any such test element $d = (ca_1 \ldots a_n)^N$ we have

$$dz^q \in (w_1, \ldots, w_n)^{[q]}U^{-1}S$$

for all powers q of p, and following the proof of Theorem 3.3, we get

$$dz^q \in U^{-1}I_1{}^q + \ldots + U^{-1}I_n{}^q$$

for all nonnegative powers q of p.

This implies that d is a TI closure test element for $U^{-1}I_1, \ldots, U^{-1}I_n$, and since this holds for all U, we conclude that d is a locally stable TI closure test element for $I_1, \ldots, I_n$ in R. $\square$

For ideals in a finitely generated algebra over a field of characteristic p, we are able to compute explicit TI closure test elements. Using a theorem of Lipman and Sathaye, Hochster and Huneke described specific tight closure test elements for such rings.

Theorem 3.11 (Corollary 1.5.5 [HH2]). *Let k be a field of characteristic p and let R be a d-dimensional geometrically reduced domain over k (meaning that $\overline{k} \otimes_k R$ is reduced) that is finitely generated as a k-algebra. Let $R = k[u_1, \ldots, u_m]/(g_1, \ldots, g_s)$ be a presentation of R as a homomorphic image of a polynomial ring. Then the $(m-d) \times (m-d)$ minors of the Jacobian matrix $(\delta g_i/\delta u_j)$ are contained in the test ideal of R, and remain so after localization and completion. Thus, any element of the Jacobian ideal generated by all these minors that is in R^o is a completely stable test element.*

Theorem 3.12. *Let k be a field of characteristic p and let R be a d-dimensional geometrically reduced domain over k that is finitely generated as a k-algebra. Let $R = k[u_1, \ldots, u_m]/(g_1, \ldots, g_s)$ be a presentation of R as a homomorphic image of a polynomial ring. Take ideals $I_1, \ldots, I_n$ of R where for each $i = 1, \ldots, n$, I_i is minimally generated by the elements $f_1^i, \ldots, f_{m_i}^i$ of R. Then*

$$I_1{}^{m_1-1} \ldots I_n{}^{m_n-1} \mathcal{J}_{m-d} \subseteq \tau(I_1, \ldots, I_n) \ ,$$

where $\mathcal{J}_{m-d}$ is the ideal generated by the $(m-d) \times (m-d)$ minors of the Jacobian matrix $\mathcal{J}(R) = (\frac{\delta g_i}{\delta u_j})$.

$$
\left(\begin{array}{c cccccccccccccc}
\begin{matrix}\frac{\delta g_1}{\delta u_1} & \cdots & \frac{\delta g_1}{\delta u_m}\\ \vdots & \ddots & \vdots \\ \frac{\delta g_s}{\delta u_1} & \cdots & \frac{\delta g_s}{\delta u_m}\end{matrix} & \multicolumn{14}{c}{0} \\
 & x_1^1 & 0 & \cdots & 0 & w_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & x_{m_1}^1 & 0 & \cdots & 0 & 0 & \cdots & w_1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
 & 0 & x_1^2 & \cdots & 0 & 0 & \cdots & 0 & w_2 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A & 0 & x_{m_2}^2 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & w_2 & \cdots & 0 & \cdots & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & 0 & 0 & \cdots & x_1^n & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & w_n & \cdots & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 & 0 & 0 & \cdots & x_{m_n}^n & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & w_n
\end{array}\right)
$$

Figure 1: Part of the Jacobian matrix

Proof. Let S be the ring

$$S = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}].$$

Then there is a surjective map to S from the polynomial ring

$$H = k[u_1, \ldots, u_m, w_1, \ldots, w_n, x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^n, \ldots, x_{m_n}^n]$$

mapping x_i^j to $\frac{f_i^j}{w_j}$. If for $1 \leq i \leq n$ and $1 \leq j \leq m_i$, F_j^i is an element of the polynomial ring $k[u_1, \ldots, u_m]$ whose image in R is f_j^i, the kernel $\wp$ of this map is generated by

$$g_1, \ldots, g_s, w_1x_1^1 - F_1^1, \ldots, w_1x_{m_1}^1 - F_{m_1}^1, \ldots, w_nx_1^n - F_1^n, \ldots, w_nx_{m_n}^n - F_{m_n}^n$$

and possibly other polynomials. So S is isomorphic under this map to $H/\wp$. A part of the Jacobian matrix for this presentation of S will then look like the matrix shown in Figure 1.

Now $\dim R = d$, and so $\dim S = d+n$. If $m' = m_1 + \ldots + m_n$, then $m+n+m'-(n+d) = m+m'-d$, and so we take the ideal $\mathcal{I}$ of the $(m+m'-d) \times (m+m'-d)$ minors of this matrix. We note that $\mathcal{I}$ is contained in the ideal generated by the $(m+m'-d) \times (m+m'-d)$ minors of the Jacobian matrix of S, and so from Theorem 3.11 we can conclude that $\mathcal{I}$ is in the (tight closure) test ideal for S. Corollary 3.9 then implies that $\mathcal{I} \cap R \subseteq \tau(I_1, \ldots, I_n)$.

To find elements of $\mathcal{I}$ that are in R, we make the following partition of the matrix:

Step 1. We take the $m' \times m'$ minor of the lower right corner of the matrix by taking the first n columns, and for each i, $i = 1, \ldots, n$, removing one of the columns (say the s_ith column) involving a w_i. The outcome is $x_{s_1}^1 \ldots x_{s_n}^n {w_1}^{m_1-1} \ldots {w_n}^{m_n-1}$.

Step 2. The upper left corner of this matrix is just $\mathcal{J}(\mathcal{R})$. By taking the $(m-d) \times (m-d)$ minors of the upper left corner we obtain $\mathcal{J}_{m-d}$.

So

$$x_{s_1}^1 \ldots x_{s_n}^n {w_1}^{m_1-1} \ldots {w_n}^{m_n-1} \mathcal{J}_{m-d} \subseteq \mathcal{I}.$$

It follows that:

$$(\frac{I_1}{w_1})^{m_1-2} \ldots (\frac{I_n}{w_n})^{m_n-2} x_{s_1}^1 \ldots x_{s_n}^n {w_1}^{m_1-1} \ldots {w_n}^{m_n-1} \mathcal{J}_{m-d} \subseteq \mathcal{I},$$

and so

$$F^1_{s_1} \dots F^n_{s_n} {I_1}^{m_1-2} \dots {I_n}^{m_n-2} \mathcal{J}_{m-d} \subseteq \mathcal{I}, \tag{5}$$

since for each i, $x^i_{s_i} w_i = F^i_{s_i} \in I_i$.

By repeating Step 1, and for each $i = 1, \dots, n$ allowing s_i to vary between 1 and m_i, we see that the inclusion in (5) will hold for $1 \leq s_i \leq m_i$ and $i = 1, \dots, n$. It follows that

$$I_1 \dots I_n {I_1}^{m_1-2} \dots {I_n}^{m_n-2} \mathcal{J}_{m-d} = {I_1}^{m_1-1} \dots {I_n}^{m_n-1} \mathcal{J}_{m-d} \subseteq \mathcal{I}.$$

$\square$

In practice, k being of characteristic zero or perfect and R being reduced ensures that R is geometrically reduced (see tensor products over fields in [ZS1]).

Example 3.13. Let $R = k[x, y, z]$ be a polynomial ring over a perfect field k of characteristic p. Let $I = (f, g)$ and $J = (h)$ be ideals of R. Then every element of I is a TI closure test element for the ideals I and J.

3.4 Test Exponents and the Localization of TI Closure

A recent advance in tight closure theory is the development of *test exponents*. Hochster and Huneke show in [HH4] that the existence of test exponents for tight closure is roughly equivalent to tight closure commuting with localization. This result suggests that the study of test exponents could provide a breakthrough in the localization problem for tight closure.

The situation is similar for TI closure: below we define TI closure test exponents, and establish that TI closure commuting with localization is related to the existence of test exponents, although the result we get for TI closure is somewhat weaker than the corresponding result for tight closure in [HH4]. Nevertheless, Theorem 3.17 below strengthens our previous statement (Theorem 3.6) on the equivalence of tight closure commuting with localization with TI closure commuting with localization.

We begin by stating some relevant facts from tight closure theory.

Definition 3.14 ([HH4]). Let R be a reduced Noetherian ring of positive prime characteristic p. Let c be a fixed test element for R, and let I be an ideal of R. Then $q = p^e$ is called a *test exponent for c and I*, if whenever $cu^Q \in I^{[Q]}$ for some $u \in R$ and $Q \geq q$, then $u \in I^*$.

Thus the existence of test exponents reduces the process of checking whether an element u of the ring is in the tight closure of an ideal I, to just checking if $cu^Q \in I^{[Q]}$ for *some* large Q, rather than *all* large Q. It is easy to see that the existence of a test exponent for an ideal forces that ideal to commute with localization (see Proposition 2.3 of [HH4]). The converse of this statement is however difficult to prove. Here, we produce a parallel definition for TI closure test exponents. It will easily follow that the existence of such an exponent for a set of ideals will force the TI closure of those ideals to commute with localization. For the converse, we exploit the fact that TI closure can be described in terms of tight closure and apply results of [HH4] to achieve the desired statement.

Definition 3.15. Let R be a Noetherian ring of prime characteristic p, and let $I_1, \dots, I_n$ be a set of ideals in R. Let c be a fixed test element for $I_1, \dots, I_n$ in R. Then $q = p^e$ is called a *test exponent for* $c, I_1, \dots, I_n$, if whenever $cu^Q \in {I_1}^Q + \dots + {I_n}^Q$ for some $u \in R$ and $Q \geq q$, then $u \in (I_1, \dots, I_n)^{\underline{*}}$.

Theorem 3.16. *Let R be a Noetherian ring of prime characteristic p, and let $I_1, \ldots, I_n$ be ideals of R. Suppose that c is a locally stable test element for $I_1, \ldots, I_n$, and suppose that c, $I_1, \ldots, I_n$ have a test exponent q. Then TI closure commutes with localization for $I_1, \ldots, I_n$.*

Proof. Let U be a multiplicative set in R, and let $z/1 \in (U^{-1}I_1, \ldots, U^{-1}I_n)^{\underline{*}}$. We have

$$cz^q/1 \in U^{-1}{I_1}^q + \ldots + U^{-1}{I_n}^q,$$

and so for some $u \in U$, $ucz^q \in {I_1}^q + \ldots + {I_n}^q$, hence $c(uz)^q \in {I_1}^q + \ldots + {I_n}^q$. Since q is a test exponent, this implies that $uz \in (I_1, \ldots, I_n)^{\underline{*}}$, and so

$$z/1 \in U^{-1}(I_1, \ldots, I_n)^{\underline{*}}.$$

□

Theorem 3.17. *Let R be a Noetherian ring of prime characteristic p that is either essentially of finite type over an excellent local ring or R_{red} is F-Finite. Let $I_1, \ldots, I_n$ be ideals of R. Suppose that the tight closure of the ideal $(w_1, \ldots, w_n)$ in the ring $S = R[w_1, \ldots, w_n, I_1/w_1, \ldots, I_n/w_n]$ commutes with localization at all primes ideals in* $\operatorname{Ass}(S/(w_1, \ldots, w_n)^*)$*. Let c be a locally stable test element for $I_1, \ldots, I_n$ in R, which is also a locally stable test element for the ring S (such test elements exist; see the statement and proof of Theorem 3.10). Then $c, I_1, \ldots, I_n$ have a test exponent.*

Proof. Since tight closure of the ideal $(w_1, \ldots, w_n)$ commutes with localization at all primes in $\operatorname{Ass}(S/(w_1, \ldots, w_n)^*)$, Theorem 2.4 of [HH4] implies that $c, (w_1, \ldots, w_n)$ have a test exponent q.

Now suppose $cu^Q \in {I_1}^Q + \ldots + {I_n}^Q$ for some $u \in R$ and $Q \geq q$. Then cu^Q belongs to the ideal $(w_1, \ldots, w_n)^{[Q]}$ in S, which implies that $u \in (w_1, \ldots, w_n)^*$. Hence

$$u \in (w_1, \ldots, w_n)^* \cap R = (I_1, \ldots, I_n)^{\underline{*}},$$

which proves that q is a test exponent for $c, I_1, \ldots, I_n$. □

We are now able to strengthen one direction of the statement of Theorem 3.6.

Corollary 3.18. *If R, S, $I_1, \ldots, I_n$ are as in Theorem 3.17, then the TI closure of $I_1, \ldots, I_n$ commutes with localization at any multiplicative set in R, if the tight closure of $(w_1, \ldots, w_n)$ in S commutes with localization at all primes that belong to* $\operatorname{Ass}(S/(w_1, \ldots, w_n)^*)$.

3.5 *TI* Closure in Equal Characteristic Zero

The translation of TI closure into tight closure enables us to extend TI closure to rings of characteristic zero in an effective way. The definition of TI closure in characteristic zero was introduced in [Ho2], but due to the difficulty of working with the positive characteristic definition, several questions remained unanswered (see Section 1.3). In this section, we address some of those questions for affine algebras over fields of characteristic zero. We show that TI closure in equal characteristic zero has similar features to tight closure in equal characteristic zero. We also introduce *universal test elements* for TI closure based on universal test elements for tight closure.

We begin by recalling the definition of TI closure in characteristic zero:

Definition 3.19. Let R be a finitely generated algebra over a field K of characteristic zero, and let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a set of ideals of R. We say that an element x of R is in the *tight integral closure of $\mathcal{I}$*, denoted by $\mathcal{I}^{\underline{*}}$, if there exist descent data $(D, R_D, I_{1,D}, \ldots, I_{n,D})$, such that for every maximal ideal m of D, if $k = D/m$, then $x_k \in {\mathcal{I}_k}^{\underline{*}}$, where $\mathcal{I}_k$ denotes the set of ideals $\{I_{1,k}, \ldots, I_{n,k}\}$, with $I_{t,k} = k \otimes_k I_{t,D}$.

See Definition 1.13 for the definition of descent data. Similar to the positive characteristic situation, we can describe TI closure in equal characteristic zero in terms of tight closure.

Theorem 3.20. *Let R be an affine algebra over a field K of characteristic zero, which can be presented as*

$$R = K[u_1, \dots, u_m]/(g_1, \dots, g_s)$$

where $u_1, \dots, u_m$ are indeterminates and $g_1, \dots, g_s$ are polynomials in $K[u_1, \dots, u_m]$. Suppose $I_1, \dots, I_n$ are ideals of R, and $x \in R$. Then $x \in (I_1, \dots, I_n)^{\underline{*}}$ if and only if

$$x \in \left((w_1, \dots, w_n)R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}]\right)^*$$

where $w_1, \dots, w_n$ are indeterminates.

Proof. The main point of the proof is that one can construct descent data that work for the ideals $I_1, \dots, I_n$ in R as well as for the ideal $(w_1, \dots, w_n)$ in

$$S = R[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}].$$

Suppose that for each i, $I_i = (f_1^i, \dots, f_{s_i}^i)$ in R and for every i and j, $1 \le i \le n$ and $1 \le j \le s_i$, F_j^i is a polynomial in $K[u_1, \dots, u_m]$ whose image in R is f_j^i. Pick $u \in K[u_1, \dots, u_m]$ whose image in R is x.

We can represent S as a polynomial ring

$$K[u_1, \dots, u_m, w_1, \dots, w_n, \alpha_1^1, \dots, \alpha_{s_1}^1, \dots, \alpha_1^n, \dots, \alpha_{s_n}^n]$$

modulo the ideal J which is generated by $g_1, \dots, g_s$, polynomials of the form $w_i\alpha_j^i - F_j^i$ for $1 \le i \le n$ and $1 \le j \le s_i$, and possibly other polynomials (see the proof of Theorem 3.12 for a more detailed description of this isomorphism).

We can now construct descent data D for $(w_1, \dots, w_n)$ and x in S by adjoining the coefficients of all the generators of J in K and the coefficients of u in K to the ring of integers $\mathbf{Z}$. We then replace D by a localization at a single element to assure that it satisfies the properties of descent data (Lemma of Generic Freeness, [HR]). This D will also work as descent data for $I_1, \dots, I_n$ and x in R, since it is an enlargement of some basic descent data that one would construct, and once some descent data work, every enlargement of them work as well (see Section 1.2).

With this construction, we have that $x \in (I_1, \dots, I_n)^{\underline{*}}$ if and only if for every maximal ideal m of D, setting $k = D/m$

$$x_k \in (I_{1,k}, \dots, I_{n,k})^{\underline{*}}$$

in R_k. Since k is a finite and therefore perfect field, R_k will be F-finite, and so equivalently

$$x_k \in \left((w_1, \dots, w_n)R_k[w_1, \dots, w_n, \frac{I_{1,k}}{w_1}, \dots, \frac{I_{n,k}}{w_n}]\right)^*.$$

This is equivalent to $x \in ((w_1, \dots, w_n)S_k)^*$, and since this holds for all residue class fields k of D, we conclude that equivalently $x \in ((w_1, \dots, w_n)S)^*$. ◻

The description of TI closure as tight closure in characteristic zero yields the following:

Theorem 3.21 (independence of choice of descent). *Let K be a field of characteristic zero, R a finitely generated K-algebra, $I_1, \ldots, I_n$ ideals of R and $u \in R$. Let $(D, I_{1,D}, \ldots, I_{n,D})$ be descent data for R, $I_1, \ldots, I_n$ and u. If $u \in (I_1, \ldots, I_n)^{\underline{*}}$, then for almost all maximal ideals m of D (i.e. for m in a dense open subset of* MaxSpec*D), if $k = D/m$, then $u_k \in (I_{1,k}, \ldots, I_{n,k})^{\underline{*}}$ in R_k.*

Proof. By joining finitely many elements of K to D and localizing at an element, we obtain descent data (A, S_A, γ_A) for

$$S = R[w_1, \ldots, w_n, \frac{I_1}{w_1}, \ldots, \frac{I_n}{w_n}],$$

$\gamma = (w_1, \ldots, w_n)$ and u. We can localize both A and D at an element of D so that A will be free and hence faithfully flat over D ([HR]).

Let $m \in \mathrm{MaxSpec}D$ and let $k = D/m$. Take $m' \in \mathrm{MaxSpec}A$ that lies over m, and let $k' = A/m'$. Since tight closure is independent of the choice of descent, $u_{k'} \in {\gamma_{k'}}^*$ in $S_{k'}$. On the other hand $S_{k'} = k' \otimes_k S_k$ (where $S_k = R_k[w_1, \ldots, w_n, \frac{I_{1,k}}{w_1}, \ldots, \frac{I_{n,k}}{w_n}]$). The map $k \longrightarrow k'$ is a finite separable extension of fields, since both k and k' are finite. It follows now from [HH3] Theorem $7.29a^o$ that $u_k \in {\gamma_k}^*$ in S_k, which implies that $u_k \in (I_{1,k}, \ldots, I_{n,k})^{\underline{*}}$ in R_k. □

We now introduce the notion of universal test element for TI closure in the case where R is an affine domain over a field of characteristic zero.

Definition 3.22. Let R be an affine algebra over a field of characteristic zero that is a domain and let $I_1, \ldots, I_n$ be ideals of R. Let $(D, R_D, I_{1,D}, \ldots, I_{n,D})$ be descent data. Then an element $c_D \in R_D^o$ is called a *universal test element for* $I_{1,D}, \ldots, I_{n,D}$ if for every $u \in (I_1, \ldots, I_n)^{\underline{*}}$, and almost all $m \in \mathrm{MaxSpec}D$, if $k = D/m$, then

$$c_k {u_k}^q \in {I_{1,k}}^q + \ldots + {I_{n,k}}^q,$$

for all positive powers q of p, where p is the characteristic of k.

Similar to the situation in the positive characteristic case, we can explicitly calculate universal test elements for the TI closure of a set of ideals. We first state the analogous theorem for tight closure.

Theorem 3.23 ([HH2] 2.4.10). *Let $A \supseteq \mathbf{Z}$ be a domain finitely generated over $\mathbf{Z}$ with fraction field $\mathcal{F}$, and let R_A be a finitely generated A-algebra. Suppose that $R_{\mathcal{F}}$ is an absolute domain of dimension d, that is, $\overline{\mathcal{F}} \otimes_{\mathcal{F}} R_{\mathcal{F}}$ (where $\overline{\mathcal{F}}$ is the algebraic closure of the field $\mathcal{F}$) is a domain. Let*

$$R_A = A[u_1, \ldots, u_m]/(g_1, \ldots, g_s).$$

Then every element of the ideal generated by the $(m-d) \times (m-d)$ minors of the Jacobian matrix $(\delta g_i/\delta u_j)$ is a universal test element of R_A over A.

Similar to the positive characteristic situation, we can deduce

Theorem 3.24. *Let K be a field of characteristic zero and let R be an equidimensional finitely generated reduced algebra over K. Take ideals $I_1, \ldots, I_n$ of R where for each $i = 1, \ldots, n$, I_i is minimally generated by the elements $f_1^i, \ldots, f_{m_i}^i$ of R. Let $(D, R_D, I_{1,D}, \ldots, I_{n,D})$ be descent data such that if $\mathcal{F}$ is the fraction field of D, then $R_{\mathcal{F}}$ is an absolute domain of dimension d. Suppose that R_D is presented as*

$$R_D = D[u_1, \ldots, u_m]/(g_1, \ldots, g_s).$$

Then every nonzero element of

$$I_{1,D}{}^{m_1-1} \dots I_{n,D}{}^{m_n-1} \mathcal{J}_{m-d}$$

is a universal test element for $I_{1,D}, \dots, I_{n,D}$*, where* $\mathcal{J}_{m-d}$ *is the ideal generated by the* $(m-d) \times (m-d)$ *minors of the Jacobian matrix* $(\delta g_i/\delta u_j)$.

Proof. Let $u \in (I_1, \dots, I_n)^{\underline{*}}$. Then for almost all maximal ideals m of R_D, if $k = D/m$, $u_k \in (I_{1,k}, \dots, I_{n,k})^{\underline{*}}$. Equivalently, by Theorem 3.20, $u_k \in ((w_1, \dots, w_n)S_k)^*$, where

$$S_D = R_D[w_1, \dots, w_n, \frac{I_1}{w_1}, \dots, \frac{I_n}{w_n}].$$

If c is a universal test element for S_D over S, then c_k is a test element for S_k for almost all $m \in \text{MaxSpec}D$, $k = D/m$ (see [HH2] or Theorem 1.15). If c happens to be in R_D, it will follow that c_k is a TI closure test element for the ideals $I_{1,k}, \dots, I_{n,k}$ for almost all $m \in \text{MaxSpec}D$, $k = D/m$, and so c will be a universal test element for $I_{1,D}, \dots, I_{n,D}$.

As in Theorem 3.12 we use the D-algebra structure of S_D and construct part of its Jacobian matrix, and take appropriate minors of the Jacobian to generate universal test elements for $I_{1,D}, \dots, I_{n,D}$ in R_D.

With the presentation of R_D as a finitely generated D-algebra given above, S_D will be isomorphic to the polynomial ring

$$D[u_1, \dots, u_m, w_1, \dots, w_n, x_1^1, \dots, x_{m_1}^1, \dots, x_1^n, \dots, x_{m_n}^n]$$

modulo the ideal generated by

$$g_1, \dots, g_s, w_1x_1^1 - F_1^1, \dots, w_1x_{m_1}^1 - F_{m_1}^1, \dots, w_nx_1^n - F_1^n, \dots, w_nx_{m_n}^n - F_{m_n}^n$$

and possibly other polynomials, where F_j^i is an element of the polynomial ring $D[u_1, \dots, u_m]$ whose image in R_D is f_j^i, $1 \leq i \leq n$, $1 \leq j \leq m_i$. A part of the Jacobian matrix of S_D will then look like the matrix shown in Figure 1 on page 25.

We are given that $\dim R_{\mathcal{F}} = d$, and so $\dim S_{\mathcal{F}} = d + n$. If $m' = m_1 + \dots + m_n$, then $m + n + m' - (n + d) = m + m' - d$, and so to obtain universal test elements via Theorem 3.23, we are interested in the ideal generated by the $(m + m' - d) \times (m + m' - d)$ minors of this matrix, which is contained in the ideal generated by the $(m + m' - d) \times (m + m' - d)$ minors of the Jacobian matrix of S_D.

Following the exact same steps as in the proof of Theorem 3.12, we can see that this ideal will contain the ideal

$$\mathcal{H} = I_{1,D}{}^{m_1-1} \dots I_{n,D}{}^{m_n-1} \mathcal{J}_{m-d}.$$

Since $\mathcal{H} \subseteq R_D$, every element of $\mathcal{H}$ gives a universal test element for $I_{1,D}, \dots, I_{n,D}$.

□

References

[B] Bourbaki, N. *Elements of Mathematics, Commutative Algebra*, Springer Verlag, 1989.

[BH] Bruns, W. and Herzog, J. *Cohen-Macaulay Rings*, vol. 39, Cambridge studies in advanced mathematics, revised edition, 1998.

[E] Eisenbud, D. *Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Mathematics, No. 150, Springer Verlag, New York 1995.

[EGA] A. Grothendieck. *Èlèments de Gèomètrie Algèbrique* II. I.H.E.S Publ. math. **4** (1965).

[F] Faridi, S. *Closure operations on ideals*, Thesis, University of Michigan, 2000.

[Ha] Hara, H. *A characterization of rational singularities in terms of injectivity of Frobenius maps*, Amer. J. Math. 120 (1998), no. 5, 981-996.

[Har] Harris, J. *Algebraic geometry. A first course*, Graduate Texts in Mathematics, No. 133. Springer-Verlag, New York, 1992.

[HH1] Hochster, M. and Huneke, C. *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. vol. 3 (1990), 31-116.

[HH2] Hochster, M. and Huneke, C. *Tight closure in equal characteristic zero*, preprint.

[HH3] Hochster, M. and Huneke, C. *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. 346 (1994), 1-62.

[HH4] Hochster, M. and Huneke, C. *Localization and test exponents for tight closure*, Michigan Math. J. 48 (2000), 305–329.

[HH5] Hochster, M. and Huneke, C. *Tight closure and strong F-regularity*, Mm. Soc. Math. France (N.S.) No. 38, (1989), 119–133.

[Ho1] Hochster, M. *The notion of tight closure in equal characteristic zero*, Appendix in *Tight closure and its applications* by C. Huneke, CBMS Regional Conference Series in Mathematics, 88.

[Ho2] Hochster, M. *The tight integral closure of a set of ideals*, J. Algebra 230 (2000), no. 1, 184–203.

[Ho3] Hochster, M. *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. (2) 96 (1972), 318–337.

[HR] Hochster, M. and Roberts, J.L. *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. 13 (1974), 115-175.

[Hu] Huneke, C. *Tight closure and its applications*, CBMS Regional Conference Series in Mathematics, 88.

[L1] Lipman, J. *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. No. 36 1969 195–279.

[L2] Lipman, J. *Relative Lipschitz-saturation*, Amer. J. Math. 97 (1975), no. 3, 791–813.

[M] Matsumura, H. *Commutative ring theory*, second edition, Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge-New York, 1989.

[Sm1] Smith, K.E. *F-Rational rings have rational singularities*, American Journal, 119, 159-180 (1997).

[Sm2] Smith, K.E. *Tight Closure commutes with localization in binomial rings*, Proc. Amer. Math. Soc. 129 (2001), no. 3, 667–669.

[V] Vasconcelos, W. *The Arithmetic of Blowup Algebras*, Cambridge University Press, 1994.

[Va] Valla, G. *Certain graded algebras are always Cohen-Macaulay*, J. Algebra 42(1976), 537-548.

[ZS1] Zariski, O. and Samuel, P. *Commutative Algebra*, Vol. 1, GTM 28, Springer Verlag, New York, 1975.

[ZS2] Zariski, O. and Samuel, P. *Commutative Algebra*, Vol. 2, Van Nostrand, Princeton, 1960.