Weak subintegral closure of ideals and connections with reductions and valuations

Marie A. Vitulli

University of Oregon
Eugene, OR 97403

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Background

Definition

An extension $A \subseteq B$ is *weakly subintegral* if

1. B is integral over A;
2. $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a bijection; and
3. residue field extensions are purely inseparable.

If rings are reduced and of finite type over $k = \overline{k}$ with $\mathrm{char}(k) = 0$ then (3) follows from (1)-(2). In this case, the residue field extensions are isomorphisms.

Definition

1. The *weak normalization* ${}^*_B A$ of A in B is the largest weakly subintegral extension of A in B.
2. If B is the normalization of a reduced Noetherian ring A then we write ${}^*A$ in lieu of ${}^*_B A$ and call ${}^*A$ the *weak normalization* of A.

Definition

An element $b \in B$ is said to be *weakly subintegral over A* provided that there exist $q \in \mathbb{N}$ and $a_i \in A$ $(1 \leq i \leq 2q+1)$ such that b satisfies the equations

$$F_n(T) = T^n + \sum_{i=1}^{n} \binom{n}{i} a_i T^{n-i} = 0 \quad (q+1 \leq n \leq 2q+1) \quad (1)$$

Reid, Roberts, and Singh proved that $b \in B$ is weakly subintegral over A if and only if $A \subseteq A[b]$ is a weakly subintegral extension.

NOTE: Let $F(T) = F_{2q+1}(T)$

- $F_{2q}(T) = (2q+1)F'(T)$;
- $F_n(T) = (2q+1)\cdots(n+1)F^{(2q+1-n)}(T)$
  for $(q+1 \leq n \leq 2q+1)$.

For a rational function h on an algebraic variety V let V_h denote the sets of points where h is regular and let $\Gamma_h \subset V_h \times \mathbb{C}$ denote the graph of $h : V_h \to \mathbb{C}$.

Proposition (Gaffney-Vitulli)

Let $V \subset \mathbb{C}^m$ be an irreducible algebraic variety. Let $A = \mathbb{C}[V]$ and let $h \in \overline{A}$. Then, $h \in {}^*A \Leftrightarrow$ there exists an affine variety $Y \subset \mathbb{C}^{m+1}$ such that:

1. *the projection onto the first m factors $p : Y \to \mathbb{C}^m$ is a finite morphism;*
2. *the restriction of p to $p^{-1}(V)$ is a homeomorphism; and*
3. $\Gamma_h \subset p^{-1}(V)$.

Definition

Consider $I \subset A \subset B$.

- We say $b \in B$ is *weakly subintegral over* I provided that there exist $q \in \mathbb{N}$ and $a_i \in I^i$ $(1 \leq i \leq 2q+1)$ such that $b^n + \sum_{i=1}^{n} \binom{n}{i} a_i b^{n-i} = 0 \quad (q+1 \leq n \leq 2q+1)$.
- We let ${}_B^*I = \{b \in B \mid b \text{ is weakly subintegral over } I\}$ and call ${}_B^*I$ the *weak subintegral closure of* I *in* B.
- The *weak subintegral closure of* I is the weak subintegral closure of I in A and is denoted by ${}^*I$.

Fact. ${}_B^*I$ is an ideal of ${}_B^*A$.

Theorem

For $I \subseteq A \subseteq B$, let R denote the Rees ring $A[It]$ and let $S = B[t]$. Then,

$$ {}_S^*R = \oplus_{n\geq 0} \, {}_B^*(I^n)t^n. $$

In particular, ${}_B^*(I^n)$ contains each element of B that is weakly subintegral over ${}_B^*(I^n)$, for all $n \geq 0$.

Corollary

Let A be a reduced ring with finitely many minimal primes and I a regular ideal of A. Let Q denote the total quotient ring of A and $R = A[It]$. Then

$$ {}^*R = \oplus_{n\geq 0} \, {}_Q^*(I^n)t^n. $$

Local Characterizations

- Joint work with T. Gaffney
- Assume A is a Noetherian ring.

Notation

For an ideal $I \subseteq A$ and $a \in A$

$$\operatorname{ord}_I(a) = \sup\{n \mid a \in I^n\}.$$

Next we let

$$\overline{v}_I(a) = \lim_{n\to\infty} \frac{\operatorname{ord}_I(a^n)}{n}.$$

(the *asymptotic Samuel function of* I) Let

$$I_> = \{a \in A \mid \overline{v}_I(a) > 1\}.$$

Example

$$\begin{aligned} I &= (x^2, y^2) \subset k[x,y] \\ v(x^a y^b) &= a+b \quad (v \text{ is the only Rees valuation of } I) \\ v(I) &= 2 \\ \overline{v}_I(f) &= \frac{v(f)}{v(I)} = \frac{v(f)}{2} > 1 \Leftrightarrow \\ v(f) &> 2 \Leftrightarrow \\ v(f) &\geq 3 \end{aligned}$$

Thus $I_> = \mathfrak{m}^3 \subset \overline{I} = \mathfrak{m}^2$.

Fact. $I_>$ is a subideal of $\overline{I}$.

Notation

For a non-nilpotent ideal I, let

$\mathcal{RV}(I) = \{(V_1, \mathfrak{m}_1), \dots, (V_r, \mathfrak{m}_r)\}$: Rees valuation rings of I, and

$$\{v_1, \dots, v_r\} : \text{ the corresponding Rees valuations.}$$

For $\mathbb{N}$-graded ring R let

$$R_+ = \oplus_{n>0} R_n,$$

$$\mathrm{Proj}(R) = \{P \in \mathrm{Spec}(R) \mid P \text{ homogeneous, } R_+ \not\subset P\}.$$

The following is an algebraic version of a result by LeJeune-Teissier.

Proposition

Let I be a nonzero proper ideal in a reduced local ring $(A, \mathfrak{m}, k)$, $R = A[It]$, $S = \overline{R}$, and $a \in A$. Then,

$$a \in \overline{I} \Leftrightarrow a \in IS_{(\mathfrak{q})} \ \forall \mathfrak{q} \in \mathrm{Proj}(S) \text{ s.t. } \mathfrak{q} \cap A = \mathfrak{m}.$$

There is analog of this for weak subintegral closure.

Proposition

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Connections with Reductions

- Joint work with T. Gaffney
- We still assume A is Noetherian

If $I = \overline{I}$ then $I_>$ is a subideal of I. Recall that

$$\overline{v}_I(a) = \min_j \left\{ \frac{v_j(a)}{v_j(I)} \right\},$$

where $v_j(I) = \min\{v_j(b) \mid b \in I\}$ and the v_j are the Rees valuations of I.

The ideal $I_>$

Lemma

Let I be an ideal of a Noetherian ring A. Then,

$$I_> = \bigcap_i \mathfrak{m}_i I V_i \cap A.$$

In particular, $I_>$ is an integrally closed ideal.

Proof. Let $a \in A$. Notice that $a \in I_> \Leftrightarrow v_j(a) > v_j(I)$ $(j = 1, \ldots, r)$. Since $(V_j, \mathfrak{m}_j)$ is a dvr the latter is true if and only if $a \in \mathfrak{m}_j I V_j$ for all $(V_j, \mathfrak{m}_j) \in \mathcal{RV}(I)$.

Conjecture of D. Lantz

The following was first conjectured by D. Lantz in the case of an $\mathfrak{m}$-primary ideal in 2-dimensional regular local ring $(A, \mathfrak{m})$.

Proposition

Let I be an ideal of a Noetherian ring A. Then, $I_> \subseteq {}^*I$.

Proof. Suppose that $a \in I_>$. Then we must have $\mathrm{ord}_I(a^n) > n$ for all $n \gg 0$. In particular, $a^n \in I^n$ for all $n \gg 0$. This immediately implies that $a \in {}^*I$.

Notation

For an ideal I we write

$$\mathcal{MR}(I)$$

for the set of minimal reductions of I.

Corollary

Let I be an ideal of a Noetherian ring A. Then,

$$I_> \subseteq \bigcap_{J \in \mathcal{MR}(I)} {}^*J.$$

Proof. Observe that if J is any reduction of I then $\overline{v}_J = \overline{v}_I$ and hence $J_> = I_>$. The assertion immediately follows from preceding Proposition.

Theorem

Let $(A, \mathfrak{m}, k)$ *be a local ring of dimension* d *such that* $k = \overline{k}$ *and* $\mathrm{char}(k) = 0$*. Suppose that* $I = \overline{I}$ *is an* $\mathfrak{m}$*-primary ideal. If* J *is any minimal reduction of* I*, then*

$$J + I_{>} = {}^{*}J.$$

Lemma

Let $I_1 \subseteq I_2$ and $J \subseteq \mathfrak{m}$ be ideals in a local ring $(A, \mathfrak{m})$. If $I_1 + JI_2$ is a reduction of I_2, then I_1 is a reduction of I_2.

Proposition

Let $J \subseteq I$ be ideals in a Noetherian ring. If $J + (I_> \cap I)$ is a reduction of I, then J is a reduction of I.

Corollary

If I is $\mathfrak{m}$-primary ideal in the local ring $(A, \mathfrak{m}, k)$ of dimension d, then $I/(I_> \cap I)$ is a k-vector space and $\dim_k(I/I_> \cap I) \geq d$.

Corollary

Let $(A, \mathfrak{m}, k)$ be a local ring of dimension d such that $k = \overline{k}$ has characteristic 0. Suppose that $I = \overline{I}$ is an $\mathfrak{m}$-primary ideal.

1. *If $\dim_k(I/I_>) = d$, then ${}^*J = I$ for every reduction J of I.*
2. *If $\dim_k(I/I_>) > d$, then $\bigcap_{J \in \mathcal{MR}(I)} {}^*J = I_>$.*

Proof of 1. Assume $\dim_k(I/I_>) = d$ and let $J \in \mathcal{MR}(I)$. Then, $J/(J \cap I_>) = I/I_>$ we must have $J + I_> = I$. Since $J + I_> \subseteq {}^*J$ we also have ${}^*J = I$.

Proof of 2. Assume $\dim_k(I/I_>) = D > d$. Choose $g_1, \ldots, g_D$ in I whose images form a k-basis for $I/I_>$.
The set of minimal reductions of $(g_1, \ldots, g_D)$ can be identified with a dense Zariski-open subset of the space of d-planes in $I/I_>$, which we identify with affine D-space. Intersecting over all minimal reductions J of $(g_1, \ldots, g_D)$ we get $\cap^* J/I_> = \cap(J + I_>)/I_>$ is the zero subspace.
Hence the intersection of the ideals $J + I_>$ over all minimal reductions of $(g_1, \ldots, g_D)$ is $I_>$.
Since every minimal reduction of $(g_1, \ldots, g_D)$ is a minimal reduction of I the result follows.