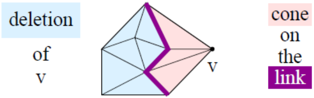
Recent Applications of Geometric Vertex Decomposition

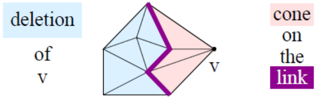
Sergio Da Silva

Combinatorial Algebra meets Algebraic Combinatorics
January 21, 2023

Geometric Vertex Decomposition



$$
\begin{array}{ccc}
 & I & \\
N_{y,I} & & C_{y,I}
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
 & \mathrm{in}(I) & \\
I_{\mathrm{del}_\Delta} & & I_{\mathrm{star}_\Delta}
\end{array}
$$



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I: $N_{y,I}$, $C_{y,I}$

$\mathrm{in}(I)$: I_{del_Δ}, I_{star_Δ}

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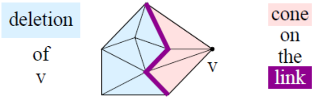

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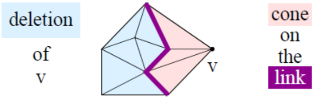

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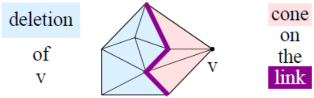

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- We can also build Gröbner bases via linkage

Geometric Vertex Decomposition

- Set $R = k[x_1, \ldots, x_n]$ and fix $y = x_i$ and a lex order with $y > x_j$.
- Let $I = \langle g_1, \ldots, g_m \rangle$ and write $g_i = y^{d_i} q_i + r_i$.

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Definition

The ideal I is *geometrically vertex decomposable* if I is unmixed with

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and $C_{y,I}$ and $N_{y,I}$ are GVD in $k[x_1, \dots, \hat{x}_i, \dots, x_n]$.

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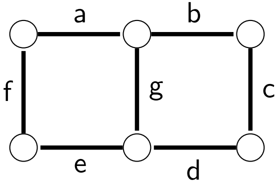

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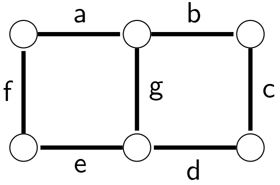

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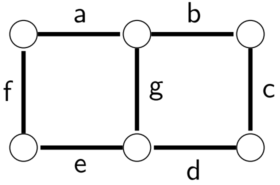

$$(g, e, f, a) \quad \rightarrow \quad fg - ae \in I_G$$

$$(g, e, f, a, g, d, c, b) \quad \rightarrow \quad cfg^2 - abde \in I_G$$

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Note that $C_{g,I}$ is already GVD, and setting $J = N_{g,I}$:

$$C_{f,J} = \langle bd \rangle \qquad N_{f,J} = \langle 0 \rangle$$

Structure Results

Joint work with Michael Cummings, Jenna Rajchgot, and Adam Van Tuyl:

- $N_{y,I_G} = I_{G\setminus y}$ and $C_{y,I_G} = \bigcap_{i=1}^{d}(M_i + I_{G\setminus E_i})$.

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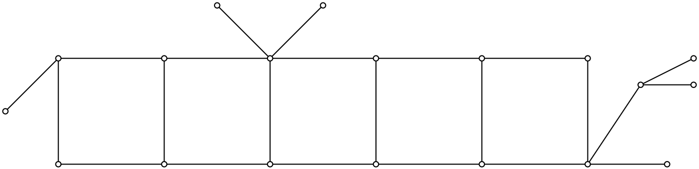
- Gluing on even cycles preserves the GVD property.



Families of Graphs

- I_G is GVD if G is a bipartite graph (building on proof of Constantinescu and Gorla).

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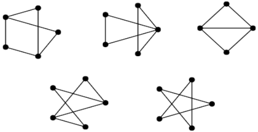

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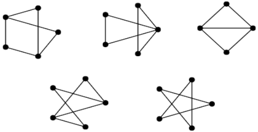
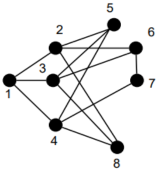


- Let G be a gap-free graph such that the graph complement $\bar{G}$ is not gap-free. Then I_G is glicci.

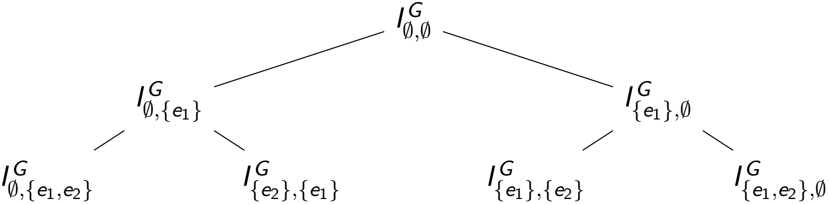


The Square-free Degeneration Case

Goal: Classify which toric ideals of graphs are GVD.

Conjecture

Let G be a finite simple graph with toric ideal I_G. If $\mathrm{in}_<(I_G)$ is square-free with respect to a lex ordering $<$, then I_G is GVD.



Height of I_G

In joint work with Jenna Rajchgot and Emma Naguit:

If I_G is a GVD, then $\mathrm{height}(I_G) = \#$ of "boundary" non-degenerate one-step GVDs of I_G.

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$$\mathrm{height}(I_G) = \begin{cases} |E| - |V| & \text{if } G \text{ is not bipartite} \\ |E| - |V| + 1 & \text{if } G \text{ is bipartite} \end{cases}$$

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Conjecture

Given any toric ideal I_G of a graph G, there exists at least one variable y and some order $<$ for which I_G is square-free in y. That is, there is some $<$ where $\mathrm{in}_<(I_G)$ is square-free in y.

What about the graph G?

- Suppose that I_G is GVD. Consider the graph deletions

$$G \setminus y_1, G \setminus \{y_1, y_2\}, \ldots$$

  corresponding to variables from GVD.
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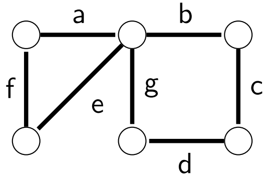
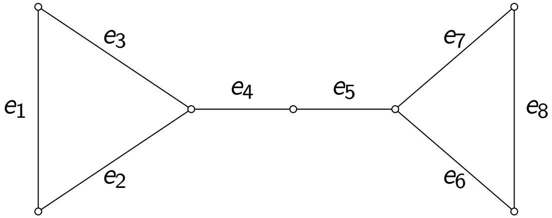
Theorem

Let G be a finite simple graph which is not bipartite such that I_G is GVD. Suppose y defines a degenerate GVD of I_G and is not a bridge of G. Then $G \setminus y$ is bipartite.



Geometric Vertex Decomposition by Substitution

What about the non-square-free case?



$$I_G = \langle e_1 e_4^2 e_6 e_7 - e_2 e_3 e_5^2 e_8 \rangle$$

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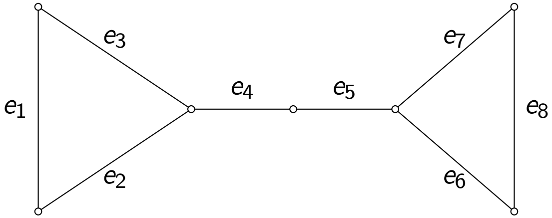


$$I_G = \langle e_1 e_4^2 e_6 e_7 - e_2 e_3 e_5^2 e_8 \rangle$$

Here I_G is not GVD. Make the substitution $y = e_4^2$:

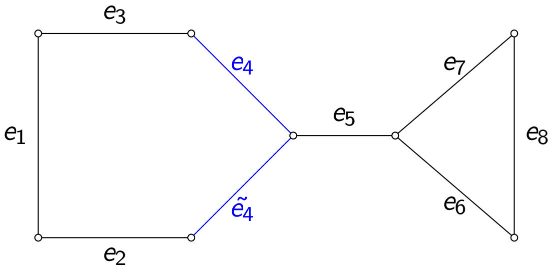
$$\langle y e_1 e_6 e_7 - e_2 e_3 e_5^2 e_8 \rangle \subseteq \mathbb{C}[e_1, e_2, e_3, y, e_5, e_6, e_7, e_8]$$

This is now GVD, but not the toric ideal of a graph.

Associated Graph

In joint work with Agnieszka Nachman:

Goal 1: Formalize how to associate graphs after substituting:



$$I_G = \langle e_1 e_4 \tilde{e}_4 e_6 e_7 - e_2 e_3 e_5^2 e_8 \rangle$$

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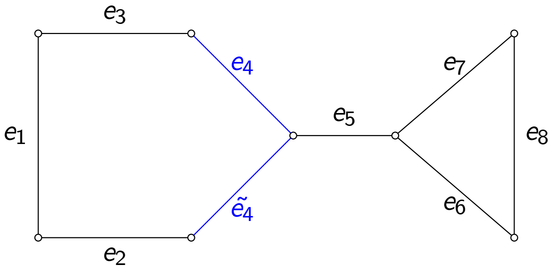
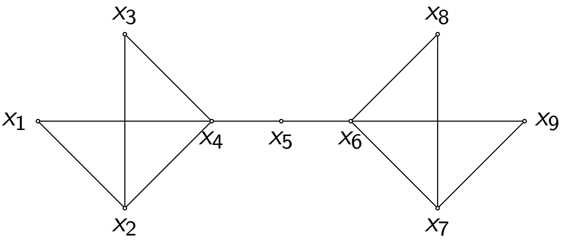


$$I_G = \langle e_1 e_4 \tilde{e}_4 e_6 e_7 - e_2 e_3 e_5^2 e_8 \rangle$$

Setting $\tilde{e}_4 = 1$ gives the ideal from the previous slide.

Goal 2: Find families of graphs which are not GVD, but are after substitution.



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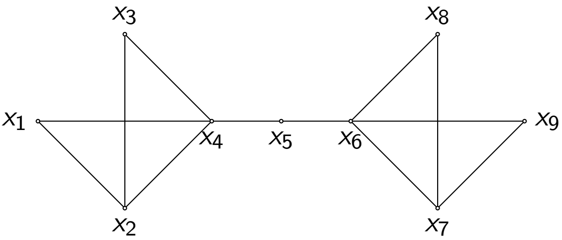

Conjecture

Let G_1 and G_2 be two graphs which are not bipartite, and suppose that I_{G_1} and I_{G_2} are GVD. Construct a new graph H by joining an odd cycle of G_1 to an odd cycle of G_2 by a path of length > 2. Then I_H is not GVD, but is up to substitution.

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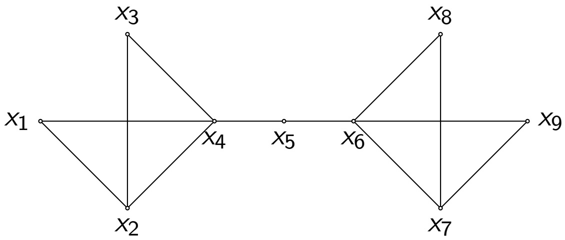

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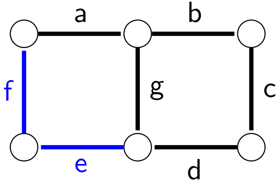
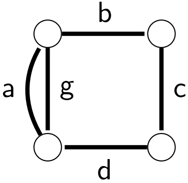
Result: Conjecture holds when G_1 and G_2 are bipartite with exactly one odd cycle glued on.

Edge Contractions

Goal 3: Find graph operations which preserve the list of primitive closed even walks.





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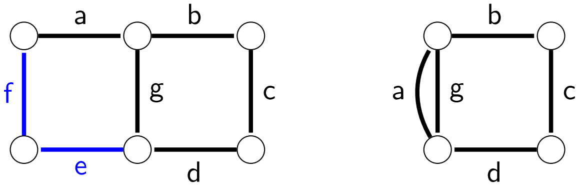

$$\langle ace - bdf, ae - fg, bd - cg\rangle \longrightarrow \langle ac - bd, a - g, bd - cg\rangle$$

Edge Contraction Results

Theorem

Choose a vertex v of G and contract all edges $e_1, \ldots, e_k$ incident to v. The set of primitive closed even walks of the contracted graph G_v is equal to the set of primitive closed even walks of G with $e_1 = \cdots = e_k = 1$.

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Example

Havel-Hakimi theorem can be used to compute when a given list of non-negative integers is the degree sequence of a graph.

Question: When is a homogeneous ideal the toric ideal of some graph G?

Apply the theorem to all possible subsets of variables set to 1. If we cannot determine whether the resulting ideal is the toric ideal of graph, continue the process.

Other Results

- With M. Harada, regular nilpotent Hessenberg varieties in the w_0-chart are GVD.
- With M.Cummings, M. Harada, and J. Rajchgot, regular nilpotent Hessenberg varieties in each Schubert cell are GVD. Provides a computational proof that regular nilpotent Hessenberg varieties have an affine paving.
- M. Cummings and A. Van Tuyl developed a Macaulay2 package for computing GVDs and related quantities.

Going Forward: There is a real need to optimize the general algorithm to be able to compute examples quickly.

Thank you!