Computing Fundamental Groups of Weighted Hassett Spaces

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Moduli Space $\overline{M_{0,n}}(\mathbb{R})$

We denote the moduli space of stable genus zero curves with $n \geq 3$ distinct marked points, up to automorphism, as $\overline{M_{0,n}}(\mathbb{R})$. In addition, we let $M_{0,n}(\mathbb{R}) \subseteq \overline{M_{0,n}}(\mathbb{R})$ be the subspace consisting of the automorphism classes of **smooth** curves. More formally, elements of $M_{0,n}(\mathbb{R})$, under equivalence of an action by $\mathrm{Aut}(\mathbb{RP}^1)$, are of the form $(\mathbb{RP}^1; x_1, x_2, x_3, \ldots, x_n)$, where $x_i \in \mathbb{RP}^1$ are distinct. Using the fact that any 3 distinct points in $\mathbb{RP}^1$ respectively map to any 3 distinct points via a *unique* $\varphi \in \mathrm{Aut}(\mathbb{RP}^1)$, we may set $x_1 = 0, x_2 = 1, x_3 = \infty$, and obtain

$$M_{0,n}(\mathbb{R}) = \{(\mathbb{RP}^1; 0, 1, \infty, x_4, x_5, \ldots, x_n) : x_i \in \mathbb{R} \setminus \{0,1\} \text{ distinct }\}$$

which gives a one-to-one correspondence between $M_{0,n}(\mathbb{R})$ and the configuration space $(\mathbb{R} \setminus \{0,1\})^{n-3} \setminus \cup_{4 \leq i < j \leq n}\{x_i = x_j\}$. The space, $\overline{M_{0,n}}(\mathbb{R})$, is known as the *Deligne-Mumford-Knudsen* compactification of $M_{0,n}(\mathbb{R})$.

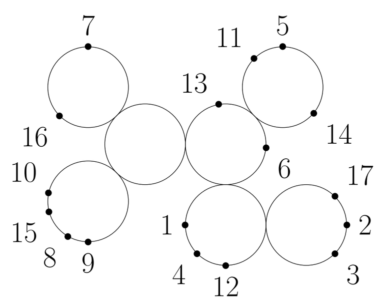

Figure 1: A boundary stratum in $\overline{M_{0,17}}(\mathbb{R})$.

Up to equivalence, an element in $\overline{M_{0,n}}(\mathbb{R})$ is a connected curve C that is a finite union of curves C_i, $1 \leq i \leq k$, with each C_i isomorphic to $\mathbb{RP}^1$; together with n distinct marked points $x_1, \ldots, x_n \in C$ such that each x_i is on a unique component; and for $i \neq j$, $C_i \cap C_j$ is either empty or consists of exactly one point, a nodal singularity. Furthermore, we require that the graph with vertex set $\{C_1, \ldots, C_k, x_1, \ldots, x_n\}$, and edge set $\{\{C_i, C_j\} : C_i \cap C_j \neq \varnothing\} \cup \{\{x_i, C_j\} : x_i \in C_j\}$, is a tree where each internal vertex has degree at least three. We consider this tree with its leaves labelled (x_i is labelled i), together with a natural plane embedding that is well defined up to a dihedral ordering[Dev00] at each internal vertex. We call the subset of elements associated to a particular tree and plane embedding, a boundary stratum in $\overline{M_{0,n}}(\mathbb{R})$.

Cactus Group

Definition 1. *The n-th order cactus group [Eti+10], J_n, is the group generated by elements $s_{p,q}, 1 \leq p < q \leq n$ with the following relations:*

(a) $s_{p,q}^2 = 1$ for each $p < q$;
(b) $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$ if $p < q < m < r$;
(c) $s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q}$ if $p \leq m < r \leq q$.

for convenience, we set $s_{p,p} = 1$.

The fundamental group, $\pi_1(\overline{M_{0,n+1}}(\mathbb{R}))$, is isomorphic to a particular subgroup of J_n. Furthermore, there is a short exact sequence of group homomorphisms

$$1 \to \pi_1(\overline{M_{0,n+1}}(\mathbb{R})) \to J_n \to S_n \to 1$$

where S_n denote the symmetric group on n symbols. The map, $J_n \to S_n$, sends $s_{p,q}$ to the involution that reverses the integers in $[p,q]$.

Some Notation:

- $\sigma_{p,q,r} := s_{p,r}s_{p,q}s_{q+1,r}$; $1 \leq p \leq q < r \leq n$
- $b_{p,q,r,m} := \sigma_{p,q,r}^{-1}\sigma_{p+r-q,r,m}^{-1}\sigma_{p,q,m}$; $1 \leq p \leq q < r < m \leq n$

Theorem 1 ([Eti+10])**.** *The group, $\pi_1(\overline{M_{0,n+1}}(\mathbb{R}))$ is generated by the elements $b_{p,q,r,m}$, as a normal subgroup of J_n.*

Weighted Hassett Space $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$

A weighted variant of the moduli space $\overline{M_{0,n}}(\mathbb{R})$ is constructed by Hassett [Has03]. Let $\mathcal{A} = (a_1, a_2, \ldots, a_n) \in (0,1]^n$ be a collection of weight data with $\sum a_i > 2$. We consider genus zero curves with n (not necessarily distinct) smooth marked points, $(C; x_1, \ldots, x_n)$, where C is a curve that has the tree structure as in the case of $\overline{M_{0,n}}(\mathbb{R})$. Intuitively, we assign a weight a_i to the i-th marked point, x_i. Let $\mathcal{N} \subseteq C$ denote the set of nodes in C. Let $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$ be the moduli space consisting of, up to automorphism, the elements $(C; x_1, \ldots, x_n)$ satisfying

1. For each component C_i of C, $\left(\sum_{\{j:\, x_j \in C_i\}} a_j\right) + |\mathcal{N} \cap C_i| > 2$; and
2. For each smooth point $y \in C$, $\sum_{\{j:\, x_j = y\}} a_j \leq 1$.

We refer to the spaces $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$ as **Hassett spaces**. It follows that if each $a_i = 1$, then $\overline{M_{0,\mathcal{A}}}(\mathbb{R}) = \overline{M_{0,n}}(\mathbb{R})$. Blankers et al.[BB22] gives a construction of Hassett spaces using **simplicial complexes** on $[n] := \{1, \ldots, n\}$: Subsets of $\mathcal{P}([n])$ which contain singletons and are closed under taking subsets. The weight data $\mathcal{A}$ corresponds to the simplicial complex $\{I \subseteq [n] : \sum_{i \in I} a_i \leq 1\}$, the sets of weights that may coincide in $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$.

Construction via blowups of $\mathbb{RP}^{n-3}$

The unweighted moduli space, $\overline{M_{0,n}}(\mathbb{R})$, can be constructed via an iterated sequence of blow-ups applied to $\mathbb{RP}^{n-3}$. This is Kapranov's [Kap92, Section 4.3] construction, and its steps are as follows:

1. Let $W_0 := \mathbb{RP}^{n-3}$. Choose $n-1$ points $q_1, \ldots, q_{n-1} \in \mathbb{RP}^{n-3}$ in general position.
2. Blow up the points $q_i \in W_0$ in any order. Denote W_1 as the resulting variety.
3. Consider $k = 2, 3, .., n-4$ in increasing order. For each k-subset $I \in \binom{[n-1]}{k}$ considered in any order, blow up the (strict transform of) the $k-1$-dimension subspace $S_I := \langle q_i : i \in I\rangle$ in W_{k-1}. Denote the resulting variety as W_k.
4. It follows that $W_{n-4} \simeq \overline{M_{0,n}}(\mathbb{R})$.

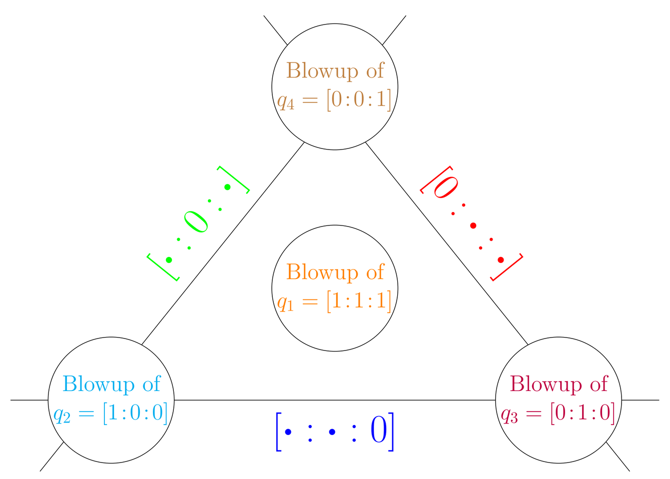

Figure 2: $\overline{M_{0,5}}(\mathbb{R})$ as a blowup of four points in $\mathbb{RP}^2$.

Convention Give $\mathbb{RP}^{n-3}$ the homogeneous coordinates $\mathbf{x} = [x_2 : x_3 : \ldots : x_{n-1}]$. Put $q_1 = [1 : 1 : \ldots : 1]$, and for $2 \leq i \leq n$, $q_i = [\underbrace{0 : \ldots : 0}_{i-1} : 1 : \underbrace{0 : \ldots : 0}_{n-2-i}]$. If $a_n = 1$, we have an inclusion $D \subseteq \mathbb{RP}^{n-3} \hookrightarrow \overline{M_{0,\mathcal{A}}}(\mathbb{R})$,

$$[x_2 : \ldots : x_{n-1}] \mapsto (\mathbb{RP}^1; x_1, x_2, \ldots, x_n)$$

with $x_1 = 0$ and $x_n = \infty$. In particular,

$$D = \left\{\mathbf{x} \,\middle|\, \sum_{i:\, x_i = y} a_i \leq 1 \quad \forall y \in \mathbb{RP}^1\right\}.$$

Suppose $n = 4$ and each $a_i = 1$. We blow up $q_1, q_2, q_3, q_4 \in \mathbb{RP}^2$ to form $\overline{M_{0,5}}(\mathbb{R})$. The points in $\mathbb{RP}^2$ away from the exceptional divisors (E_i of q_i) are elements of $M_{0,5}(\mathbb{R})$. Each E_i correspond to a non-smooth boundary stratum in $\overline{M_{0,5}}(\mathbb{R})$. For example, E_1 correspond to the stable curves where one of the components contain x_1, x_5, exactly one node, and no other x_j.

Assume $a_n = 1$ so that $x_n \neq x_j$ for each $j < n$. To obtain the Hassett space $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$, we perform the above steps, blowing up the space in the same order with a slight modification: Let $\mathcal{K} \subseteq \mathcal{P}([n-1]) \cup \{\{n\}\} \subseteq \mathcal{P}([n])$ denote the simplicial complex that $\mathcal{A}$ correspond to. We blow up S_I if and only if $[n-1] \setminus I \notin \mathcal{K}$.

Computing Fundamental Groups

The main goal of this project is to compute a combinatorial presentation of $\pi_1(\overline{M_{0,\mathcal{A}}}(\mathbb{R}))$ for certain weight data $\mathcal{A}$. We take a recursive approach, using the fact that the exceptional divisors obtained by blowups are naturally products of Hassett spaces of smaller dimension. We attempt to perform compuations in increasing order of n, and use Seifert van-Kampen to relate the fundamental groups of tubular neighbourhoods of exceptional divisors (and space away from the blowups) to that of the entire space. We also use the proposition below, which follows from the Whitney approximation theorem and an extended version of the transversality homotopy theorem.

Proposition 1. *Let X be a path connected manifold of finite dimension. Let $Z \subseteq X$ be an embedded submanifold. Assume $\mathrm{codim}_X(Z) \geq 3$. Then the inclusion $Z \hookrightarrow X$ induces an isomorphism $\pi_1(Z) \xrightarrow{\simeq} \pi_1(X)$.*

We roughly describe the recursive procedure of computing $\pi_1(\cdot)$. Consider a manifold X, that results after a sequence of blowups of finitely many closed subspaces of $\mathbb{RP}^{n-3}$. Let $Z \subseteq X$ be the strict transform of another closed subspace of $\mathbb{RP}^{n-3}$. Let $Y := \mathrm{Bl}_Z(X)$ be the blowup of Z in X. Suppose full presentations are known for $\pi_1(X)$ and $\pi_1(Z)$. We compute $\pi_1(Y)$ as follows: Take U to be a tubular neighbourhood of Z, and V to be Y with the exception divisor of Z removed, so $V \simeq X \setminus Z$. We use out knowledge of $\pi_1(X)$ and $\pi_1(Z)$ to obtain presentations for $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$. Finally, using Seifert Van-Kampen, we obtain a presentation for $\pi_1(Y)$.

Example Results

If X is $\mathbb{RP}^2$ blown up at k points, then $\pi_1(X) = \langle s_1, \ldots, s_{k+1} \mid s_1^2 s_2^2 \ldots s_{k+1}^2 = 1\rangle$. In particular, $\pi_1(\overline{M_{0,5}}(\mathbb{R})) = \langle s_1, \ldots, s_5 \mid s_1^2 \ldots s_5^2 = 1\rangle$. If $n \geq 6$ and X is $\mathbb{RP}^{n-3}$ blown up at k points, then $\pi_1(X) = (\mathbb{Z}/2\mathbb{Z})^{*(k+1)}$ is a free product of $k+1$ copies of $\mathbb{Z}/2\mathbb{Z}$. Consider the weight data $\mathcal{A}$ with $a_n = 1$. We have thus computed $\pi_1(\overline{M_{0,\mathcal{A}}}(\mathbb{R}))$ in the case where either $n = 5$; or $n > 5$ and $1 < \sum_{i=1}^{n-1} a_i \leq 1 + a_j$ for each $j < n$. We improve these slightly.

Theorem 2. *Assume $n \geq 7$. Let X denote the blowup of $\mathbb{RP}^{n-3}$ at q_2, q_3. Blow up the strict transform of the line $\langle q_2, q_3\rangle$ in X, and denote the resulting space as Y, and the exceptional divior as $E_{2,3}$. Then $\pi_1(X) = \langle l, l_2, l_3 \mid l^2 = l_2^2 = l_3^2 = 1\rangle$ and*

$$\pi_1(Y) = \langle \alpha, \beta, l, l_2, l_3 \mid \alpha^2 = l^2 = l_2^2 = l_3^2 = 1, \alpha\beta = \beta\alpha = l_3 l l_2\rangle$$

where the generators are illustrated by the figure below, $l_2 = {\gamma_2}^{-1}\iota_2\gamma_2$, and $l_3 = \gamma_3\iota_3{\gamma_3}^{-1}$.

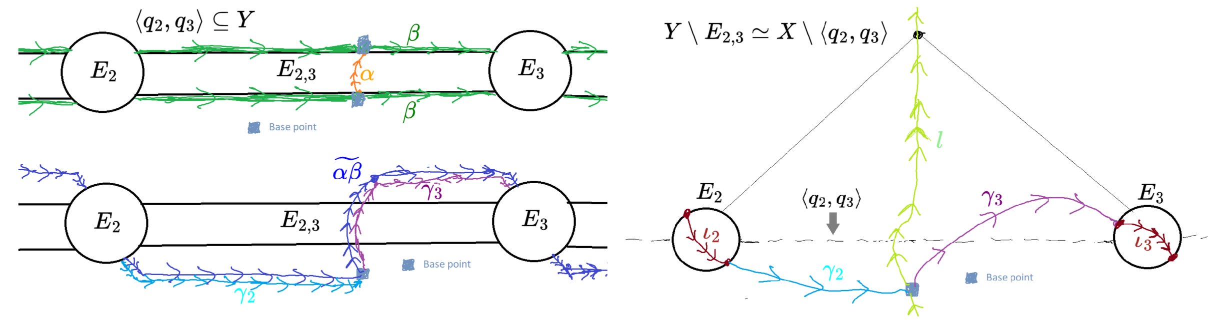

Figure 3: Illustrations for Theorem 2

Proof Sketch. Take U to be a tubular neighbourhood of $E_{2,3}$, and $V = Y \setminus E_{2,3} \simeq X \setminus \langle q_2, q_3\rangle$. Knowing that $U \cap V$ is a double cover of $E_{2,3}$, and that $\alpha\beta$ lifts to a loop $\widetilde{\alpha\beta} \in \pi_1(U \cap V)$, we determine that $\pi_1(U \cap V) = \langle\widetilde{\alpha\beta}\rangle \simeq \mathbb{Z}$, and $\widetilde{\alpha\beta} = l_3 l l_2 \in \pi_1(V) = \pi_1(X)$. □

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