Equations for $\overline{M}_{0,n}$

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Moduli space of n distinct points on $\mathbb{P}^1$

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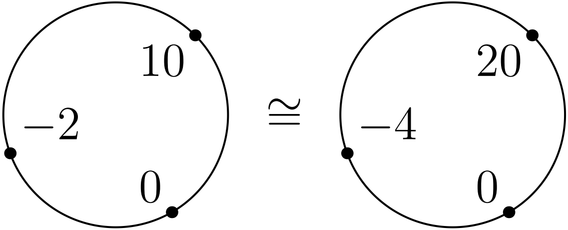

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&\cong \{(t_1, \ldots, t_{n-3}, 0, 1, \infty) : t_i \neq t_j, 0, 1, \infty\} \\
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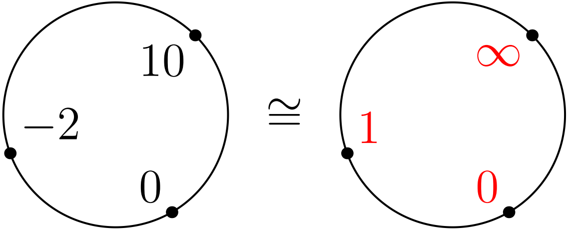

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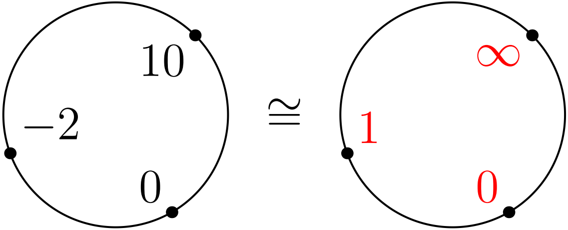
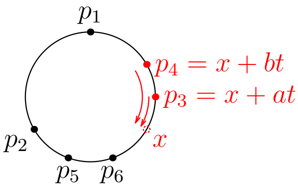


- This is not compact (points can't collide).
- $\overline{M}_{0,n}$: a nice compactification that "simulates" collisions. (Deligne–Mumford–Knudsen)

Stable curves and collisions

- Nodes represent collisions:



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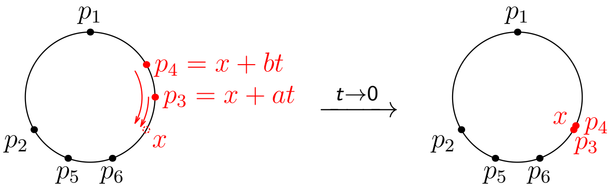

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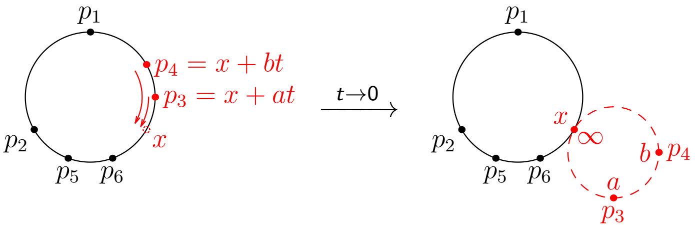

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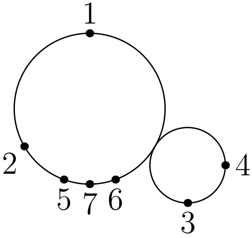
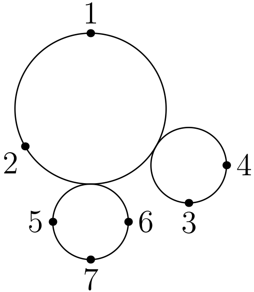
  - $p_1, \ldots, p_n \in C$: distinct smooth points





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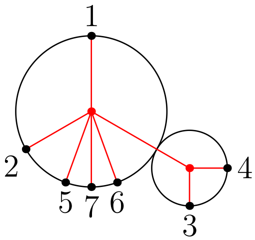

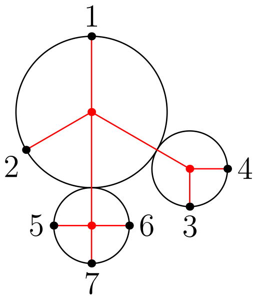

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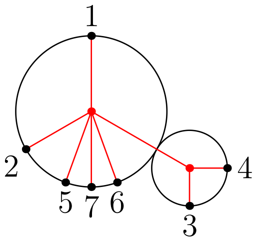
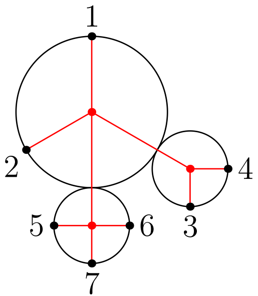
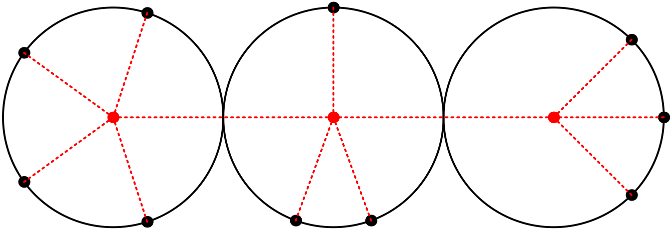




  - "stable": ≥ 3 **marked points and/or nodes** on each $\mathbb{P}^1$
    $\Leftrightarrow$ no nontrivial automorphisms of $(C, (p_1, \ldots, p_n))$.

Stratification of $\overline{M}_{0,n}$

- Strata of $\overline{M}_{0,n}$ are indexed by **at-least trivalent trees**
  - $X_T^\circ = \{(C, p_\bullet) : \text{dual tree of } C = T\}$, $X_T = \overline{X_T^\circ}$
  - $\operatorname{codim}(X_T) = \#\text{internal edges}(T)$



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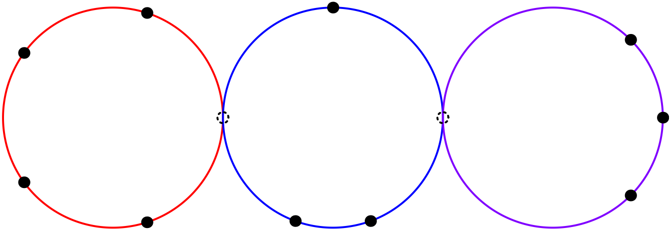

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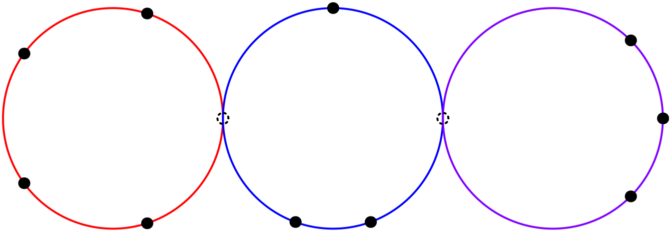

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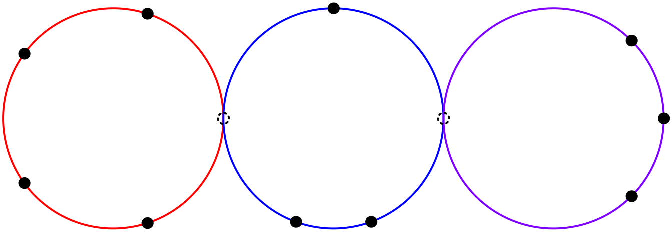

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- Fractal structure!
  $\partial\overline{M}_{0,n} = \overline{M}_{0,n} \setminus M_{0,n} =$ union of products of $\overline{M}_{0,n'}$.
  - Combinatorics of **labeled trees**
  - Operads (category theory), $H^*(\overline{M}_{0,n})$, rep theory, ...
  - and algebraic geometry of course!

Analogy: $\overline{M}_{0,n}$ vs $Gr(k,n)$

The **Grassmannian** is the moduli space of planes:

$$\mathrm{Gr}(k,n) = \{\text{subspaces } S \subseteq \mathbb{C}^n : \dim S = k\}.$$

	$\mathrm{Gr}(k,n)$	$\overline{M}_{0,n}$
$\mathrm{Aut}(M)$	k-planes in $\mathbb{C}^n$	n-pointed curves
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Maps to $\mathbb{P}^n$	Plücker coordinates	Kapranov coordinates

Kapranov maps

- Labels: $S = \{a, b, c, 1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$.
- i-th *Kapranov map* $\psi_i : \overline{M}_{0,S} \to \mathbb{P}^n$ by "zooming in on p_i":

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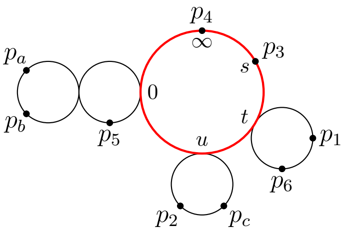

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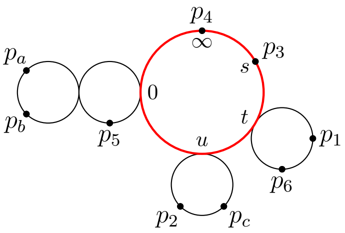

$$\xmapsto{\psi_4} \quad [z_b : z_c : \cdots \widehat{z_4} \cdots : z_6] \in \mathbb{P}^6$$

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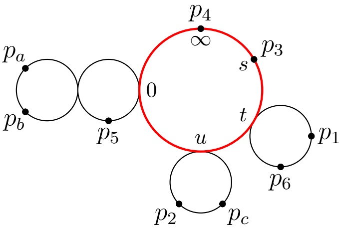

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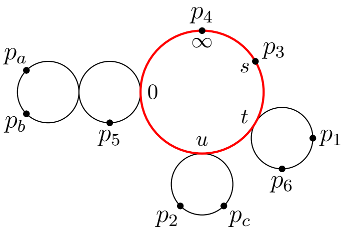

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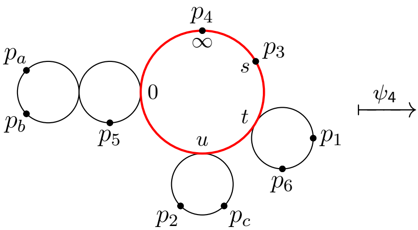

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- Hyperplane section: the i-th *psi class* $\in H^*(\overline{M}_{0,n+3})$.

Log-canonical embedding

- Forgetful map: $\pi_i : \overline{M}_{0,S} \to \overline{M}_{0,S\setminus\{i\}}$: forget p_i, then stabilize if necessary.

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- Composing with the Segre map gives the *log-canonical embedding*: $\overline{M}_{0,n+3} \hookrightarrow \mathbb{P}^{(n+1)!-1}$.

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Example

$\overline{M}_{0,5} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is a surface of bi-degree $(1, 2)$.

Aside: Multidegrees and combinatorics

The map $\Omega_n : \overline{M}_{0,n+3} \hookrightarrow \mathbb{P}^n \times \cdots \times \mathbb{P}^1$ has nice combinatorics:

- Cavalieri–Gillespie–Monin (2021):
  Total degree (sum of multidegrees) of $\Omega_n(\overline{M}_{0,n+3})$ is:

  $$(2n-1)!! = \# \text{ labelled trivalent trees on } n+2 \text{ leaves.}$$

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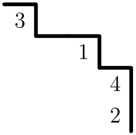

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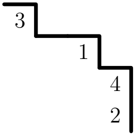
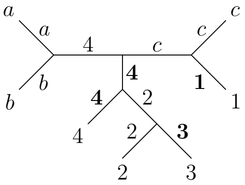
- Gillespie–Griffin–L (2022): Enumeration by boundary points on $\overline{M}_{0,n+3}$ (*lazy tournaments*).





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Conjecture (Monin–Rana 2017)

The image of $\Omega_n : \overline{M}_{0,n+3} \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$ is cut out by the 2×2 minors of several $2 \times k$ matrices (for various $2 \leq k \leq n$).

- Shown for $n \leq 8$ using Macaulay2.

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- Coordinates:
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Example

$\overline{M}_{0,5} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ (coordinates $[X_b : X_c], [Y_b : Y_c : Y_1]$) is:

$$\det \begin{bmatrix} X_b(Y_b - Y_1) & X_c(Y_c - Y_1) \\ Y_b & Y_c \end{bmatrix} = 0.$$

Combinatorial algebra

Theorem (Gillespie–Griffin–L 2022)

Monin–Rana's equations cut out $\Omega_n(\overline{M}_{0,n+3}) \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$ *for all* n.

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Sketch of proof:

0. Monin–Rana: $\Omega_n(\overline{M}_{0,n+3}) \subseteq \mathbb{V}(\mathrm{MR}_n)$.
1. Set-theoretic equality: $\Omega_n(\overline{M}_{0,n+3}) = \mathbb{V}(\mathrm{MR}_n)$ as sets.
2. Scheme-theoretic equality: tangent spaces agree.

Each step has combinatorics + algebra.

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Focus on Step 1.

Set-theoretic equality: **Graph coloring**

Let $x \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$ satisfy the Monin–Rana equations.

By induction: $\mathrm{pr}_n(x) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-1}$

$$= \Omega_{n-1}(C_{n-1}, p_\bullet) \text{ for some } (C_{n-1}, p_\bullet) \in \overline{M}_{0,n+2}.$$

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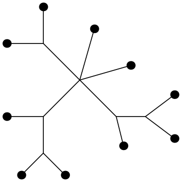

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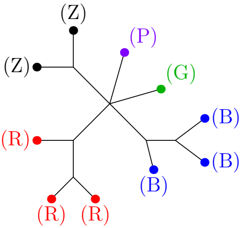

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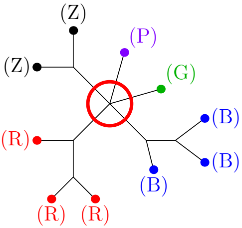
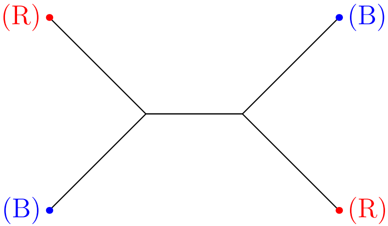


- Graph coloring of the dual tree of C_{n-1}.
- Monin–Rana equations $\Rightarrow$ "strong separation property".
- Identifies a unique vertex ($\leftrightarrow$ component $\mathbb{P}^1 \subseteq C_{n-1}$).

Noncrossing colorings

Roughly, we show that if a 2×2 minor of $MR_{i,k}$ is nonzero, then there's a "crossing coloring" of this form:



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- λ (up to coordinate change) says where to insert p_n on the $\mathbb{P}^1$.
- Gives $(C_n, p_\bullet)$ such that $\Omega_n(C_n, p_\bullet) = x$.

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- We linearize the Monin–Rana equations near each $x = \Omega_n(C)$ for $C \in \overline{M}_{0,n+3}$.

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Scheme-theoretic equality: **Tangent spaces**

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- We decompose the tangent space according to "branches" of the dual tree near $p_n \in C$.
- We show: $\dim T_x \Omega_n(\overline{M}_{0,n+3}) = \dim T_x \mathbb{V}(\mathrm{MR}_n)$.

Some questions for all of you

- Minimal generators for the ideal?
  (Recall: $I_d = J_d$ for $d \gg 0 \leftrightarrow \mathrm{Proj}(R/I) \cong \mathrm{Proj}(R/J)$.)
- Minimal free resolution?

- Equations for $\overline{M}_{0,n}$ in other embeddings? e.g.
  $\overline{M}_{0,n} \hookrightarrow (\mathbb{P}^1)^{\binom{n}{4}}$?

- Equations for variations on $\overline{M}_{0,n}$?
  Losev–Manin space LM_n (permutohedral variety)?
  Hassett spaces?

Thank you!