Computing Higher Direct Images of Toric Morphisms

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Overview

Higher direct images are a relative generalization of sheaf cohomology. They are used in derived categories, have applications to virtual resolutions, etc.. We would like an effective method for computing higher direct images.

Central Challenge

How do we frame this as a finite problem?

1. General varieties are hard to describe.
2. Morphisms between varieties can be very complicated.
3. Sheaves are fundamentally infinite objects.

Main Result

We give an effective algorithm for computing a local presentation of the higher direct images of line bundles for toric morphisms.

Higher Direct Images

Definition. Let $f : X \to Y$ be a morphism of projective varieties and $\mathcal{F}$ a quasi-coherent sheaf on X. For an integer i, the i-th higher direct image of $\mathcal{F}$ is the sheaf on Y defined by

$$R^i f_* \mathcal{F}(U) := H^i(f^{-1}(U), \mathcal{F})$$

for affine opens $U \subset Y$.

In the special case that Y is a point, higher direct images recover sheaf cohomology.

In practice, morphisms between projective varieties are difficult to describe. We specialize to the case of toric varieties and torus-equivariant morphisms, as these are well-suited for computation.

Toric Morphisms

Definition. Let X and Y be toric varieties with polyhedral fans $\Sigma_X \subset N_X$ and $\Sigma_Y \subset N_Y$ respectively. A morphism $f\colon X \to Y$ is called toric if:

Algebra

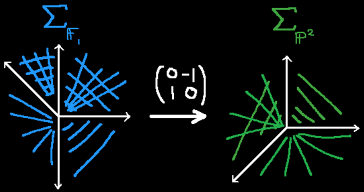
We have the commutative diagram

$$\begin{array}{ccc} T_X \times X & \longrightarrow & X \\ \big\downarrow {\scriptstyle f|_{T_X} \times f} & & \big\downarrow {\scriptstyle f} \\ T_Y \times Y & \longrightarrow & Y \end{array}$$

where $T_X \cong (\mathbb{C}^*)^{\dim X}$ and $T_Y \cong (\mathbb{C}^*)^{\dim Y}$.

Combinatorics

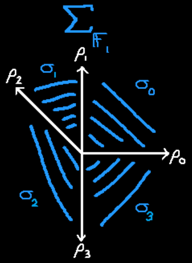
There is a corresponding linear map $\overline{f}\colon N_X \to N_Y$ such that for every cone $\sigma \in \Sigma_X$, the image $\overline{f}(\sigma)$ is contained in some cone $\tau \in \Sigma_Y$.



Cohomology of Toric Varieties

Assume X is a smooth projective toric variety of dimension n. The distinguished open cover $\{U_\sigma \mid \sigma \in \Sigma_X(n)\}$ allows one to compute sheaf cohomology via Čech cohomology.

Example. Let $X = \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and $\mathcal{F}$ a sheaf on X. We construct the "reduced" Čech complex.



$$0 \to \begin{array}{c} H^0(U_{\sigma_0},\mathcal{F}) \\ \oplus \\ H^0(U_{\sigma_1},\mathcal{F}) \\ \oplus \\ H^0(U_{\sigma_2},\mathcal{F}) \\ \oplus \\ H^0(U_{\sigma_3},\mathcal{F}) \end{array} \to \begin{array}{c} H^0(U_{\rho_0},\mathcal{F}) \\ \oplus \\ H^0(U_{\rho_1},\mathcal{F}) \\ \oplus \\ H^0(U_{\rho_2},\mathcal{F}) \\ \oplus \\ H^0(U_{\rho_3},\mathcal{F}) \end{array} \to H^0(U_{\{0\}},\mathcal{F}) \to 0$$

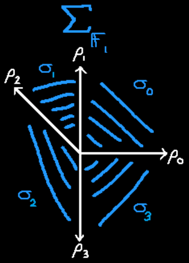
Cohomology of Toric Varieties

Suppose X has Picard rank r. The Cox ring of X is $R := \mathbb{C}[x_1, x_2, \dots, x_{n+r}]$. For $\mathcal{L} \in \operatorname{Pic} X$ and $\sigma \in \Sigma_X$, we have

$$R[x_i^{-1} \mid \rho_i \not\subset \sigma]_{\mathcal{L}} \cong H^0(U_\sigma, \mathcal{L}),$$

which allows one to compute sheaf cohomology of all line bundles simultaneously.

Example. Let $X = \mathbb{F}_1$.



$$0 \longrightarrow \begin{array}{c} R_{x_2x_3} \\ \oplus \\ R_{x_0x_3} \\ \oplus \\ R_{x_0x_1} \\ \oplus \\ R_{x_1x_2} \end{array} \longrightarrow \begin{array}{c} R_{x_1x_2x_3} \\ \oplus \\ R_{x_0x_2x_3} \\ \oplus \\ R_{x_0x_1x_3} \\ \oplus \\ R_{x_0x_1x_2} \end{array} \longrightarrow R_{x_0x_1x_2x_3} \longrightarrow 0$$

Strategy for Higher Direct Images

Let $\pi \colon X \to Y$ be a surjective toric morphism and $\mathcal{L}$ a line bundle on X. Fix a maximal cone $\tau \in \Sigma_Y$ and set $S = H^0(U_\tau, \mathcal{O}_Y)$.

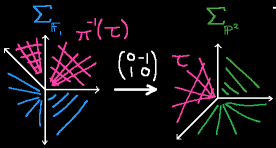
The preimage $\pi^{-1}(U_\tau)$ is a smooth toric variety, so we can compute $H^i(f^{-1}(U_\tau), \mathcal{L})$ using reduced Čech cohomology.

Problem. This computation yields an R-module, which is then an S-module. However, R is infinitely generated as an S-module, and we want a finite generating set over S.

Solution. The monomial lattice of S embeds into the monomial lattice of R as a linear subspace. We "slice" the reduced Čech complex by this linear subspace, to obtain finitely many polyhedra in the monomial lattice of S.

Example of Higher Direct Images

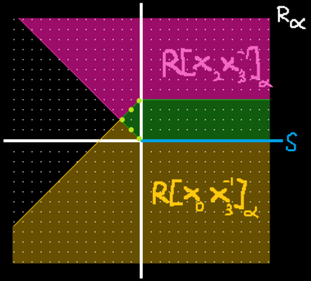
Let $\pi : \mathbb{F}_1 \to \mathbb{P}^2$ be the blowdown map and $\mathcal{L} \in \operatorname{Pic} \mathbb{F}_1$.



The reduced Čech complex for $\pi^{-1}(U_\tau)$ is

$$0 \to \begin{matrix} R_{x_2 x_3} \\ \oplus \\ R_{x_0 x_3} \end{matrix} \to R_{x_0 x_1 x_3} \to 0.$$

We want to replace the terms in this complex by S-modules.



We take the α-graded slice of the complex. The terms in the complex can be represented by polyhedra in the monomial lattice. We know that the cohomology will be finitely generated as an S-module.

Further Work

1. We want to glue together this local data to obtain a global formula for the higher direct images of a line bundle.
2. Using spectral sequence arguments, we want to extend a global formula to any coherent sheaf on X.
3. By experimenting on many toric morphisms, we want to try finding a vanishing pattern for higher direct images of coherent sheaves, ideally generalizing the known results for vanishing of sheaf cohomology.

Thank you!