An involution on the nilpotent commutator of a nilpotent matrix

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Abstract

We consider the irreducible variety $\mathcal{N}_B$ of nilpotent elements of the commutator $\mathcal{C}_B$ of a nilpotent $n \times n$ Jordan matix B having Jordan blocks given by the partition P of n, over a field K. Fix a homomorphism: $\pi : \mathcal{C}_B \rightarrow M_B$, where M_B is a product of matrix algebras over K, with kernel the Jacobson radical $\mathfrak{J}_B$ of $\mathcal{C}_B$. The inverse image of the subvariety U_r corresponding to strictly upper triangular matrices, is a maximal nilpotent subalgebra $\mathcal{U}_B$ of $\mathcal{C}_B$. R. Basili gave a specific homomorphism π, and parametrization of $\mathcal{U}_B$, that has been used by several. We describe an involution on $\mathcal{U}_B$, that is a generalized transpose. This involution underlies some of the symmetries we reported last year, in matrices related to the vanishing of elements of A^k, A generic in $\mathcal{U}_B$.

We pose several questions related to the open one of determining the Jordan partition of the generic element of $\mathcal{N}_B$.

CONTENTS

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Introduction. Several groups are studying similar problems.

- V. Baranovsky, R. Basili, and A. Premet.

Let $R = K\{x, y\}$, pow.series; K alg. closed. $V = K^n$;

$NP_n(K)$ = pairs (A, B) of nilpotent $n \times n$ matrices, $[A, B] = 0$.

$NP'_n(K)$ = pairs having a cyclic vector $v \in V$.

Fibration: $\tau : NP'_n(K) \times V \quad \to \text{Hilb}^n R$:

$\tau : (A, B, v) \to K[A, B]$, Artinian algebra.

Briançon's Thm. (1977). Reproved/extended by M. Granger (1983)

$\text{Hilb}^n K[x, y]$ is irred, over field K, char $K = 0$. (I.-also char $K > n$).

Thm. (Baranovsky, 2001)

$\text{Hilb}^n(R)$ irreducible $\Leftrightarrow NP_n(K)$ irreducible.

$\therefore NP_n(K)$ irred. char $K = 0$ or char $K = p > n$.

Thm. [Basili 2003], char $K = 0$ or $p > n/2$; [Pre] all alg. cld K:

$NP_n(K)$ is irred (proven directly). $\therefore \text{Hilb}^n(R)$ irred. $\forall K$.

- R. Basili-I.: Goal: Understand $\tau^{-1} Z_H, H$ a fixed Hilbert function.

Let $\mathcal{N}_B$ =$\{$ nilpotent $A \mid [A, B] = 0\}$. Subgoal: Understand $Q(P)$ = largest Jordan block partition of $A \in \mathcal{N}_B$.

Thm. $Q(P) = P$ iff parts of P differ pairwise by ≥ 2.

Thm. $A \in \mathcal{N}_B$ and $\exists$ cyclic $v \Rightarrow$ general $A + tB$ has partition $P(H)$,

($P(H)$ = partition giving lengths of the rows of bar graph of H.)

• T. Košir, P. Oblak: Goal: Understand $Q(P)$. Motivation: PDE.

Thm. (P. Oblak, 2006): Finds index of $Q(P)$. (Later also by BI).

Thm. (T. Košir and P. Oblak): $Q(Q(P)) = Q(P)$ ($Q(P)$ "stable").

• G. McNinch: Pencils of nilpotent matrices (char p Lie groups).

Thm. $A \in \mathcal{N}_B \Rightarrow A, B \in$ Jacobson radical of $A + tB$ for t generic (char $K = 0$ or most p, no need for cyclic v.)

• D.I. Panyushev: Goal: Understand Premet, (from Lie theory).

Thm. Determines "self large" orbits for any Lie group $\mathcal{G}$, char $K = 0$.

(In special case $G = gl(n)$, "self large"= stable). Pencils, char $K = 0$.

• T. Harima and J. Watanabe. Strong Lefschetz properties (SLP) of $x \in \mathcal{A}$, Artin algebra. Def x has SLP if m_{x^i} has max rank for $H, i > 0$.

Thm. $\mathcal{A}$ any graded Artinian $x \in \mathcal{A}_1$ generic $\Rightarrow$ x has a strong lefschetz property, under suitable (strong) conditions.

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Goal of present work:

A. Describe an involution ι on the fibres of π.

B. Contribute to understanding algebra structure of $\mathcal{U}_B$, maximal nilpotent subalgebra of $\mathcal{N}_B$: we give bases for $(U_B)^i$.

C. Generalize the problem of finding $Q(P)$.

1 What is $Q(P)$, maximal nilpotent orbit in $\mathcal{C}_B$?

Let K = algebraically closed field, $M_n(K) = n \times n$ matrices.

$\mathcal{N}(n, K) = \{\text{nilpotent } A \in M_n(K)\}$.

Fix $B \in \mathcal{N}(n, K)$ in Jordan form, of partition $P = (\lambda_1, \ldots, \lambda_t)$.

$\mathcal{C}_B = \mathcal{A} \in \mathcal{M}(n, k) \mid [A, B] = 0. \quad N_B = \mathcal{C}_B \cap \mathcal{N}(n, K)$.

Problem 1.1. Find $\mathcal{Q}(P) = \{\text{Jordan partitions of } A \in \mathcal{N}_B\}$.

Thm 1.2. *$\mathcal{N}_B$ is irreducible. $\mathcal{Q}(P)$ has a maximum, $Q(P)$.*

Ex 1.3. $P = (4)$, so B is *regular* (single Jordan block).

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

When $a \neq 0$, $A^3 \neq 0$ and $P(A) = (4)$.

When $a = 0, b \neq 0, A^3 = 0,\ P(A) = (2, 2)$

When $a = b = 0, c \neq 0$, then $P(A) = (2, 1, 1)$.

When $a = b = c = 0$ then $P(A) = (1, 1, 1, 1)$.

$(3, 1)$ *cannot occur for $P(A), A \in \mathcal{N}_B, P(B) = (4)$.*

1.1 The morphism $\pi : \mathcal{C}_B \to M_B$. (semisimple part)

R. Basili [Basili 2000] using [Tur, Ait] parametrized $\mathcal{N}_B$:

Ex 1.4. Let $P = (3,3,2), B = J_P$. Then $A \in \mathcal{C}_B$ satisfies:

$$A = \left(\begin{array}{ccc|ccc|cc} \underline{a_{11}^1} & a_{11}^2 & a_{11}^3 & \underline{a_{12}^1} & a_{12}^2 & a_{12}^3 & a_{13}^1 & a_{13}^2 \\ 0 & a_{11}^1 & a_{11}^2 & 0 & a_{12}^1 & a_{12}^2 & 0 & a_{13}^1 \\ 0 & 0 & a_{11}^1 & 0 & 0 & a_{12}^1 & 0 & 0 \\ \hline \underline{a_{21}^1} & a_{21}^2 & a_{21}^3 & \underline{a_{22}^1} & a_{22}^2 & a_{22}^3 & a_{23}^1 & a_{23}^2 \\ 0 & a_{21}^1 & a_{21}^2 & 0 & a_{22}^1 & a_{22}^2 & 0 & a_{23}^1 \\ 0 & 0 & a_{21}^1 & 0 & 0 & a_{22}^1 & 0 & 0 \\ \hline 0 & a_{31}^2 & a_{31}^3 & 0 & a_{32}^2 & a_{32}^3 & \underline{\alpha_{33}^1} & a_{33}^2 \\ 0 & 0 & a_{31}^2 & 0 & 0 & a_{32}^2 & 0 & \alpha_{33}^1 \end{array}\right)$$

with entries in the ring $\mathbb{Z}[a_{11}^1, \dots, a_{33}^2]$ in 21 variables. Let

$\mathfrak{J}$ = Jacobson rad. of $\mathcal{C}_B$, $M_B = \mathcal{C}(B)/\mathfrak{J}$ semisimple quotient.

Set $\mathcal{A}(3) = \begin{pmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{pmatrix}, \quad \mathcal{A}(2) = (\alpha_{33}^1),$

Morphism: $\pi : \mathcal{C}_B \to M_B : A \to (\mathcal{A}(3), \mathcal{A}(2))$.

Here $\mathcal{U}_B = \pi^{-1}$ (strictly upper triangular 2×2 matrices , 0).

1.2 Maximal nilpotent subalgebra $\mathcal{U}_B$ of $\mathcal{C}_B$.

Let the partition $P = (p_1^{r_1}, \dots, p_s^{r_s}), p_1 > \cdots > p_s$.

$M_B = M_{r_1}(K) \times \cdots \times M_{r_s}(K)$,

$N_r(B) = N_{r_1}(K) \times \cdots \times N_{r_s}(K)$. We have $\mathcal{N}_B = \pi^{-1}(N_r(B))$.

$U_r(B) = U_{r_1}(K) \times \cdots \times U_{r_s}(K)$. Let $\mathcal{U}_B = \pi^{-1}(U_r(B))$.

Lemma. $\mathcal{U}_B =$ a maximal nilpotent subalgebra of $\mathcal{C}_B$.

Def. *Digraph* $\mathcal{D}(A)$ of a matrix $A \in M_n(K)$: Directed graph:

Vertices $= \{1, 2, \dots n\}$; An arrow from i to j iff $A_{ij} \neq 0$.

Lemma. For A generic in $\mathcal{U}_B$, $\mathcal{D}(A)$ has no loops. Also

$$\forall k \in \mathbf{N}, \forall i, j \mid 1 \leq i, j \leq n, (A^k)_{ij} = 0 \Rightarrow (A^{k+1})_{ij} = 0.$$

Question. Is the rank of $A^k, k = 1, 2, \dots$ an invariant of $\mathcal{D}(P)$? Is this rank the same as that for a generic matrix of zeros and variables with the same digraph [Pol, KnZe]?

Thm.(P. Oblak): Yes, for $\min\{k \mid A^k = 0\}$: index of $Q(P)$.

P. Oblak determined this index [Ob1].

Thm [BI1, Pan]. $Q(P) = P \Leftrightarrow$ parts differ pairwise by ≥ 2.

(BI: P "stable" if $Q(P) = P$. Panyushev: P "self large")

Thm.(T. Kosir and P. Oblak)[KO]: $Q(Q(P)) = Q(P)$.

Proof: Show $K[A, B]$ is Gorenstein if $A \in \mathcal{N}_B$ is generic.

Ex 1.5. $P = (3, 1)$. Choose A generic in $\mathcal{U}(B) = \pi^{-1}(0, 0)$.

$$B = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad A = \left(\begin{array}{ccc|c} 0 & a & b & f \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & d & 0 \end{array}\right).$$

Then $A^2 = \alpha E_{13}, \alpha = a^2 + df$. If $\alpha \neq 0, P(A) = (3, 1)$

When $\alpha = 0, P(A) = (2, 2)$ or $(2, 1, 1)$ or $(1, 1, 1, 1)$.

Ex 1.6. $P = (3, 1, 1)$. $\mathcal{U}_B = \pi^{-1}\left(0, \left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)\right)$

$$B = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad A = \left(\begin{array}{ccc|cc} 0 & a & b & f & g \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & e & 0 & c \\ 0 & 0 & d & 0 & 0 \end{array}\right).$$

Here $A^3 = cdf\, E_{13}$ so $Q(P) = (4,1)$.

Also, $A^3 = 0$ iff $P(A) \leq (3,1,1)$,

Note: the $\mathcal{C}_B$ orbit of (3,1,1) in $\mathcal{U}_B$ is *reducible*, though its $\mathcal{C}_B$ orbit in $\mathcal{N}_B$ is *irreducible*.

We have

$$A^2 = \left(\begin{array}{ccc|cc} 0 & 0 & \alpha & 0 & cf \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & cd & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad A^3 = \left(\begin{array}{ccc|cc} 0 & 0 & cdf & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad A^4 = 0.$$

where $\alpha = a^2 + dg + ef$.

When $cdf \neq 0$ we have $P(A) = (4,1) = Q(P)$

When $cdf = 0$ but cd or ef or $\alpha \neq 0$ we have $\operatorname{rank} A^2 = 1$, and $P(A) = (3,2)$ if $\operatorname{rank} A = 3$ or $(3,1,1)$ if $\operatorname{rank} A = 2$.

When $A^2 = 0$, $P(A) = (2,2,1), (2,1,1,1)$ or $(1,1,1,1,1)$.

$$\mathcal{Q}(P) = \overline{(4,1)} = \{(4,1), (3,2), (3,1,1), (2,1,1,1), (1,1,1,1,1)\}.$$

2 An involution ι on $\mathcal{C}_B$ that restricts to U_B

2.1 An involution on partitioned matrices

Ex 2.1. The involution $\sigma_s(2,3)$ takes $M_5(R) \to M_5(R)$:

$$\left(\begin{array}{cc|ccc} a & b & \alpha_4' & \alpha_5' & \alpha_6' \\ c & d & \alpha_1' & \alpha_2' & \alpha_3' \\ \hline \alpha_3 & \alpha_6 & e & f & g \\ \alpha_2 & \alpha_5 & h & i & j \\ \alpha_1 & \alpha_4 & k & l & m \end{array}\right) \text{ to } \left(\begin{array}{cc|ccc} d & b & \alpha_4 & \alpha_5 & \alpha_6 \\ c & a & \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_3' & \alpha_6' & m & j & g \\ \alpha_2' & \alpha_5' & l & i & f \\ \alpha_1' & \alpha_4' & k & h & e \end{array}\right).$$

Definition 2.2. The action of $\sigma_s(a,b)$ on $M_{a+b}(R)$:

i. reflects the entries in the $a \times a$ block at the upper left, and in the $b \times b$ block in the lower right, about their non-main diagonals.

ii. Sends the $b \times a$ block in the lower left into the $a \times b$ block at upper right by transpose followed by reversing the order of rows, then reversing the order of columns.

2.2 The involution ι for $\mathcal{C}_B$, $P = (p^a, q^b)$

. We let $P = (p^a, q^b) = (p, \ldots, p; q, \ldots, q), p > q$.. Let $t = (a+b), n = ap + bq$. Let $M = \begin{pmatrix} M(1,1) & M(1,2) \\ M(2,1) & M(2,2) \end{pmatrix}$.

Replace the entries of $M \in M_{a+b}(K)$ by small blocks, forming $M' \in M_n(K)$.. These blocks are $M(1,1)$, $p \times p$ in the upper left M; $M(2,1)$, $p \times q$ in the upper right; $M(2,1)$, $q \times p$ in the lower left; and $M(2,2)$, $q \times q$ in the lower right. The small blocks have the circulant form found in $A \in \mathcal{U}_B$:

i. The matrices that comprise the entries of $M(2,1)$ have the first $p-q$ columns zero, followed by a circulant $q \times q$ subblock $C(2,1)_{uv}, 1 \leq u \leq b, 1 \leq v \leq a$.

ii. The matrices that comprise the entries of $M(1,2)$ have the last $p-q$ rows zero, preceded by a $q \times q$ matrix $B(1,2)_{uv}, 1 \leq u \leq a, 1 \leq v \leq b$.

Note: We use that circulant $q \times q$ matrices commute.

Def. For $P = (p^a, q^b)$, We define $\sigma_{s,P}$ on $\mathcal{C}_B, B = J_P$.

a. apply the involution $\sigma_s(a, b)$ to M, permuting the small blocks. However, in applying $\sigma_s(a, b)$ we must

b. replace each $q \times p$ entry $M(2,1)_{uv} = (0, C_{uv}), 1 \leq u \leq a, 1 \leq v \leq b$ of M_{21} by the $p \times q$ matrix $\begin{pmatrix} C_{uv} \\ 0 \end{pmatrix}$, and

c. replace each $p \times q$ entry $M(1,2)_{uv} = \begin{pmatrix} B_{uv} \\ 0 \end{pmatrix} 1 \leq u \leq b, 1 \leq v \leq a$ of M_{21} by the $q \times p$ matrix $(0, B_{uv})$.

This definition extends to $\iota = \sigma_{s,P} : \mathcal{C}_B \to \mathcal{C}_B$ for all P. Let $K[X_P]$ the ring of variables, entries of $A_{gen} \in \mathcal{C}_B$; define $\sigma : K[X_P] \to K[X_P]$ by the action of $\sigma_{s,P}$ on A_{gen}.

Lem 2.3. *We have for $U, V \in \mathcal{C}_B$, $\iota =$ general transpose:*

$$\iota(UV) = \iota(V) \cdot \iota(U); \quad U \in \mathcal{U}_B \Rightarrow \iota(U) \in \mathcal{U}_B. \tag{2.1}$$

We have for $U, V \in$ subring $K[A_{gen}] \subset \mathcal{C}_B$:

$$\iota(U) = \sigma(U), \text{ and } \iota(UV) = \iota(U)\ \iota(V). \tag{2.2}$$

and similarly for $K[A], A$ generic in $\mathcal{U}_B$.

Ex 2.4. Let $P = (3^2, 1^3)$. Then a generic $A \in C_B$ satisfies

$$A = \left(\begin{array}{ccc|ccc|ccc} \alpha_{11} & a_1 & a_2 & d & d_2 & d_3 & f_4 & f_5 & f_6 \\ 0 & \alpha_{11} & a_1 & 0 & d & d_2 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{11} & 0 & 0 & d & 0 & 0 & 0 \\ \hline \alpha_{21} & c & c_2 & \alpha_{22} & a_3 & a_4 & f & f_2 & f_3 \\ 0 & \alpha_{21} & c & 0 & \alpha_{22} & a_3 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{21} & 0 & 0 & \alpha_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & e_3 & 0 & 0 & e_6 & \beta_{11} & s & s_2 \\ 0 & 0 & e_2 & 0 & 0 & e_5 & \beta_{21} & \beta_{22} & t \\ 0 & 0 & e & 0 & 0 & e_4 & \beta_{31} & \beta_{32} & \beta_{33} \end{array}\right),$$

$$\pi(A) = \left(\begin{pmatrix} \alpha_{11} & d \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \begin{pmatrix} \beta_{11} & s & s_2 \\ \beta_{21} & \beta_{22} & t \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \right).$$

Then $\sigma_{s,P}$ reflects $\pi(A)$ about the non-main diagonals. and

$$\sigma_{s,P} : a_1 \to a_3, a_2 \to a_4; e \to f, e_i \to f_i, 2 \le i \le 6.$$

2.3 The vanishing-order matrix Pow(P); the matrix Powxe(P)

Def. $X_P = \{x_{ij} \mid \text{both } A_{ij} \neq 0, A_{ij}^2 = 0, A \text{ generic in } \mathcal{U}_B\}/\text{mod}$ Hankel relations $\}$. (i.e. We identify equal circulant entries)

$M_{X_1}(P) = n \times n$ matrix with

$$M_{X_1}(P)_{ij} = \begin{cases} x_{ij} \in X_P \text{ if } A \text{ generic in } \mathcal{U}_B \text{ has entry } A_{ij} \in X_P \\ 0 \text{ otherwise.} \end{cases} \tag{2.3}$$

$\text{Powxe}(P) = M_{X_1} + (M_{X_1})^2 + \cdots$.

$\text{Powx}(P)_{ij} =$ highest degree term of $\text{Powxe}(P)_{ij}$,

$\text{Pow}(P)$ integer matrix, $\text{Pow}(P)_{ij} =$ degree of $\text{Powx}(P)_{ij}$.

Ex 2.5. $P = (3)$,

$$M_{X_1} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{Powxe}(P) = \begin{pmatrix} 0 & a & a^2 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

For $P = (3,1,1)$, the generic $A \in U(B)$ and Powxe(P) are

$$A = \left(\begin{array}{ccc|cc} 0 & \underline{a} & b & \underline{f} & g \\ 0 & 0 & \underline{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & e & 0 & \underline{c} \\ 0 & 0 & \underline{d} & 0 & 0 \end{array}\right). \tag{2.4}$$

$$\text{Powxe}(P) = \left(\begin{array}{ccc|cc} 0 & a & cdf + a^2 & f & cf \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & cd & 0 & c \\ 0 & 0 & d & 0 & 0 \end{array}\right). \tag{2.5}$$

Here $\sigma : d \longrightarrow f, e \longrightarrow g$ and $\iota(\text{Powxe}(P) = \sigma(\text{Powxe}(P))$.

Also $\sigma(cdf + a^2) = cdf + a^2$ - entry fixed by ι;

$$\text{and } \iota \text{ takes } \begin{pmatrix} cd \\ d \end{pmatrix} \text{ to } \begin{pmatrix} f & cf \end{pmatrix} = \sigma \begin{pmatrix} d & cd \end{pmatrix},$$

2.4 Constructing Powxe(P), an example.

Ex 2.6. For $P = (3^2, 1^3)$. $A \in \mathcal{U}_B$ and Pow = Pow(P) are

$$A = \left(\begin{array}{ccc|ccc|ccc} 0 & a_1 & a_2 & d & d_2 & d_3 & f_4 & f_5 & f_6 \\ 0 & 0 & a_1 & 0 & d & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\ \hline 0 & c & c_2 & 0 & a_3 & a_4 & f & f_2 & f_3 \\ 0 & 0 & c & 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & e_3 & 0 & 0 & e_6 & 0 & s & s_2 \\ 0 & 0 & e_2 & 0 & 0 & e_5 & 0 & 0 & t \\ 0 & 0 & e & 0 & 0 & e_4 & 0 & 0 & 0 \end{array}\right), \ \mathrm{Pow} = \left(\begin{array}{ccc|ccc|ccc} 0 & 2 & 5 & 1 & 3 & 6 & 2 & 3 & 4 \\ 0 & 0 & 2 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 4 & 0 & 2 & 5 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 & 0 & 4 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \end{array}\right).$$

Here the variables X_1 of M_{X_1} are $\{c, d, e, f, s, t\}$ and correspond to the entries 1 of Pow(P).

We have for $P = (3^2, 1^3) = (3,3,1,1,1)$, Powxe$(P)$ is

$$\left(\begin{array}{ccc|ccc|ccc}
0 & cd & defst + c^2d^2 & d & cd^2 & \underline{d^2efst} + c^2d^3 & df & dfs & dfst \\
0 & 0 & cd & 0 & d & cd^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
\hline
0 & c & efst + c^2d & 0 & cd & defst + c^2d^2 & f & fs & fst \\
0 & 0 & c & 0 & 0 & cd & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & est & 0 & 0 & dest & 0 & s & st \\
0 & 0 & et & 0 & 0 & det & 0 & 0 & t \\
0 & 0 & e & 0 & 0 & de & 0 & 0 & 0
\end{array}\right).$$

Here $Q(P)$ has two parts (by an R. Basili result, as $P = p^a, q^b, p > q+1$ has $r_P = 2$); the highest nonzero power of a generic $A \in \mathcal{U}_B$ is $A^6 = \underline{d^2efst}\, E_{16}$, hence $Q(P) = (7,2)$.

Here Powxe(P) shows the symmety

$$\iota(\text{Powxe}(P)) = \sigma(\text{Powxe}(P),$$

and is evidently simply constructed from $M_{X_1} \in \mathcal{U}_B$. [BI2].

2.5 Pow(P) and a basis for ${\mathcal{U}_B}^i$.

[1] Let $P = (p_1^{r_1}, \ldots, p_t^{r_t}), p_1 > \cdots > p_t$, and let A be a generic element of $\mathcal{U}_B$. If the entry $A_{ij} \neq 0$ and $A_{ij} \neq A_{i-1,j-1}$ we denote it by x_{ij}, and the set of all such by X_P (one variable for each small Hankel diagonal). Let $s_i = r_1 + \cdots + r_{i+1}$. Considering $\pi : \mathcal{C}_B \to M_B$, $\dim_K(U_B) = \#\, X_P$ satisfies

$$\#\, X_P = \sum_i \left(i r_i \left(r_i + 2s_i\right) - r_i \left(\frac{r_i + 1}{2}\right)\right). \tag{2.6}$$

Let $S_P = \{i \mid r_i > 0\}$, and $\forall i \in S_P$, $j_i = r_i + \max\{r_{i-1}, r_{i+1}\}$ (jump index), $s = \sum r_i$, and recall $t = \#\, S_P$. We denote by

$$X_k = \{x_{ij} \in X_P \mid A_{ij}^k \neq 0 \text{ but } A_{ij}^{k+1} = 0\} \tag{2.7}$$

Thus, X_k comprise the distinct variables from X_P corresponding to entries k of Pow(P). We have [BI2, Sec. 3.1]

$$\#\, X_1 = s + 2(t-1) - \#\, \{i \mid j_i > r_i\} \tag{2.8}$$

[1]This section, an algebraic interpretation of some of the results in [BI2], was inspired by our discussions at the ‘CA meets AC’ conference January 08 with J. Weyman and T. Koŝir.

We let $\mathcal{B}_P = I + \mathcal{U}_B$, and filter it by the ideals

$$\mathcal{B}_P \supset U_B \supset {U_B}^2 \supset \cdots \supset {U_B}^{e_P} \supset 0.$$

Here $e_P = i(Q(P)) - 1, i(Q(P))$ = index of $Q(P)$, the largest part. We set $U_B^0 = \mathcal{B}_P$. Denote by $E = \langle \{e_{ij}, 1 \leq i, j \leq n\} \rangle$, the n^2-dim vector space. For $x_{ij} \in X_P$, let $v_{ij} \in E$ satisfy $v_{ij} = \sum' e_{uv}$ where $\sum'$ is over $\{uv \mid A_{uv} = x_{ij}\}$. Let $V_k = \{v_{ij}, \mid x_{ij} \in X_k\}$, and $\langle V_k \rangle \subset E$ their span, $V = \sum_{k=1}^{e_P} V_k$.

Thm. We have the internal direct sums

A. $\mathcal{B}_P = \oplus_{k=0}^{e_P} \langle V_i \rangle \cong \oplus_{k=0}^{e_P} {U_B}^k / {U_B}^{k+1}$;

B. for $i \geq 0$, $(U_B)^i = \oplus_{k \geq i} \langle V_k \rangle$.

C. Also, for $1 \leq i \leq e_P, \quad 0 : (U_B)^i = (U_B)^{e_P - i}$.

Proof Outline. We write e_{ij} also for the corresponding element of U_B, provided $x_{ij} \in X_P$. (So $U_B \subset V$). Let $u \in {U_B}^k \subset E$ have nonzero component on some e_{ij} (with $x_{ij} \in X_k$). Then we achieve v_{ij} as a product of k elements $v_1 \times \cdots \times v_k, v_i \in V_1$.

□

Ex 2.7. $P = (3, 11)$.

$$A = \left(\begin{array}{ccc|cc} 0 & \underline{x_{12}} & x_{13} & \underline{x_{14}} & x_{15} \\ 0 & 0 & \underline{x_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & x_{43} & 0 & \underline{x_{45}} \\ 0 & 0 & \underline{x_{53}} & 0 & 0 \end{array}\right).$$

Here $v_{12} = e_{12} + e_{23}, v_{33} = e_{33}, \ldots, v_{53} = e_{53}$.

$U_B/{U_B}^2 = V_1 = \langle v_{12}, v_{14}, v_{45}, v_{53}\rangle$.

${U_B}^2/{U_B}^3 = V_2 = \langle v_{15}, v_{43}\rangle$ and ${U_B}^3 = V_3 = \langle v_{13}\rangle$.

The action of ι extends to V, and each V_i is ι-invariant.

Remark. There is symmetry here and for some other (not all) P in the "U_B-Hilbert functions", when stratified by large matrix blocks", corresponding to $3, (3, 1), 1$. Here $H_{U_B}(V_1) = (1, 2, 1), H_{U_B}(V_2) = (0, 2, 0)$.

Problem. Let A_i = generic element of ${U_B}^i$. We have, evidently, rank $A_i \geq$ rank A^i. Compare these ranks.

3 What is $Q_S(P)$ – maximal nilpotent orbit in $\pi^{-1}(M_S(B))$?

3.1 Nilpotent multi-orbits $M_S(B) \subset M(B)$.

Definition 3.1. Let $P = (p_1^{r_1}, \ldots, p_k^{r_k}), p_1 > \ldots > p_k$. Let $\langle r_i \rangle$ = POS of partitions of r_i. Let $S = (S_1, \ldots, S_k), S_i \in r_i, 1 \leq i \leq k$. Let $\mathfrak{S}(P) = \{S \in \langle r_1 \rangle \times \cdots \times \langle r_k \rangle\}$.

$M_S(B)$ = nilpotent multi-orbit in $M_{r_1}(K) \times \cdots \times M_{r_k}(K)$ determined by S.

Since $M_S(B)$ is irreducible and $\pi^{-1}(M_S(B))$ is fibred over $M_S(B)$ by an affine space isomorphic to the Jacobson radical $\mathfrak{J}$ of $\mathcal{C}_B$, we have $\pi^{-1}(M_S(B))$ is irreducible.

We denote by $Q_S(B)$ the partition giving the Jordan blocks of a generic element of $\pi^{-1}(M_S(B))$.

Ex 3.2. When $S = ((r_1), \ldots, (r_k))$ (each S_i a single Jordan block), then $M_S(B) = M(B), Q_S(B) = Q(B)$.

Let $0 = S_0 = ((1^{r_1}), \ldots, (1^{r_k}))$ then $M_S(B) = \{(0, \ldots, 0)\}$, and $Q_0(B)$ is the maximal partition for an element of $\mathfrak{J}$.

Observation. When the distinct parts of P differ by two or more, then $Q_0(P) = P$; otherwise, $Q_0(P) \neq P$.

For $P = (2, 1^3), S = ((1), (1^3))$, then $Q_0(B) = (3, 1, 1) \neq P$.

Problem: Find $Q_S(B)$ for each S. Interpolates between $Q(P)$, and the generic orbit for $\mathcal{A} \in \mathfrak{J}$, the Jacobson radical.

Lem 3.3 (Lifting). *i. Let $\sigma \in Gl_{r_1}(K) \times \cdots \times Gl_{r_k}(K)$ and $M, M' \in M(B)$, and let $A \in \mathcal{C}_B$ with $\pi(A) = M$. Then there is a unit $\sigma' \in \mathcal{C}_B$ such that $\pi(\sigma'(A)) = A'$.*

ii. $Q_S(P) = P(A)$ for A generic in $\pi^{-1}(J_{S_1}, \ldots, J_{S_k})$.

That is, in finding $Q_S(P)$ we may assume that $\pi(A)$ has components each in Jordan block form.

3.2 The partition $Q_S(P)$

Def: For a fixed P denote by $\mathfrak{Q}(P)$ the POS

$$\mathfrak{Q}(P) = \{Q_S(P) \quad \forall S \in (\mathfrak{P}(r_1) \times \cdots \times \mathfrak{P}(r_k))\},$$

Lem 3.4. *: $S \to Q_S(P)$ is a map of POS: $\mathfrak{S}(P) \to \mathfrak{Q}(P)$.*

For a partition $(S_1 = (s_{11}, \ldots, s_{1t})$, we let $m(S_1) = (ms_{11}, \ldots ms_{1t})$.

Ex 3.5 (Observation). Let $P = (m^a) = (m, \ldots, m)$, and let S_1 be a partition of (a). Then $Q_{S_1}(P) = m(S_1)$.

Ex 3.6 (Observation). [$Q_S(P)$ for hooks] Let $P = (p, 1^b) \mid p > 1$. Then the map $S \to Q_S(P) : \mathfrak{S} \to \mathfrak{Q}(P)$, is an isomorphism of lattices.

$Q_0(P) = P$ if $p \geq 3$; $\quad Q_0(P) = (3, 1^{b-1})$ if $p = 2$.

Let $S = ((1), R), T \in \mathcal{P}(B)$. Then $Q_S(P)$ is obtained by "adding" T to $Q_0(P)$: add $T_i - 1$ to $Q_0(P)_i, i = 1, 2, \ldots$ until the sum n is attained.

Ex $P = (2, 1^4)$ (see Ex 3.7B). $Q_0(P) = (3, 1, 1)$. $S = (2, 2)$

$$Q_S(P) = (2, 2) + (3, 1, 1, 1) = (3 + 2 - 1, 1 + 2 - 1) = (4, 2)$$

Ex 3.7. Hooks, $p = 2$.

A. $P = (2, 1^3)$; $\mathfrak{S} = \langle 1 \rangle \times \langle 3 \rangle$.

$Q_S(P)$	S
(5)	(3)
$(4,1)$	$(2,1)$
$(3,1,1)$	$(1,1,1)$

B. $P = (2, 1^4)$; $\mathfrak{S} = \langle 1 \rangle \times \langle 4 \rangle$.

$Q_S(P)$		S	
(6)		(4)	
$(5,1)$	$(4,2)$	$(3,1)$	$(2,2)$
$(4,1,1)$		$(2,1,1)$	
$(3,1^3)$		(1^4)	

Ex 3.8. Hook: $p = 3$. $P = (3, 1^4)$; $\mathfrak{S} = \langle 1 \rangle \times \langle 4 \rangle$.

$Q_S(P)$		S
$(6,1)$	$\longleftarrow$	(4)
$(5,1,1)$ $(4,2,1)$	$\longleftarrow$	$(3,1)$ $(2,2)$
$(4,1,1,1)$	$\longleftarrow$	$(2,1,1)$
$(3,1^4)$	$\longleftarrow$	(1^4)

Ex 3.9. $P = (2^2, 1^3)$; $\mathfrak{S} = \langle 2 \rangle \times \langle 3 \rangle$.

$Q_S(P)$		S
(7)	$\longleftarrow$	$(2) \times (3)$
$(5,2)$	$\longleftarrow$	$(1,1) \times (3)$ $(2) \times (2,1)$
$(4,3)$	$\longleftarrow$	$(1,1) \times (2,1)$
$(4,2,1)$	$\longleftarrow$	$(2) \times (1,1,1)$
$(3,3,1)$	$\longleftarrow$	$(1,1) \times (1,1,1)$

$\mathfrak{S}(P) \to \mathfrak{Q}(P)$ is *not* an isomorphism of POS.

($(1,1) \times (2,1)$ and $(2) \times (1,1,1)$ are incomparable in $\mathfrak{S}(P)$.)

3.3 Questions: the involution ι and $Q_S(P)$.

a. To what extent is $Q_S(P)$ an invariant of the digraph $\mathcal{D}(A)$, or digraph wih involution ι, for A generic in $U_S(B)$?

b. What other invariants of P are steps toward $Q_S(P)$?

c. Fix P. The condition of A being in $\pi^{-1}(J_{r_1}, \ldots, J_{r_k})$ leads to a different digraph-with-involution $\mathcal{D}'$ than $\mathcal{D}$ for A generic in $\mathcal{U}_B$. But the lengths of longest paths from $i \to j$ are unchanged, as the matrix M_{X_1} is in this fibre.

   Is the S. Poljak calculation of partitions for the generic matrices of digraphs $\mathcal{D}, \mathcal{D}'$ the same? And what is their relation to $Q(P)$?

d. Can the ranks of A^k, A generic in $\mathcal{U}_B$ be concluded from those of certain powers (or powers and sums) of M_{X_1}?

e. Fix $S = (S_1, \ldots, S_k)$. By regarding the intersection of $X_1(P)$ with $\pi^{-1}(J_{S_1}, \ldots, J_{S_k})$, one can construct variables

$X_1(S)$ and matrices $M_{X_1(S)}$. Can the ranks of powers of generic elements of the same fibres, be figured from the ranks of powers and sums of $M_{X_1(S)}$?

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