Minimal resolutions of monomial ideals

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Combinatorial Algebra Meets Algebraic Combinatorics #17

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Outline

Kaplansky's problem

Fix $I \subseteq \Bbbk[\mathbf{x}]$ monomial ideal $\mathbf{x} = x_1, \ldots, x_n$

[Kaplansky, early 1960s]. Find minimal free resolution of I

Def. Koszul simplicial complex $K^{\mathbf{b}}I = \{\sigma \in \{0,1\}^n \mid \mathbf{x}^{\mathbf{b}-\sigma} \in I\}$ at $\mathbf{b} \in \mathbb{N}^n$

[Hochster's Formula]. $\operatorname{Tor}_i(\Bbbk, I)_{\mathbf{b}} \cong \widetilde{H}_{i-1}(K^{\mathbf{b}}I; \Bbbk)$

Grading. $F_\bullet : 0 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{n-1} \leftarrow 0$
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$\mathbb{N}^n$-graded $\Rightarrow F_{i+1} \cong \bigoplus_{\mathbf{b} \in \mathbb{N}^n} \widetilde{H}_i(K^{\mathbf{b}}I; \Bbbk) \otimes_{\Bbbk} \Bbbk[\mathbf{x}](-\mathbf{b})$

Note. $F_i \leftarrow F_{i+1}$ on $F^{\mathbf{b}}_{i+1}$ determined by action on $\widetilde{H}_i K^{\mathbf{b}} I$

Note. $(F^{\mathbf{a}}_i)_{\mathbf{b}} = \begin{cases} \widetilde{H}_{i-1}K^{\mathbf{a}}I & \text{if } \mathbf{a} \preceq \mathbf{b} \\ 0 & \text{otherwise} \end{cases}$

Kaplansky's problem $\Leftrightarrow$ find maps $\widetilde{H}_{i-1}K^{\mathbf{a}}I \leftarrow \widetilde{H}_i K^{\mathbf{b}}I$ for $\mathbf{a} \prec \mathbf{b}$ whose induced maps $F^{\mathbf{a}}_i \leftarrow F^{\mathbf{b}}_{i+1}$ constitute a free resolution of I.

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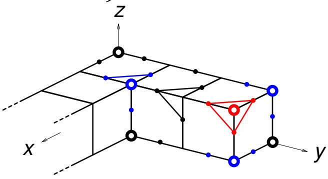

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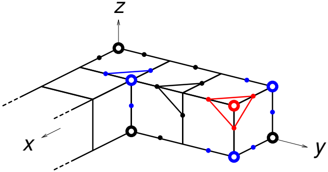

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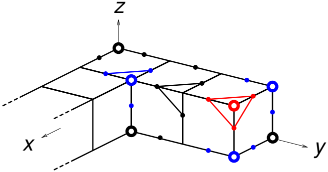

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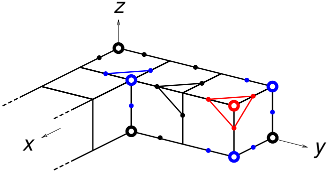

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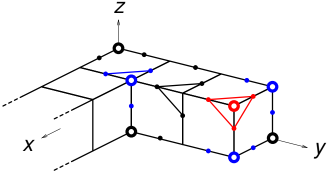

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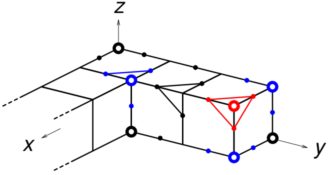

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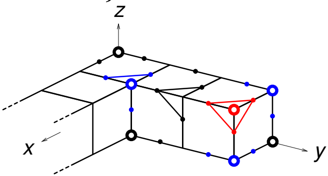

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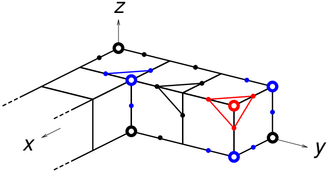

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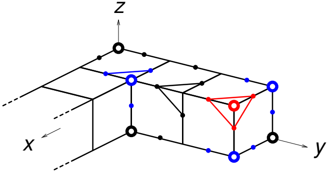

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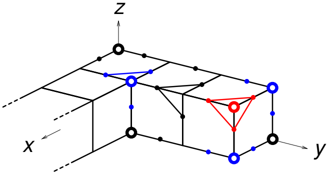

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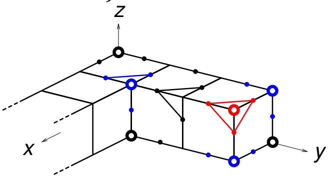

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Kaplansky's problem $\Leftrightarrow$ find maps $\widetilde{H}_{i-1} K^{\mathbf{a}} I \leftarrow \widetilde{H}_i K^{\mathbf{b}} I$ for $\mathbf{a} \prec \mathbf{b}$ whose induced maps $F_i^{\mathbf{a}} \leftarrow F_{i+1}^{\mathbf{b}}$ constitute a free resolution of I.

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Fix $I \subseteq \Bbbk[\mathbf{x}]$ monomial ideal $\qquad \mathbf{x} = x_1, \ldots, x_n$

[Kaplansky, early 1960s]. Find minimal free resolution of I

Def. Koszul simplicial complex $K^{\mathbf{b}}I = \{\sigma \in \{0,1\}^n \mid \mathbf{x}^{\mathbf{b}-\sigma} \in I\}$ at $\mathbf{b} \in \mathbb{N}^n$

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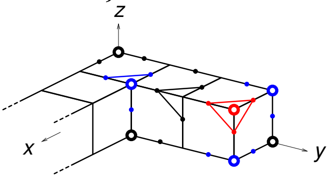
Cor. $\qquad \beta_{i,\mathbf{b}}(I) = \qquad \dim_{\Bbbk}$



Grading. $F_\bullet : \; 0 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{n-1} \leftarrow 0$
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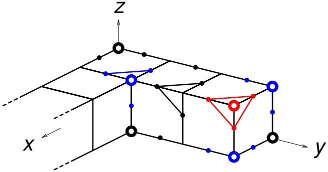

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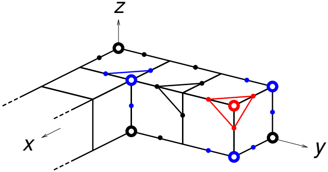

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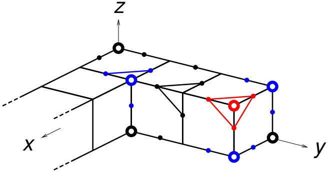

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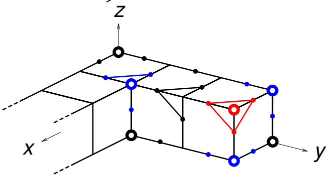

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Kaplansky's problem

Wish List.
- universal
- canonical
- closed form
- combinatorial
- minimal

Past progress

- [Taylor 1966] not minimal
- [Lyubeznik 1988] not minimal or canonical
- Wall resolutions [Eagon 1990] not proved combinatorial or universal
- stable ideals [Eliahou–Kervaire 1990] not universal
- hull resolutions [Bayer–Sturmfels 1998] not minimal
- [Bayer–Peeva–Sturmfels 1998, M–Sturmfels–Yanagawa 2000]
  - generic monomial ideals: not universal
  - degenerate Scarf resolutions: not minimal or canonical
- [Yuzvinsky 1999] not combinatorial (and claimed not canonical)
- shellable monomial ideals [Batzies–Welker 2002] not universal
- trivariate monomial ideals [M 2002] not canonical
- order complex of Betti poset [Tchernev–Varisco 2015] not minimal
- Buchberger resolutions [Olteanu–Welker 2016] not canonical or minimal

Subsequent development

- [Tchernev 2019] not closed-form (algorithmically combinatorial)

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Sylvan matrices

Obstacle. Express maps $\widetilde{H}_{i-1}K^{\mathbf{a}}I \leftarrow \widetilde{H}_iK^{\mathbf{b}}I$ for $\mathbf{a} \prec \mathbf{b}$ canonically

Suffices. $\widetilde{H}_i$ given as cycles $\widetilde{Z}_i \subseteq \widetilde{C}_i$

Def. For each $\mathbf{a} \prec \mathbf{b}$, the sylvan matrix for $F_i \leftarrow F_{i+1}$ has block $D^{\mathbf{ab}}$ of the form

$$\widetilde{H}_{i-1}K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \xleftarrow{\begin{array}{c|ccc} & \tau_1 & \cdots & \tau_n \\ \hline \sigma_1 & & & \\ \vdots & & & \\ \sigma_m & & & \end{array}} \widetilde{H}_iK^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle$$

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$$\begin{array}{c} \quad \tau_1 \quad \cdots \quad \tau_n \quad \leftarrow \quad i\text{-faces of } K^{\mathbf{b}} \\ \begin{matrix}\sigma_1 \\ \vdots \\ \sigma_m\end{matrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \\ \widetilde{H}_{i-1}K^{\mathbf{a}} \otimes \langle \mathbf{x}^{\mathbf{a}} \rangle \longleftarrow \widetilde{H}_i K^{\mathbf{b}} \otimes \langle \mathbf{x}^{\mathbf{b}} \rangle \end{array}$$

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$(i-1)$-faces of $K^{\mathbf{a}}$ ↘ ; $\tau_1 \cdots \tau_n \leftarrow$ i-faces of $K^{\mathbf{b}}$

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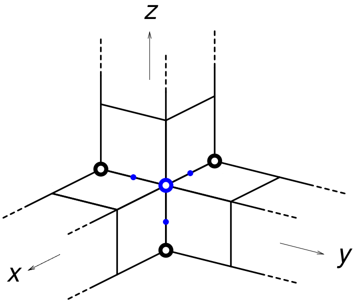
Sylvan matrices

Example 1. $I = \langle xy, yz, xz \rangle$ has Betti number $\beta_{1,111}(I) = 2$ from $K^{111}I$:



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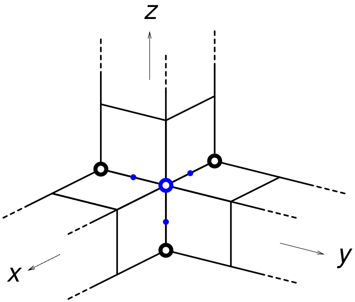

$$\begin{matrix} \widetilde{H}_{-1}K^{110} \otimes \langle xy \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{101} \otimes \langle xz \rangle \\ \oplus \\ \widetilde{H}_{-1}K^{011} \otimes \langle yz \rangle \end{matrix} \xleftarrow{\begin{matrix} & x & y & z \\ \varnothing & [0 & 0 & 1] \\ \varnothing & [0 & 1 & 0] \\ \varnothing & [1 & 0 & 0] \end{matrix}} \widetilde{H}_0 K^{111} \otimes \langle xyz \rangle$$

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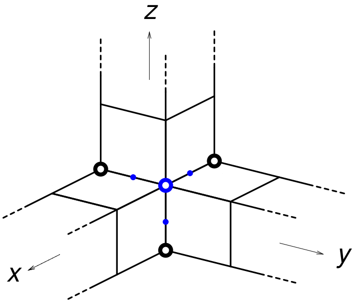

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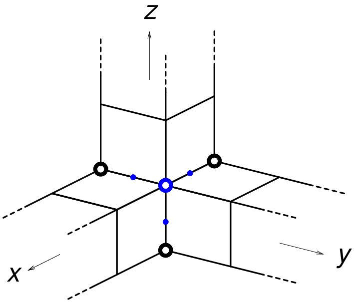

$$
\begin{array}{rcccl}
 & \tilde{H}_{-1}K^{110}\otimes\langle xy\rangle & & \begin{array}{cccc} & x & y & z \\ \varnothing & [0 & 0 & 1] \\ \varnothing & [0 & 1 & 0] \\ \varnothing & [1 & 0 & 0] \end{array} & (x-z)\otimes xyz \\
 & \oplus & & & \\
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 & \oplus & & & \\
\varnothing\otimes x\cdot yz & \tilde{H}_{-1}K^{011}\otimes\langle yz\rangle & & &
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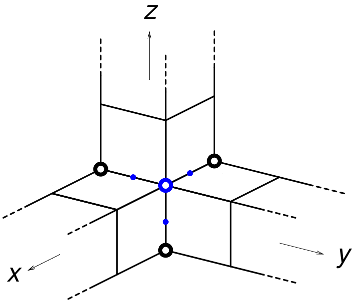
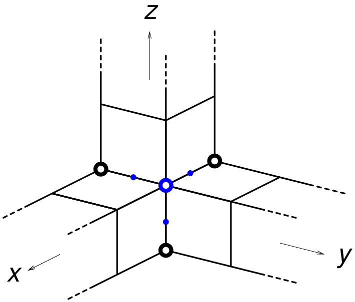


$$
\begin{array}{ccc}
 & \widetilde{H}_{-1}K^{110} \otimes \langle xy \rangle & \\
 & \bigoplus & \\
 & \widetilde{H}_{-1}K^{101} \otimes \langle xz \rangle & \xleftarrow{\quad \begin{array}{c} \quad x \quad y \quad z \\ \begin{array}{c} \varnothing \\ \varnothing \\ \varnothing \end{array} \begin{bmatrix} [0 & 0 & 1] \\ [0 & 1 & 0] \\ [1 & 0 & 0] \end{bmatrix} \end{array} \quad} \ \widetilde{H}_0 K^{111} \otimes \langle xyz \rangle \\
 & \bigoplus & \\
\varnothing \otimes x \cdot yz & \widetilde{H}_{-1}K^{011} \otimes \langle yz \rangle &
\end{array}
$$

$(x - z) \otimes xyz$

Sylvan matrices

Example 1. $I = \langle xy, yz, xz \rangle$ has Betti number $\beta_{1,111}(I) = 2$ from $K^{111}I$:



$$-\varnothing \otimes z \cdot xy \qquad \widetilde{H}_{-1}K^{110} \otimes \langle xy \rangle$$

$$\oplus$$

$$\widetilde{H}_{-1}K^{101} \otimes \langle xz \rangle \xleftarrow{\begin{array}{c} \quad x \quad y \quad z \\ \begin{array}{c}\varnothing\\ \varnothing\\ \varnothing\end{array} \begin{bmatrix} [0 & 0 & 1] \\ [0 & 1 & 0] \\ [1 & 0 & 0] \end{bmatrix} \end{array}} \widetilde{H}_0 K^{111} \otimes \langle xyz \rangle \qquad (x - z) \otimes xyz$$

$$\oplus$$

$$\varnothing \otimes x \cdot yz \qquad \widetilde{H}_{-1}K^{011} \otimes \langle yz \rangle$$

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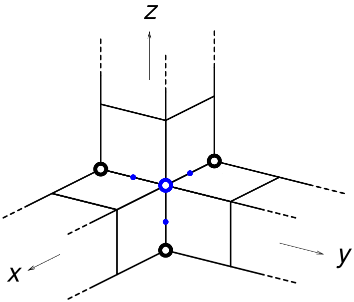

$$\begin{array}{rc c c}
 & & \begin{matrix} x & y & z \end{matrix} & \\
-\varnothing \otimes z \cdot xy & \widetilde{H}_{-1}K^{110} \otimes \langle xy \rangle & \begin{matrix} \varnothing \\ \varnothing \\ \varnothing \end{matrix} \begin{bmatrix} [0 & 0 & 1] \\ [0 & 1 & 0] \\ [1 & 0 & 0] \end{bmatrix} & (x - z) \otimes xyz \\
 & \bigoplus & & \\
 & \widetilde{H}_{-1}K^{101} \otimes \langle xz \rangle & \longleftarrow & \widetilde{H}_0 K^{111} \otimes \langle xyz \rangle \\
 & \bigoplus & & \\
\varnothing \otimes x \cdot yz & \widetilde{H}_{-1}K^{011} \otimes \langle yz \rangle & &
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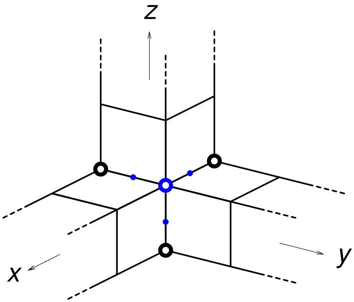
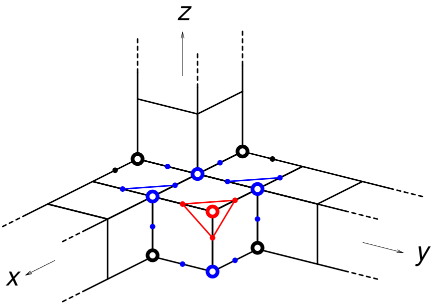


$$\begin{array}{ccccc}
-\varnothing \otimes z \cdot xy & \widetilde{H}_{-1}K^{110} \otimes \langle xy \rangle & & & \\
\oplus & \oplus & & \begin{array}{c} \; \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c}\varnothing\\ \varnothing\\ \varnothing\end{array}\begin{bmatrix} [0 & 0 & 1] \\ [0 & 1 & 0] \\ [1 & 0 & 0] \end{bmatrix} \end{array} & (x - z) \otimes xyz \\
0 & \widetilde{H}_{-1}K^{101} \otimes \langle xz \rangle & \longleftarrow & & \widetilde{H}_0 K^{111} \otimes \langle xyz \rangle \\
\oplus & \oplus & & & \\
\varnothing \otimes x \cdot yz & \widetilde{H}_{-1}K^{011} \otimes \langle yz \rangle & & &
\end{array}$$

Sylvan matrices

Example 2. $I = \langle yz, xz, xy^2, x^2y \rangle$



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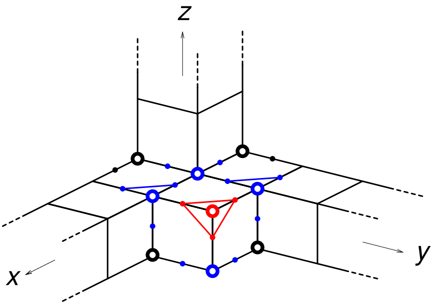

$$
\begin{matrix}
\widetilde{H}_{-1} K^{011} \otimes \langle yz \rangle \\
\oplus \\
\widetilde{H}_{-1} K^{101} \otimes \langle xz \rangle \\
\oplus \\
\widetilde{H}_{-1} K^{120} \otimes \langle xy^2 \rangle \\
\oplus \\
\widetilde{H}_{-1} K^{210} \otimes \langle x^2y \rangle
\end{matrix}
\xleftarrow{\;
\begin{matrix} \varnothing \\ \varnothing \\ \varnothing \\ \varnothing \end{matrix}
\overset{\begin{matrix} x & y & \;x & y & z & \;x & y & z & \;x & y \end{matrix}}{\left[\begin{matrix}
[0 & 0] & [\frac34 & \frac34 & 0] & [\frac14 & \frac14 & 0] & [1 & 0] \\
[0 & 0] & [\frac14 & \frac14 & 0] & [\frac34 & \frac34 & 0] & [0 & 1] \\
[1 & 0] & [0 & 0 & 1] & [0 & 0 & 0] & [0 & 0] \\
[0 & 1] & [0 & 0 & 0] & [0 & 0 & 1] & [0 & 0]
\end{matrix}\right]}
\;}
\begin{matrix}
\widetilde{H}_{0} K^{220} \otimes \langle x^2y^2 \rangle \\
\oplus \\
\widetilde{H}_{0} K^{121} \otimes \langle xy^2z \rangle \\
\oplus \\
\widetilde{H}_{0} K^{211} \otimes \langle x^2yz \rangle \\
\oplus \\
\widetilde{H}_{0} K^{111} \otimes \langle xyz \rangle
\end{matrix}
\xleftarrow{\;
\begin{matrix} x \\ y \\ x \\ y \\ z \\ x \\ y \\ z \\ x \\ y \end{matrix}
\overset{\begin{matrix} zy & yx & xz \end{matrix}}{\left[\begin{matrix}
-\frac12 & 0 & -\frac12 \\
\frac12 & 0 & \frac12 \\
0 & \frac13 & -\frac13 \\
0 & -\frac23 & -\frac13 \\
0 & \frac13 & \frac23 \\
\frac13 & \frac23 & 0 \\
\frac13 & -\frac13 & 0 \\
-\frac23 & -\frac13 & 0 \\
0 & \frac12 & 0 \\
0 & -\frac12 & 0
\end{matrix}\right]}
\;}
\widetilde{H}_{1} K^{221} \otimes \langle x^2y^2z \rangle
$$

Sylvan matrices

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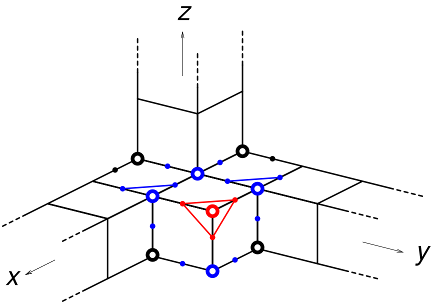

$$
\begin{array}{c}
\widetilde{H}_{-1} K^{011} \otimes \langle yz \rangle \\
\oplus \\
\widetilde{H}_{-1} K^{101} \otimes \langle xz \rangle \\
\oplus \\
\widetilde{H}_{-1} K^{120} \otimes \langle xy^2 \rangle \\
\oplus \\
\widetilde{H}_{-1} K^{210} \otimes \langle x^2y \rangle
\end{array}
\xleftarrow{\;
\begin{array}{c|cc|ccc|ccc|cc}
 & x & y & x & y & z & x & y & z & x & y \\
\hline
\varnothing & 0 & 0 & \frac34 & \frac34 & 0 & \frac14 & \frac14 & 0 & 1 & 0 \\
\varnothing & 0 & 0 & \frac14 & \frac14 & 0 & \frac34 & \frac34 & 0 & 0 & 1 \\
\varnothing & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\varnothing & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\;}
\begin{array}{c}
\widetilde{H}_{0} K^{220} \otimes \langle x^2y^2 \rangle \\
\oplus \\
\widetilde{H}_{0} K^{121} \otimes \langle xy^2z \rangle \\
\oplus \\
\widetilde{H}_{0} K^{211} \otimes \langle x^2yz \rangle \\
\oplus \\
\widetilde{H}_{0} K^{111} \otimes \langle xyz \rangle
\end{array}
\xleftarrow{\;
\begin{array}{c|ccc}
 & 1 & 1 & 1 \\
 & zy & yx & xz \\
\hline
x & -\frac12 & 0 & -\frac12 \\
y & \frac12 & 0 & \frac12 \\
\hline
x & 0 & \frac13 & -\frac13 \\
y & 0 & -\frac23 & -\frac13 \\
z & 0 & \frac13 & \frac23 \\
\hline
x & \frac13 & \frac23 & 0 \\
y & \frac13 & -\frac13 & 0 \\
z & -\frac23 & -\frac13 & 0 \\
\hline
x & 0 & \frac12 & 0 \\
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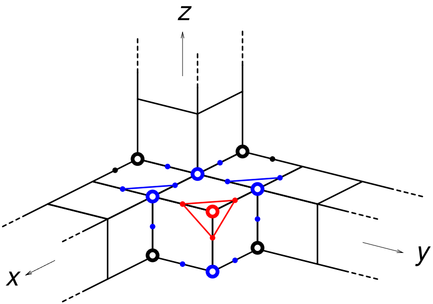

$$
\begin{array}{c}
\widetilde{H}_{-1}K^{011}\otimes\langle yz\rangle\\
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\oplus\\
\widetilde{H}_{-1}K^{210}\otimes\langle x^2y\rangle
\end{array}
\longleftarrow
\begin{array}{c|cc|ccc|ccc|cc}
 & -1 & 1 & 0 & -1 & 1 & 1 & 0 & -1 & \tfrac12 & -\tfrac12\\
 & x & y & x & y & z & x & y & z & x & y\\
\hline
\varnothing & 0 & 0 & \tfrac34 & \tfrac34 & 0 & \tfrac14 & \tfrac14 & 0 & 1 & 0\\
\varnothing & 0 & 0 & \tfrac14 & \tfrac14 & 0 & \tfrac34 & \tfrac34 & 0 & 0 & 1\\
\varnothing & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
\varnothing & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}
\quad
\begin{array}{c}
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\begin{array}{c|ccc}
 & 1 & 1 & 1\\
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\hline
x & -\tfrac12 & 0 & -\tfrac12\\
y & \tfrac12 & 0 & \tfrac12\\
\hline
x & 0 & \tfrac13 & -\tfrac13\\
y & 0 & -\tfrac23 & -\tfrac13\\
z & 0 & \tfrac13 & \tfrac23\\
\hline
x & \tfrac13 & \tfrac23 & 0\\
y & \tfrac13 & -\tfrac13 & 0\\
z & -\tfrac23 & -\tfrac13 & 0\\
\hline
x & 0 & \tfrac12 & 0\\
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\end{array}
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Canonical sylvan morphism

Theorem [Eagon–M–Ordog 2019]. If char $\Bbbk$ avoids finitely many primes, then there is a canonical sylvan homology morphism

$$\widetilde{C}_i K^{\mathbf{b}} I \xleftarrow{D^{\mathbf{ab}}} \widetilde{C}_{i-1} K^{\mathbf{a}} I,$$

satisfying
- $D(\widetilde{Z}_i K^{\mathbf{b}} I) \subseteq \widetilde{Z}_{i-1} K^{\mathbf{a}} I$

and
- $D(\widetilde{B}_i K^{\mathbf{b}} I) = 0$,

explicitly given by the sylvan matrix of $D = D^{\mathbf{ab}}$ with combinatorial entries

$$D_{\sigma\tau} = \sum_{\lambda \in \Lambda(\mathbf{a},\mathbf{b})} \frac{1}{\Delta_{i,\lambda} I} \sum_{\varphi \in \Phi_{\sigma\tau}(\lambda)} w_\varphi$$

where
- $\Lambda(\mathbf{a}, \mathbf{b}) = \{$saturated decreasing lattice paths from $\mathbf{b}$ to $\mathbf{a}\}$,
- $\Phi_{\sigma\tau}(\lambda) = \{$chain-link fences from τ to σ along $\lambda\}$,
- $w_\varphi =$ weight of φ,

and
- $\Delta_{i,\lambda} I \approx \prod_{\mathbf{c} \in \lambda} \sum \det^2$(maximal invertible submatrices of $\partial_i^{\mathbf{c}}$).

That is, $\{D^{\mathbf{ab}} \mid \mathbf{a} \prec \mathbf{b}\}$ solves Kaplansky's problem with the entire Wish List.

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Chain-link fences

Def. Fix path $\lambda \in \Lambda(\mathbf{a}, \mathbf{b})$, so $\lambda = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_\ell)$ with $\lambda_j = \mathbf{b}_{j-1} - \mathbf{b}_j$. A chain-link fence φ from an i-simplex τ to an $(i-1)$-simplex σ along λ is a sequence

```
                 τ_{ℓ-1}        ···        τ_1        τ_0 — τ
                /      \      /      \    /    \    /
σ — σ_ℓ                σ_{ℓ-1}        σ_2        σ_1
```

of faces $\tau_j \in K_i^{\mathbf{b}_j} I$ and $\sigma_j \in K_{i-1}^{\mathbf{b}_j} I$, plus a choice of hedgerow, such that

- $\tau_0 — \tau$: τ is boundary-linked to τ_0;
- $\backslash$: σ_j for $j = 1, \ldots, \ell - 1$ is a stake chain-linked to τ_j;
- $/$: σ_j for $j = 1, \ldots, \ell$ equals the facet $\tau_{j-1} - \lambda_j$ of the simplex τ_{j-1};
- $\sigma — \sigma_\ell$: σ_ℓ is cycle-linked to σ

as specified by the hedgerow. The weight of φ is

$$w_\varphi = \prod(\text{edge weights}) \prod(\text{vertex weights}).$$

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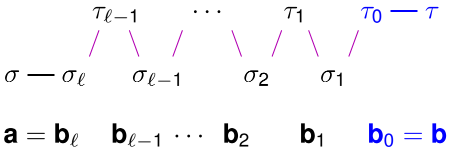

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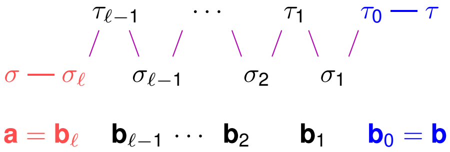

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Hedges, stakes, and shrubberies

Def. Fix a field $\Bbbk$ and a CW complex K with i-faces K_i.

1. $T_i \subseteq K_i$ is a shrubbery if $\partial T_i = \{\partial\tau \mid \tau \in T_i\}$ is a $\Bbbk$-basis for $\widetilde{B}_{i-1}$.
   e.g., $i = 1$: shrubbery $\Leftrightarrow$ spanning tree in every connected component
2. $S_{i-1} \subseteq K_{i-1}$ is a stake set if $\overline{S}_{i-1}$ maps to a $\Bbbk$-basis for $\widetilde{C}_{i-1}/\widetilde{B}_{i-1}$,
   where $\overline{S}_{i-1} = K_{i-1} \smallsetminus S_{i-1}$ $\quad$ ($\Leftrightarrow \partial^* S_{i-1}$ is a $\Bbbk$-basis for $\widetilde{B}^i$)
3. A hedge of dim i is a
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   together denoted ST_i.

Note.

1. shrubbery $T_i \Leftrightarrow$ columns of boundary matrix ∂_i span column space of ∂_i,
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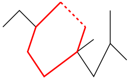

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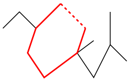

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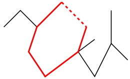

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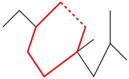

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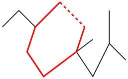

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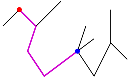

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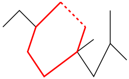

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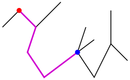

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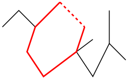

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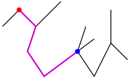

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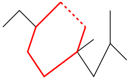

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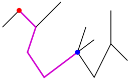

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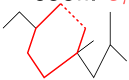

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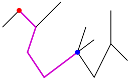

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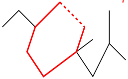

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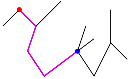

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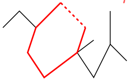

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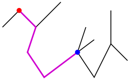

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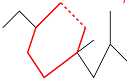

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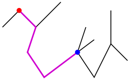
Def. σ is chain-linked to every τ appearing in s.  coeff. $c_\sigma(\tau, ST_i)$

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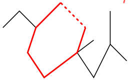

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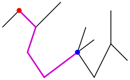

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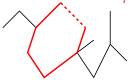

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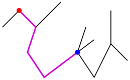

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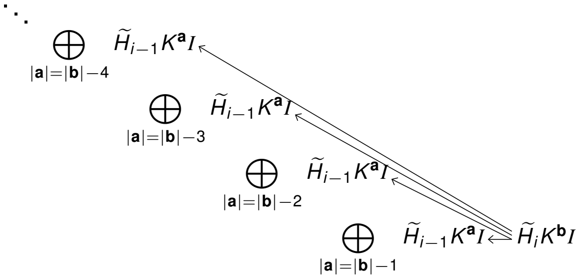

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2. Which splitting? Moore–Penrose pseudoinverse!
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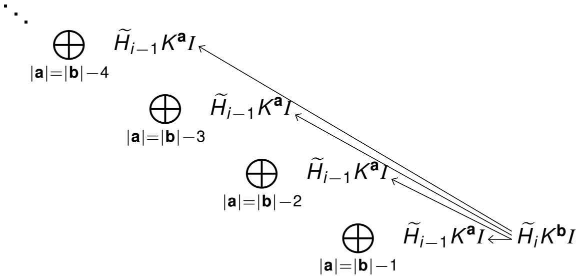

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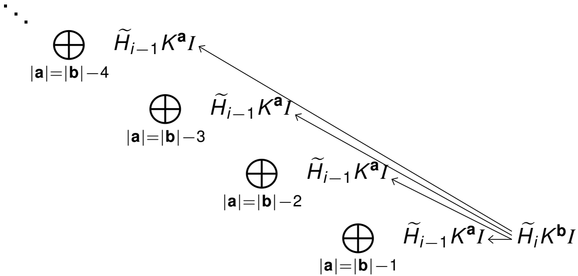

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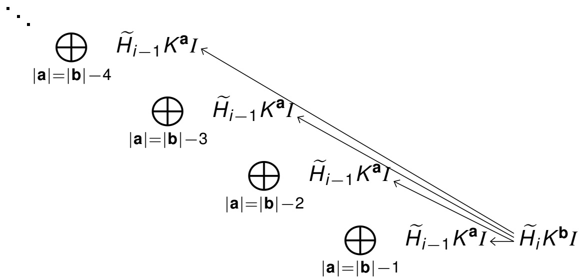

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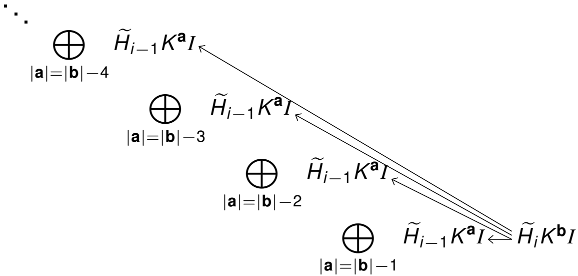

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1. Recover known resolutions (planar maps for trivariate; Eliahou–Kervaire, etc. . . .) from (noncanonical) sylvan resolutions.
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   - Koszul double complex methods on "Spanish simplicial complex"
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