Symmetric shifted ideals

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Combinatorial Algebra meets Algebraic Combinatorics
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Project history

- Started at CMO in May 2017
- Joint with:
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Betti numbers and free resolutions

Definition

Let $I \subseteq R = \Bbbk[x_1, \dots, x_n]$ be a homogeneous ideal. The numbers

$$\beta_{i,j}(I) = \dim_{\Bbbk} \mathrm{Tor}_i(I, \Bbbk)_j$$

are called the *Betti numbers* of I.

Observations:

- If $F_\bullet$ is a minimal free resolution of I, then $\beta_{i,j}(I)$ is the rank of F_i in degree j.
- A description of $F_\bullet$ is also desirable.
- Both $\beta_{i,j}(I)$ and $F_\bullet$ are (in general) hard to find.

Monomial ideals stable under permutations

Assumptions:

- $\mathfrak{S}_n$ acts on $\Bbbk[x_1, \ldots, x_n]$ permuting variables
- $I \subseteq \Bbbk[x_1, \ldots, x_n]$ be a monomial ideal such that $\mathfrak{S}_n \cdot I \subseteq I$

Lemma

The minimal monomial generating set $G(I)$ of I splits into $\mathfrak{S}_n$-orbits $\{\sigma(x^\lambda) : \sigma \in \mathfrak{S}_n\}$ for some partitions $\lambda \in \mathbb{N}^n$.

Definition

$$P(I) := \{\lambda : x^\lambda \in I\}, \qquad \Lambda(I) := \{\lambda : x^\lambda \in G(I)\}.$$

Convention: $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfies $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$.

Shifted ideals

Let $I \subset \Bbbk[x_1, \ldots, x_n]$ be an $\mathfrak{S}_n$-fixed monomial ideal.

Definition (Shifted ideal)

We say I is *shifted* if, for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in P(I)$ and $1 \leqslant k < n$ with $\lambda_k < \lambda_n$, we have $x^\lambda x_k / x_n \in I$.

Definition (Strongly shifted ideal)

We say I is *strongly shifted* if, for every $\lambda = (\lambda_1, \ldots, \lambda_n) \in P(I)$ and $1 \leqslant k < l \leqslant n$ with $\lambda_k < \lambda_l$, we have $x^\lambda x_k / x_l \in I$.

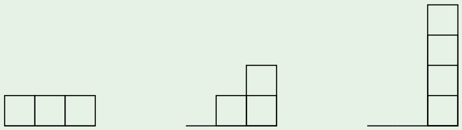
It is enough to check these conditions for every $\lambda \in \Lambda(I)$.

Example

The $\mathfrak{S}_3$-stable ideal in $\Bbbk[x_1, x_2, x_3]$

$$I = \langle x_1x_2x_3, \quad x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_3^2, x_2^2x_3, x_2x_3^2, \quad x_1^4, x_2^4, x_3^4 \rangle$$

is strongly shifted with $\Lambda(I) = \{(1,1,1), (0,1,2), (0,0,4)\}$.



$x_1^1x_2^1x_3^1$ $\quad$ $x_1^0x_2^1x_3^2$ $\quad$ $x_1^0x_2^0x_3^4$

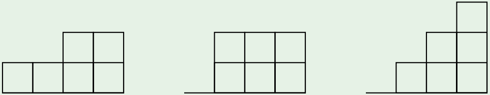
Examples of shifted ideals

Example

The $\mathfrak{S}_4$-stable ideal $I \subseteq \Bbbk[x_1, x_2, x_3, x_4]$ with

$$\Lambda(I) = \{(1,1,2,2), (0,2,2,2), (0,1,2,3)\}$$

is shifted but not strongly shifted because $(0,1,2,3) \in P(I)$ but $(1,1,1,3) \notin P(I)$.



$x_1^1 x_2^1 x_3^2 x_4^2$ $\quad$ $x_1^0 x_2^2 x_3^2 x_4^2$ $\quad$ $x_1^0 x_2^1 x_3^2 x_4^3$

Shifted ideals have linear quotients

For distinct monomials $u = \sigma(x^\lambda), v = \tau(x^\mu)$, we set $v \prec u$ if:

- $\deg(v) < \deg(u)$, or
- $\deg(v) = \deg(u)$ and $x^\mu >_{\text{lex}} x^\lambda$, or
- $\lambda = \mu$ and $v <_{\text{lex}} u$.

Theorem (BDGMNORS)

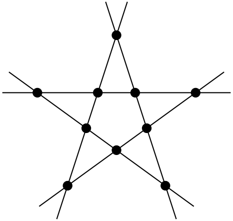
Shifted $\mathfrak{S}_n$-fixed monomial ideals have linear quotients with respect to the order $\prec$.

Star configurations

- $L_1, \dots, L_n$ linear forms in a polynomial ring
- Assume all subsets $\{L_{i_1}, \dots, L_{i_c}\}$ are linearly independent

Definition (Star configuration of codimension c)

$$I_{n,c} := \bigcap_{1 \leqslant i_1 < \dots < i_c \leqslant n} \langle L_{i_1}, \dots, L_{i_c} \rangle$$



Symbolic powers and reduction to monomials

Definition (Symbolic powers of $I_{n,c}$)

$$I_{n,c}^{(m)} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \langle L_{i_1}, \ldots, L_{i_c} \rangle^m.$$

Theorem (Geramita, Harbourne, Migliore, Nagel, 2017)

If L_i is replaced by a variable x_i, then the Betti numbers of $I_{n,c}^{(m)}$ stay the same.

Frow now on

$$I_{n,c}^{(m)} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \langle x_{i_1}, \ldots, x_{i_c} \rangle^m \subseteq \Bbbk[x_1, \ldots, x_n].$$

Previously known Betti numbers

Theorem (proved by many)

$$\beta_{i,i+n-c+1}(I_{n,c}) = \binom{n}{c-1-i}\binom{n-c+i}{i}$$

Theorem (Geramita, Harbourne, Migliore, 2013)

If $c \geqslant 2$, then

$$\beta_{i,i+j}(I_{n,c}^{(2)}) = \begin{cases} \binom{n}{c-2-i}\binom{n-c+1+i}{i}, & j = n-c+2 \\ \binom{n}{c-1}\binom{c-1}{i}, & j = 2(n-c+1) \end{cases}$$

Star configurations are strongly shifted

Proposition (BDGMNORS)

For every $m \geqslant 1$, $I_{n,c}^{(m)}$ is $\mathfrak{S}_n$-fixed and strongly shifted, with

$$P(I_{n,c}^{(m)}) = \left\{ \lambda : \sum_{i=1}^{c} \lambda_i \geqslant m \right\},$$

$$\Lambda(I_{n,c}^{(m)}) = \left\{ \lambda : \sum_{i=1}^{c} \lambda_i = m, \forall i > c \ \lambda_i = \lambda_c \right\}.$$

It follows that the ideals $I_{n,c}^{(m)}$ have linear quotients.

Betti tables of star configurations

Corollary (BDGMNORS)

1. *For every* $i \geqslant 0$,

$$\beta_{i,i+m(n-c+1)}(I_{n,c}^{(m)}) = \binom{n}{c-1}\binom{c-1}{i}.$$

2. *The Castelnuovo-Mumford regularity of* $I_{n,c}^{(m)}$ *is* $m(n-c+1)$.
3. *If* $m \geqslant 2$, *then all nonzero rows in the Betti table of* $I_{n,c}^{(m)}$ *have length* $c-1$, *with the exception of the top one.*
4. *If* $m \leqslant c$, *then for every* $i \geqslant 0$,

$$\beta_{i,i+n-c+m}(I_{n,c}^{(m)}) = \binom{n}{c-m-i}\binom{n-c+m+i-1}{i}.$$

Betti numbers of symbolic cube

Corollary (BDGMNORS)

If $c \geqslant 3$*, then* $\beta_{i,i+j}(I_{n,c}^{(3)}) =$

$$\begin{cases} \binom{n}{c-3-i}\binom{n-c+2+i}{i}, & j = n-c+3 \\ \binom{n}{c-2}\left(\binom{c-2}{i} + (n-c+1)\binom{c-1}{i}\right), & j = 2(n-c+1)+1 \\ \binom{n}{c-1}\binom{c-1}{i}, & j = 3(n-c+1) \end{cases}$$

Equivariant resolutions of shifted ideals

- $p(\lambda) := |\{k : \lambda_k < \lambda_n - 1\}|$, $r(\lambda) := |\{k : \lambda_k = \lambda_n\}|$
- if $|\lambda| = p$, S^λ is the simple $\mathfrak{S}_p$-module indexed by λ
- if $|\lambda| = p \leqslant t$, $U_t^\lambda := (S^\lambda \otimes_{\Bbbk} S^{(t-p)})\big\uparrow_{p,t-p}^{t}$
- if $|\lambda| = p$, M^λ is the permutation representation indexed by λ
- $\lambda_{\leqslant p} := (\lambda_1, \ldots, \lambda_p)$
- $N_{k,l}^\lambda := \left(\left(M^{\lambda_{\leqslant p(\lambda)}} \otimes_{\mathfrak{S}_{p(\lambda)}} U_{p(\lambda)}^{(1^k)}\right) \otimes_{\Bbbk} U_{n-p(\lambda)}^{(1^l, r(\lambda))}\right)\Big\uparrow_{p(\lambda), n-p(\lambda)}^{n}$

Theorem (BDGMNORS)

If I is a shifted ideal, then

$$\operatorname{Tor}_i(I, \Bbbk)_{i+j} \cong \bigoplus_{\lambda \in \Lambda(I), |\lambda| = j} \bigoplus_{k+l=i} N_{k,l}^\lambda.$$

Equivariant resolution of symbolic square

Theorem

If $c \geqslant 2$*, then* $\operatorname{Tor}_i(I^{(2)}_{n,c}, \mathbb{k})_{i+j} =$

$$\begin{cases} \left(S^{(1^i,n-c+2)} \otimes S^{(c-2-i)}\right)\Big\uparrow^n, & j = n-c+2 \\ \left(\left(S^{(1^i)} \otimes S^{(c-1-i)}\right)\Big\uparrow^{c-1} \otimes S^{(n-c+1)}\right)\Big\uparrow^n, & j = 2(n-c+1) \end{cases}$$

Example ($n = 6, c = 4$)

	0	1	2	3
4	$(4)\uparrow^6$	$(4,1)\uparrow^6$	$(4,1^2)$	–
5	–	–	–	–
6	$(0)\uparrow^3\uparrow^6$	$(1)\uparrow^3\uparrow^6$	$(2)\uparrow^3\uparrow^6$	$(3)\uparrow^6$