Regularity of powers of quadratic sequences and binomial edge ideals

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Quadratic sequence was introduced by K. N. Raghavan generalizing the definition of weak d-sequence introduced by Craig Huneke. [J. Alg 68(2) 471-509 (1981) Adv. Math. 46(3) 249-279 (1982)]

Raghavan studied depth R/I^n, when I is generated by a quadratic sequence.
[TAMS 343(2) 727-747 (1997)]

Let $R = \bigoplus_{n \geq 0} R_n$ be a f.g. graded algebra over a Noetherian ring. Let Λ be a finite poset and $I \subset R$ be an ideal. A set of elements $\{u_\lambda : \lambda \in \Lambda\} \subset R$ is said to be a quadratic sequence with respect to the ideal I if for every pair (Σ, λ) where Σ is a poset ideal of Λ and λ lies inside or just above Σ, $\exists$ a poset ideal Θ of Λ such that

1. $(\bar{U}_\Sigma : \bar{u}_\lambda) \cap \bar{U}_\Lambda \subseteq \bar{U}_\Theta$
2. $u_\lambda U_\Theta \subseteq (U_\Sigma + I) U_\Lambda$.

Theorem (essentially by Raghavan):

Let Λ be a finite poset and $\{u_\lambda : \lambda \in \Lambda\}$ be a set of homogeneous elements, $\deg u_\lambda = d_\lambda > 0$. If $\{u_\lambda : \lambda \in \Lambda\}$ is a quadratic sequence, then $\forall\ s \geq 1$, $\exists$ a graded filtration of R/U_Λ^s: $R/U_\Lambda^s = M_0 \supset M_1 \supset \cdots \supset M_k = (0)$ such that $\forall\ 0 \leq i \leq k-1$, $\exists$ a related ideal V_i and $0 \leq d_i \leq d(s-1)$ with $M_i/M_{i+1} \simeq R/V_i[-d_i]$

[J is a *related ideal* to the quadratic sequence if $J = U_\Lambda$ or $J = (U_\Sigma : u_\lambda) + U_\Lambda$ for some (Σ, λ)]

Theorem: $\mathrm{reg}(R/U_1^s) \leq d(s-1) + \max_{(\Sigma,\lambda)} \left\{ \mathrm{reg}\left(R/(U_\Sigma : u_\lambda) + U_1 \right) \right\}$.

Example: $R = K[x,y,z,w]$ and U denote the defining ideal of the projective monomial curve:

$$(x:y:z:w) = (u^{b+c}, u^b v^c, u^c v^b, v^{b+c})$$

with $\gcd(b,c)=1$ & $b>c$.

Morales-Simis proved that U is generated by a quadratic sequence. Using the minimal resolution they computed and our theorem, we get

$$\mathrm{reg}(R/U^s) = bs-1.$$

- $u_1, \ldots, u_n$ is said to be a d-sequence if

  - u_i is not in the ideal generated by rest of u_j's
  - for all $k \geq i+1$ & all $i \geq 0$

$$((u_1, \ldots, u_i) : u_{i+1} u_k) = (u_1, \ldots, u_i) : u_k .$$

$$\Longleftrightarrow ((u_1, \ldots, u_{i-1}) : u_i) \cap (u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1})$$

Costa [J. Alg 94(1) 256-263 (1985)]

Corollary: If $u_1, \ldots, u_n$ is a homogeneous d-seq. with $\deg(u_i) = d_i$ such that $u_1, \ldots, u_{n-1}$ is a regular sequence, then

$$\operatorname{reg}(R/U^s) \leq d(s-1) + \max\left\{\operatorname{reg}(R/U), \sum_{i=1}^{n-1} d_i - n\right\}$$

Study of $\mathrm{reg}(I^s)$ was initiated by

Bertram-Ein-Lazarsfeld when I is the
[JAMS 4(3) 587-602, 1991]

defining ideal of a smooth complex projective

variety. They showed that this is bounded

by a linear function of s.

$\vdots$

Cutkosky-Herzog-Trung, Kodiyalam
[Compositio Math 118(3) 243-261 (1999) PAMS 128(2) 407-411 (2000)]

$$\mathrm{reg}(I^s) = as + b \quad \forall\, s \gg 0.$$

Question: Given I compute or bound a & b.

- C-H-T & K: If I is generated in degree d, then $a = d$.
- In general, computing the constant term is a difficult task.
- Researchers have been trying to compute or bound a and b in terms of invariants associated with the ideal.

- For a graph G, Villarreal defined edge ideal corresponding to G [Manuscr. Math. 66(3) 277-293 (1990)]
- For G, $I(G) = \langle \{x_i x_j : \{x_i, x_j\} \in E(G)\} \rangle$
- For several classes of graphs, $\operatorname{reg}(I(G)^s)$ has been computed in terms of combinatorial invariants associated with G.
- In general, there is no formula.

  [J-Selvaraja]: $\operatorname{reg}(I(G)^s) \leq 2s + \text{co-chord}(G) - 1$

- Herzog-Hibi-Hreinsdóttir-Kahle-Rauh [Adv. in Appl. Math. 45(3), 317-333, 2010]
- Ohtani [Comm. Alg. 39(3), 905-917, 2011]

defined Binomial Edge ideal corresponding to a finite simple graph on $[n]$.

$$J_G = \langle \{ x_i y_j - x_j y_i : \{i,j\} \in E(G),\ i<j \} \rangle$$

$$\subseteq K[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

- Matsuda-Murai: $\ell(G) \leq \operatorname{reg}(S/J_G) \leq n-1$ [JCA, 2103]

- Unlike in the case of monomial edge ideals, very little is known about the interplay between algebraic invariants of J_G and combinatorial invariants of G.
- Except for the case of $G = P_n$ or $G = K_n$, nothing is known about $\operatorname{reg}(S/J_G^s)$.
- P_n is a complete intersection and J_G is a determinantal ideal.

Question: What are the almost complete int. binomial edge ideals?

- These are some trees and unicyclic graphs obtained by adding an edge between two vertices of a path or two paths.
- We proved that the Rees Algebra and the associated graded rings of these edge ideals are Cohen-Macaulay.

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- We showed that these binomial edge ideals are generated by a d-sequence.
- We first generalized the result of Matsuda and Murai on a lower bound for the regularity to all powers:

$$2s + \ell(G) - 2 \leq \operatorname{reg}(S/J_G^s) \quad \forall\, s \geq 1.$$

- Using this result and using theory of d-sequences we proved:

- $G = K_{1,n} \Rightarrow \operatorname{reg}(S/J_G^s) = 2s \quad \forall\ s \geq 1$

- $G = C_n \Rightarrow \operatorname{reg}(S/J_G^s) = 2s + n - 4 \quad \forall\ s \geq 1.$

- G = tree such that J_G is an a.c.i.

$\Rightarrow \quad 2s + iv(G) - 2 \leq \operatorname{reg}(S/J_G^s) \leq 2s + iv(G) - 1.$

- G = unicyclic graph such that J_G is an a.c.i

$\Rightarrow \quad 2s + n - 5 \leq \operatorname{reg}(S/J_G^s) \leq 2s + n - 4$

Questions

- $G = \text{tree} \implies 2s + iv(G) - 2 \leq \mathrm{reg}(S/J_G^s)$?
- Find an upper bound for $\mathrm{reg}(S/J_G^s)$ for specific classes of graphs such as bipartite, chordal, unicyclic.
- Is $\mathrm{reg}(S/J_G^{(s)}) = \mathrm{reg}(S/J_G^s)$?
- Conjecture: $G = C_n \implies J_G^{(s)} = J_G^s \quad \forall s \geq 1.$

THANK YOU