Subalgebras of a polynomial ring with minimal Hilbert function

Lisa Nicklasson

Stockholm University

Combinatorial Algebra meets Algebraic Combinatorics 2020

$R = K[x_1, \ldots, x_n]$ polynomial ring over a field K

$R = K[x_1, \ldots, x_n]$ polynomial ring over a field K

$f_1, f_2, \ldots, f_u \in R$ homogeneous of degree d, linearly independent

$R = K[x_1, \ldots, x_n]$ polynomial ring over a field K

$f_1, f_2, \ldots, f_u \in R$ homogeneous of degree d, linearly independent

$S = K[f_1, \ldots, f_u] \subseteq R$ subring generated by $f_1, \ldots, f_u$.

$R = K[x_1, \dots, x_n]$ polynomial ring over a field K

$f_1, f_2, \dots, f_u \in R$ homogeneous of degree d, linearly independent

$S = K[f_1, \dots, f_u] \subseteq R$ subring generated by $f_1, \dots, f_u$.

$$S = \bigoplus_{i \geq 0} S_i$$

S_i is the set of homogeneous polynomials of degree i in $f_1, \dots, f_u$

$R = K[x_1, \ldots, x_n]$ polynomial ring over a field K

$f_1, f_2, \ldots, f_u \in R$ homogeneous of degree d, linearly independent

$S = K[f_1, \ldots, f_u] \subseteq R$ subring generated by $f_1, \ldots, f_u$.

$$S = \bigoplus_{i \geq 0} S_i$$

S_i is the set of homogeneous polynomials of degree i in $f_1, \ldots, f_u$, a vector space over K

$R = K[x_1, \ldots, x_n]$ polynomial ring over a field K

$f_1, f_2, \ldots, f_u \in R$ homogeneous of degree d, linearly independent

$S = K[f_1, \ldots, f_u] \subseteq R$ subring generated by $f_1, \ldots, f_u$.

$$S = \bigoplus_{i \geq 0} S_i$$

S_i is the set of homogeneous polynomials of degree i in $f_1, \ldots, f_u$, a vector space over K

Hilbert function: $\mathrm{HF}_S(i) = \dim S_i$

$R = K[x_1, \ldots, x_n]$ polynomial ring over a field K

$f_1, f_2, \ldots, f_u \in R$ homogeneous of degree d, linearly independent

$S = K[f_1, \ldots, f_u] \subseteq R$ subring generated by $f_1, \ldots, f_u$.

$$S = \bigoplus_{i \geq 0} S_i$$

S_i is the set of homogeneous polynomials of degree i in $f_1, \ldots, f_u$, a vector space over K

Hilbert function: $\mathrm{HF}_S(i) = \dim S_i$ a polynomial in i

Problem

For fixed n, u and d, which $f_1, \ldots, f_u$ gives the minimal HF_S?

Problem

For fixed n, u and d, which $f_1, \ldots, f_u$ gives the $\underbrace{\textit{minimal}\ \mathrm{HF}_S}_{???}$?

Problem

For fixed n, u and d, which $f_1, \ldots, f_u$ gives the $\underbrace{\textit{minimal } \mathrm{HF}_S}_{???}$?

Proposition

Fix n, u, d, and i. To minimize $\mathrm{HF}_S(i)$, choose $f_1, \ldots, f_u$ as a strongly stable set of monomials.

Problem

For fixed n, u and d, which $f_1, \ldots, f_u$ gives the $\underbrace{\textit{minimal}\ \mathrm{HF}_S}_{???}$?

Proposition

Fix n, u, d, and i. To minimize $\mathrm{HF}_S(i)$, choose $f_1, \ldots, f_u$ as a strongly stable set of monomials.

Definition

A set W of monomials is *strongly stable* if

$$m \in W,\ x_i | m \implies \frac{x_j}{x_i} m \in W \text{ for all } j < i.$$

Problem

For fixed n, u and d, which $f_1, \ldots, f_u$ gives the $\underbrace{\textit{minimal}\ \mathrm{HF}_S}_{???}$?

Proposition

Fix n, u, d, and i. To minimize $\mathrm{HF}_S(i)$, choose $f_1, \ldots, f_u$ as a strongly stable set of monomials.

Definition

A set W of monomials is *strongly stable* if

$$m \in W,\ x_i | m \implies \frac{x_j}{x_i} m \in W \text{ for all } j < i.$$

Is there one strongly stable set $\{f_1, \ldots, f_u\}$ that minimizes $\mathrm{HF}_S(i)$ for all i?

Problem

For fixed n, u and d, which $f_1, \ldots, f_u$ gives the $\underbrace{\textit{minimal}\ \mathrm{HF}_S}_{???}$*?*

Proposition

Fix n, u, d, and i. To minimize $\mathrm{HF}_S(i)$, choose $f_1, \ldots, f_u$ as a strongly stable set of monomials.

Definition

A set W of monomials is *strongly stable* if

$$m \in W,\ x_i | m \implies \frac{x_j}{x_i} m \in W \text{ for all } j < i.$$

Is there one strongly stable set $\{f_1, \ldots, f_u\}$ that minimizes $\mathrm{HF}_S(i)$ for all i? No!

Problem

For fixed n, u and d, which $f_1, \dots, f_u$ gives the $\underbrace{\textit{minimal } \mathrm{HF}_S}_{???}$*?*

Proposition

Fix n, u, d, and i. To minimize $\mathrm{HF}_S(i)$, choose $f_1, \dots, f_u$ as a strongly stable set of monomials.

Definition

A set W of monomials is *strongly stable* if

$$m \in W,\ x_i | m \implies \frac{x_j}{x_i} m \in W \text{ for all } j < i.$$

Is there one strongly stable set $\{f_1, \dots, f_u\}$ that minimizes $\mathrm{HF}_S(i)$ for all i? No!

$\mathrm{HF}_S(i)$ is a polynomial.

Problem

For fixed n, u and d, which $f_1, \ldots, f_u$ *gives the* $\underbrace{\textit{minimal}\ \mathrm{HF}_S}_{???}$*?*

Proposition

Fix n, u, d, and i. To minimize $\mathrm{HF}_S(i)$*, choose* $f_1, \ldots, f_u$ *as a strongly stable set of monomials.*

Definition

A set W of monomials is *strongly stable* if

$$m \in W,\ x_i | m \implies \frac{x_j}{x_i} m \in W \text{ for all } j < i.$$

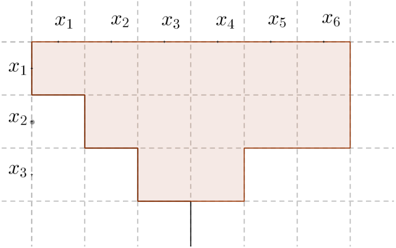
Is there one strongly stable set $\{f_1, \ldots, f_u\}$ that minimizes $\mathrm{HF}_S(i)$ for all i? No!

$\mathrm{HF}_S(i)$ is a polynomial.

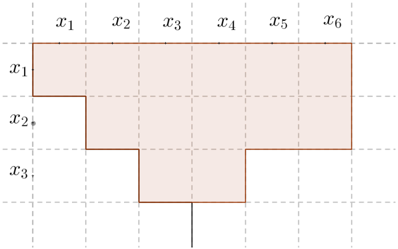
1. Minimize the degree
2. Minimize the leading coefficient

Fix $d = 2$.

Fix $d = 2$. Strongly stable sets can be drawn as shifted Ferrers diagrams.



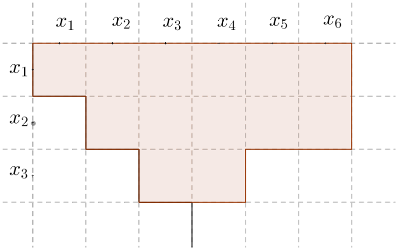
Fix $d = 2$. Strongly stable sets can be drawn as shifted Ferrers diagrams.



The degree of HF_S is $n - 1$.

1. Minimize the number of columns (variables).

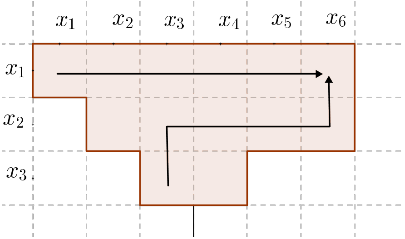
Fix $d = 2$. Strongly stable sets can be drawn as shifted Ferrers diagrams.



The degree of HF_S is $n - 1$.

1. Minimize the number of columns (variables). $\rightsquigarrow$ $\binom{n}{2} < u \leq \binom{n+1}{2}$

Fix $d = 2$. Strongly stable sets can be drawn as shifted Ferrers diagrams.

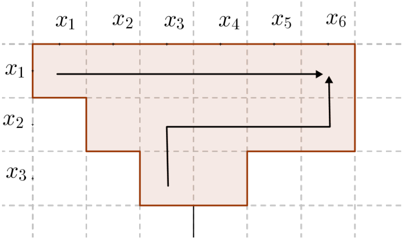


The degree of HF_S is $n-1$.

(Leading coefficient)$\cdot(n-1)!$
$=$ multiplicity of S
$=$ # maximal NE-paths in the diagram.

1. Minimize the number of columns (variables). $\rightsquigarrow$ $\binom{n}{2} < u \leq \binom{n+1}{2}$

Fix $d = 2$. Strongly stable sets can be drawn as shifted Ferrers diagrams.



The degree of HF_S is $n-1$.

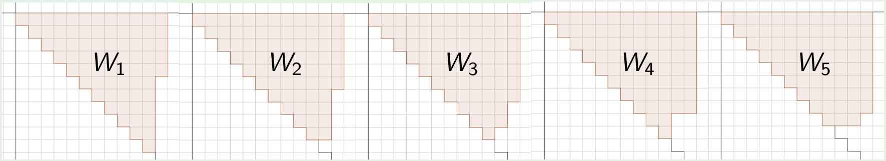
(Leading coefficient)$\cdot(n-1)!$
$=$ multiplicity of S
$=$ # maximal NE-paths in the diagram.

1. Minimize the number of columns (variables). $\rightsquigarrow$ $\binom{n}{2} < u \leq \binom{n+1}{2}$
2. Minimize the number of maximal NE-paths.

Example

$u = 71, n = 12, \quad \binom{12}{2} = 66, \binom{12+1}{2} = 78$

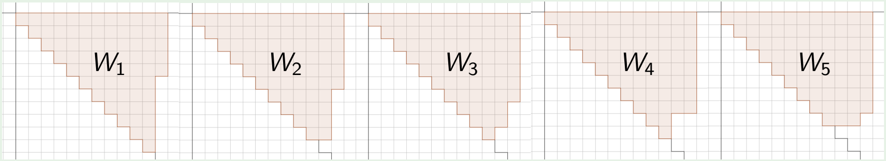
There are five strongly stable sets:



Example

$u = 71, n = 12, \quad \binom{12}{2} = 66, \binom{12+1}{2} = 78$

There are five strongly stable sets:



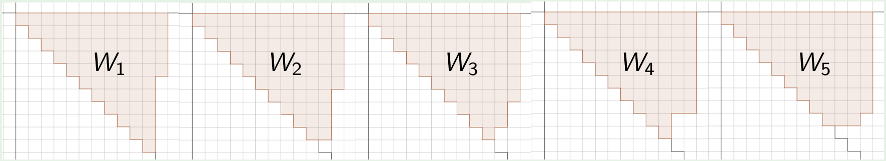
Multiplicities:

$$e(K[W_1]) = 1984, \; e(K[W_2]) = 2010, \; e(K[W_3]) = 2019,$$
$$e(K[W_4]) = 2009, \; e(K[W_5]) = 1981,$$

Example

$u = 71, n = 12, \quad \binom{12}{2} = 66, \binom{12+1}{2} = 78$

There are five strongly stable sets:



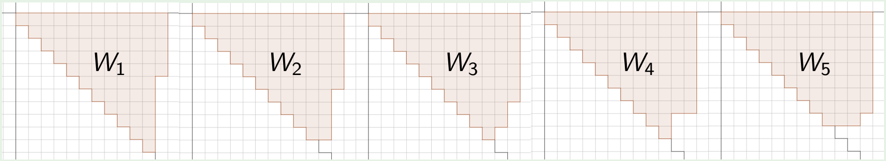
Multiplicities:

$$e(K[W_1]) = 1984,\ e(K[W_2]) = 2010,\ e(K[W_3]) = 2019,$$
$$e(K[W_4]) = 2009,\ \underline{e(K[W_5]) = 1981},$$

Example

$u = 71, n = 12, \quad \binom{12}{2} = 66, \binom{12+1}{2} = 78$

There are five strongly stable sets:



Multiplicities:

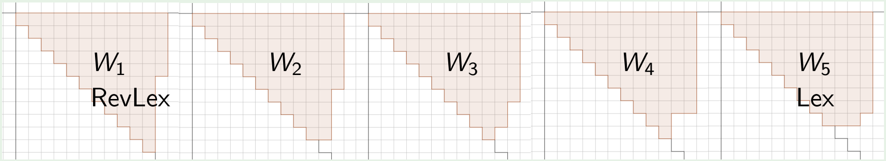
$$e(K[W_1]) = 1984, \; e(K[W_2]) = 2010, \; e(K[W_3]) = 2019,$$
$$e(K[W_4]) = 2009, \; \underline{e(K[W_5]) = 1981},$$

i	2	3	4	5	6	7
$\mathrm{HF}(K[W_1], i)$	1246	11389	70051	328771	1266005	4188859
$\mathrm{HF}(K[W_2], i)$	1256	11524	71012	333593	1285193	4253378
$\mathrm{HF}(K[W_3], i)$	1259	11565	71306	335075	1291108	4173307
$\mathrm{HF}(K[W_4], i)$	1255	11511	70922	333151	1283464	4247645
$\mathrm{HF}(K[W_5], i)$	1248	11406	70124	328965	1266265	4188404

Example

$u = 71, n = 12, \quad \binom{12}{2} = 66, \binom{12+1}{2} = 78$

There are five strongly stable sets:

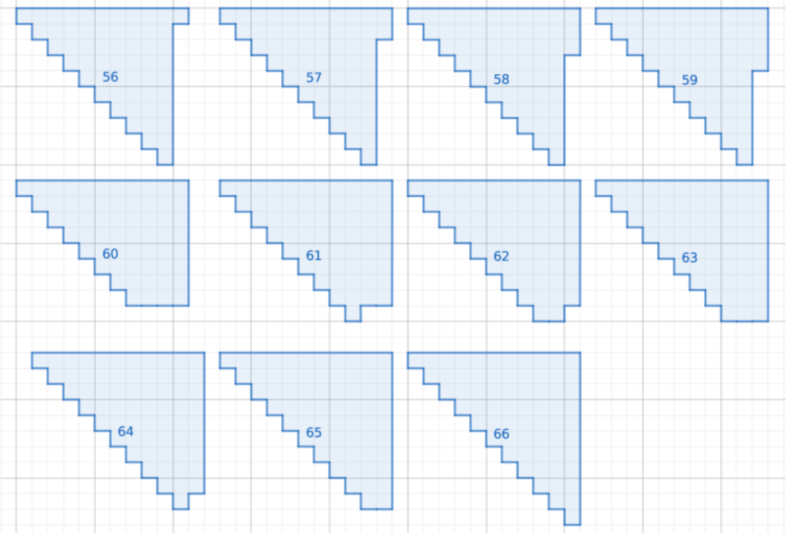


Multiplicities:

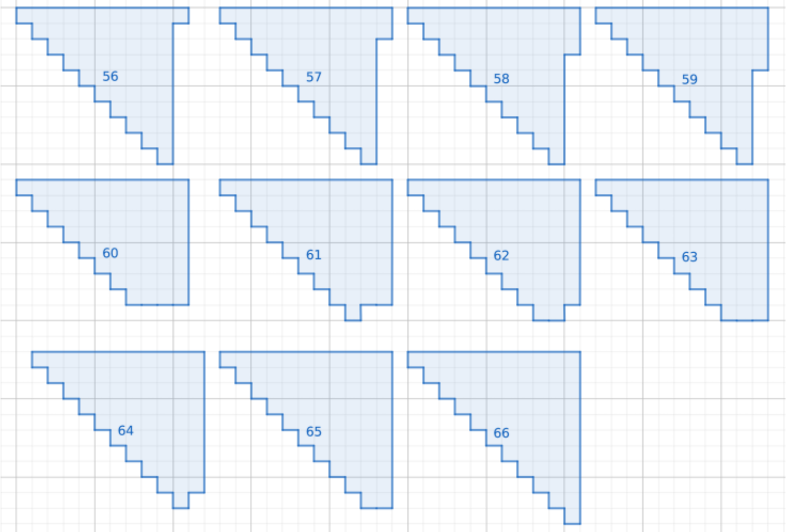
$$e(K[W_1]) = 1984, \; e(K[W_2]) = 2010, \; e(K[W_3]) = 2019,$$
$$e(K[W_4]) = 2009, \; \underline{e(K[W_5]) = 1981},$$

i	2	3	4	5	6	7
$\mathrm{HF}(K[W_1], i)$	1246	11389	70051	328771	1266005	4188859
$\mathrm{HF}(K[W_2], i)$	1256	11524	71012	333593	1285193	4253378
$\mathrm{HF}(K[W_3], i)$	1259	11565	71306	335075	1291108	4173307
$\mathrm{HF}(K[W_4], i)$	1255	11511	70922	333151	1283464	4247645
$\mathrm{HF}(K[W_5], i)$	1248	11406	70124	328965	1266265	4188404

Strongly stable sets giving the minimal HF_S for $u = 56, \dots, 66$. $(n = 11)$



Strongly stable sets giving the minimal HF_S for $u = 56, \ldots, 66$. ($n = 11$)



RevLex

Lex

Lex

Conjecture

The subalgebra generated by u homogeneous polynomials of degree two with minimal Hilbert function is generated by a Lex or RevLex segment.

Conjecture

The subalgebra generated by u homogeneous polynomials of degree two with minimal Hilbert function is generated by a Lex or RevLex segment.

The conjecture is proved in the following cases.

- The polynomials are in at most 80 variables. ($u \leq 3240$)

Conjecture

The subalgebra generated by u homogeneous polynomials of degree two with minimal Hilbert function is generated by a Lex or RevLex segment.

The conjecture is proved in the following cases.

- The polynomials are in at most 80 variables. ($u \leq 3240$)

Recall that $\binom{n}{2} < u \leq \binom{n+1}{2}$. Write $u = \binom{n}{2} + r$ where $1 \leq r \leq n$.

- $u = \binom{n}{2} + r$, for $n \geq 80$ and $1 \leq r \leq 50$ RevLex
- $u = \binom{n}{2} + r$, for $n \geq 80$ and $n - 25 \leq r \leq n$ Lex

Example

$n = 80 \quad u = \binom{80}{2} + r,\ 1 \leq r \leq 80$

r	
1–50	RevLex
51	Lex
52	Lex
53	RevLex
54	RevLex
55–80	Lex

Thank you!

References

M. Boij and A. Conca. On the Fröberg-Macaulay conjectures for algebras. *Rend. Isit. Mat. Univ. Trieste*, 50:139–147, 2018

L. Nicklasson. Subalgebras generated in degree two with minimal Hilbert function. Preprint arXiv:1911.11038