Jordan Types of Artinian Algebras with Height Two

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Combinatorial Algebra Meets Algebraic Combinatorics

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Definition

Let A be a graded Artinian k-algebra and linear form $\ell \in A_1$. The *Jordan type* of A for ℓ is a partition of $\dim_k(A)$ determining the Jordan block decomposition of the multiplication map $m_\ell : A \longrightarrow A$ and it is denoted by $P_{A,\ell}$.

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1	x	x^2	x^3
y	xy	x^{2}y	x^{3}y
y^2	xy^2	xy^2^{2}	x^{3}y^2

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- For $\ell = y$, $P_{A,y} = (3,3,3,3) = P_{A,x}^\vee$.

From now on we assume $R = k[x, y]$ and $A = R/I$ is a graded Artinian quotient of R.

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- Diagonal lengths of $P_{A,\ell}$ is given by the Hilbert function of A. [Iarrobino-Yaméogo]

Question

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Find all partitions with diagonal lengths T which occur as Jordan types of complete intersection algebras for some linear form.

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- Let $\mathcal{B}_i = (\alpha_1, \ldots, \alpha_r)$ be a k-linear basis of A_i. The matrix

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$$h^i(F) := \det\left(\mathrm{Hess}^i(F)\right)$$

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- A has the SLP with $\ell \in A_1 \iff$

$$h^i_\ell(F) \neq 0, \ \ \forall i = 0, \dots, \lfloor \frac{j}{2} \rfloor.$$

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Question:

What Jordan type partitions with diagonal length T are possible for complete intersection algebras having at lease one Hessian vanishing?

$$T = \left(1, 2, \ldots d-1, d^k, d-1, \ldots, 2, 1\right)$$

The total number of partitions with diagonal lengths T is

$$\sum_{i=1}^{d} \binom{d-1}{i-1} 2^i = 2(3^{d-1}), \quad \text{if } \; k > 1.$$

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$$\sum_{i=1}^{d} 2 \cdot 3^{i-1} + 1 = 3^{d-1}, \quad \text{if } k = 1.$$

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- $P = P_{A,\ell}$ for an Artinian complete intersection $A = R/\operatorname{Ann}(F)$ and linear form $\ell \in A_1$, and there is an ordered partition $n = n_1 + \cdots + n_c$ of an integer n satisfying $0 \leq n \leq d$ $(0 \leq n \leq d-1$ for $k = 1)$ such that $h_\ell^{n_1+\cdots+n_i-1}(F) \neq 0$, for each $i \in [1, c]$, and zero otherwise;

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- P satisfies

$$P = \left(p_1^{n_1}, \ldots, p_c^{n_c}, (d-n)^{d-n+k-1}\right), \tag{1}$$

where $p_i = k - 1 + 2d - n_i - 2(n_1 + \cdots + n_{i-1})$, for $1 \leq i \leq c$.

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where $p_i = k - 1 + 2d - n_i - 2(n_1 + \cdots + n_{i-1})$, for $1 \leq i \leq c$.

$\Rightarrow$ There are 2^d complete intersection Jordan types, if $k \geq 2$.
There are 2^{d-1} complete intersection Jordan types, if $k = 1$.

$T = (1,2,3,4,5,6,6,5,4,3,2,1)$, socle degree $= 11$

Construct Jordan type of an Artinian complete intersection algebra $A = R/\operatorname{Ann}(F)$ and $\ell \in A_1$ such that

$$h^0_\ell(F) = h^2_\ell(F) = h^3_\ell(F) = h^4_\ell(F) = 0, h^1_\ell(F) \neq 0 \text{ and } h^5_\ell(F) \neq 0.$$

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0	1	2	3	4	5
1	2	3	4	5	
2	3	4	5		
3	4	5			
4	5				
5					

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h^0	0	1	2	3	4	5	6	7	8	9	10
h^1	1	2	3	4	5	6	7	8	9	10	
h^2	2	3	4	5							
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$$P_{A,\ell} = \left(11^2, 5^4\right) .$$

Thank you!