# On different classes of Monomial Ideals associated to $\mathrm{lcm}$-lattices

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# Outlines

- **Description of the problems**
- **Basic definitions and notations**
- **Results**
- **Open problems**

# Description of the problems

- Let $K$ be a field and $S = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring in $n$ variables.
- An ideal in $S$ generated by monomials is called monomial ideal.
- Let $I \subset S$ be a monomial ideal and $\mathrm{lcm}(I)$ be its lcm lattice.

# Problem $I$

- We are interested to find classes of ideals $I$ and $J$ for which if $\mathrm{lcm}(I) \cong \mathrm{lcm}(J)$ it implies $\mathrm{lcm}(I^n)$ and $\mathrm{lcm}(J^n)$ are also isomorphic for all $n$.

# Problem $II$

- We are interested to determine the growth of the number of elements in $\mathrm{lcm}(I^n)$ as a function of $n$.

# $\mathrm{lcm}$-lattice of monomial ideals

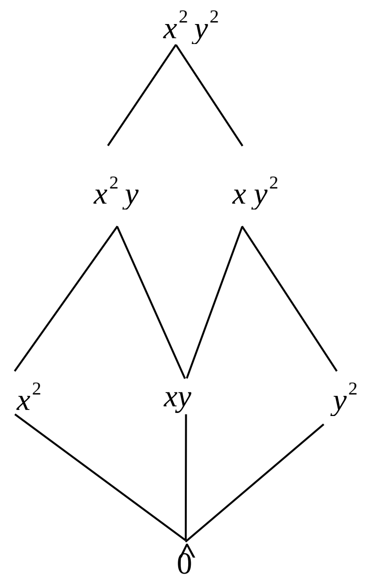
- Let $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $v = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ be two monomials, then the least common multiple $\mathrm{lcm}(u, v)$ is given by
- $\mathrm{lcm}(u, v) = x_1^{max(a_1,b_1)} x_2^{max(a_2,b_2)} \cdots x_n^{max(a_n,b_n)}$.
- Let $I =< m_1, \ldots, m_d >\subset S$ be a monomial ideal. Then $\mathrm{lcm}$-lattice $\mathrm{lcm}(I)$ of ideal $I$ is the set of all LCMs of subsets of $\{m_1, \ldots, m_d\}$ with partial ordering given by divisibility.

# $\mathrm{lcm}$-lattice of monomial ideals

- The unique maximal element is $lcm(m_1, m_2, \ldots, m_d)$ and the unique minimal element is $1$ regarded as the $\mathrm{lcm}$ of the empty set.
- $\mathrm{lcm}(I)$ with this order is a lattice.

# Example

Let $I =< x^2, xy, y^2 > \subset K[x, y]$ be a monomial ideal. Then the $\mathrm{lcm}$-lattice of $I$ is



# Power sequence

**Definition 1**

- *Let* $I \subset k[x_1, \ldots, x_n]$ *be a monomial ideal with lcm lattice* $\mathrm{lcm}(I)$.
- *Let* $\mathrm{lcm}(I)$ *has* $m$ *levels. We denote level of* $\mathrm{lcm}(I)$ *by* $l_i$ *with* $l_0 = \hat{0}$ *and* $l_m = \hat{1}$.
- *Let level* $l_j$ *of* $\mathrm{lcm}(I)$ *has* $t$ *monomials. A power sequence of a variable* $x_i$ *at level* $l_j$ *is defined as follows:*
- $l_j(x_i) : \quad \alpha_1 \lesseqgtr \alpha_2 \lesseqgtr \cdots \lesseqgtr \alpha_t.$

# Power sequence

For example, suppose level $j$ of $lcm(I)$ has the following monomials

$$xyz \qquad xz^3 \qquad x^3y^2 \qquad x^2yz$$

then the power sequence of $x$ is

$$l_j(x): \quad 1 = 1 < 3 > 2.$$

# Results(2 variables case)

**Lemma 2** *Let $I \subset K[x,y]$ be an ideal such that $\mu(I) = t$, then $\mathrm{lcm}$-lattice of $I$ has $\frac{t(t+1)}{2}$ elements.*

# Results(2 variables case)

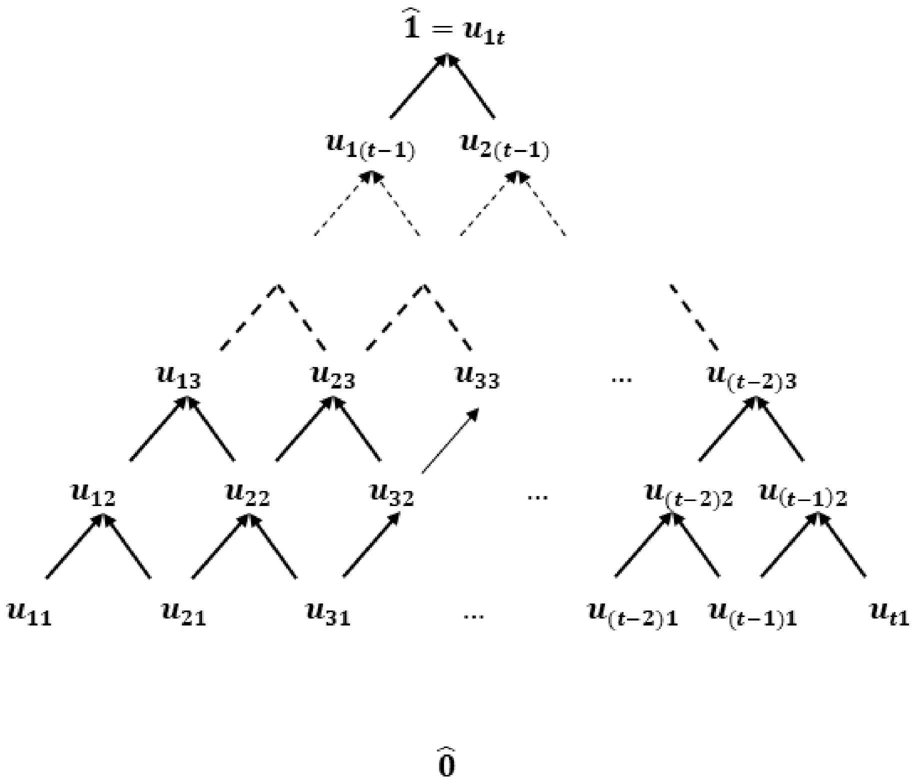



Abbildung 1: $\mathrm{lcm}(I)$ in two variable case

# Results(2 variables case)

**Corollary 3** *Let $I$ and $J$ be two monomial ideals in $K[x, y]$ such that $\mu(I) = \mu(J)$. Then*

$$\operatorname{lcm}(I) \cong \operatorname{lcm}(J).$$

# Results(2 variables case)

**Theorem 4** *Let* $k > l$ *and* $k + m > l + q$ *be numbers and* $I =< x^k y^l, x^l y^k >$, $J =< x^{k+m} y^{l+q}, x^{l+q} y^{k+m} >$ *be two monomial ideals in* $K[x, y]$ *such that*

$$lcm(I) \cong lcm(J).$$

*Then*

$$lcm(I^n) \cong lcm(J^n),$$

*for all* $n > 1$.

# Results(2 variables case)

The proof of above theorem requires following lemma.

**Lemma 5** *Let* $I =< x^{\alpha}y^{\beta}, x^{\beta}y^{\alpha} >\subset K[x,y]$ *be an ideal with* $\alpha > \beta$*. Then*

$$I^n =< x^{(n-i)\alpha+i\beta}y^{(n-i)\beta+i\alpha} \mid i = 0, 1, \ldots, n > .$$

# Results(2 variables case)

**Corollary 6** *Let $I =< x^{\alpha}y^{\beta}, x^{\beta}y^{\alpha} >\subset K[x, y]$ be an ideal with $\alpha > \beta$. Then $lcm$ lattice of $I^n$, denoted by $lcm(I^n)$, is pure and has $n + 1$ levels.*

# Results(2 variables case)

**Corollary 7** *Let* $I =< x^\alpha y^\beta, x^\beta y^\alpha > \subset K[x, y]$ *be an ideal with* $\alpha > \beta$*. Then* $lcm$ *lattice of* $I^n$ *has* $\frac{(n+1)(n+2)}{2}$ *elements.*

# Results(2 variables case)

**Lemma 8** *Let* $I =< x^{\alpha}y^{\beta}, x^{\beta}y^{\alpha} >\subset K[x,y]$ *be a monomial ideal with* $\alpha > \beta$*. Let* $u_i \in lcm(I^{n-1})$ *be monomial at level* $i$ *of* $lcm(I^{n-1})$ *for some* $i \in \{1, \ldots, n\}$*. Then* $u_i^{\alpha} \in lcm(I^n)$ *at level* $i$ *of* $lcm(I^n)$*.*

# Results(3 variables case)

**Lemma 9** *Let* $I =< x, y, z >$ *be a monomial ideal in* $K[x, y, z]$*, then*

$$\mu(I^n) = \frac{(n+1)(n+2)}{2}$$

# Results

**Corollary 10** *Let $I \subset k[x_1, \ldots, x_n]$ be a monomial ideal such that $\mu(I) = t$. Let*

$$I = P_1 \cap P_2 \cap \cdots \cap P_r$$

*be irreducible primary decomposition of $I$ such that supp$(P_i) \subseteq \{x_{i_1}, x_{i_2}\}$ for only one component $P_i$ and $|$supp$(P_j)| < 2$ for all other components different from $P_i$. Then number of elements in the $\mathrm{lcm}(I)$ is*

$$\frac{t(t+1)}{2}.$$

# Results

**Lemma 11** *Let* $I \subset K[x_1, \ldots, x_n]$ *and* $J \subset K[x_1, x_2]$ *be two monomial ideals such that* $\mu(I) = \mu(J)$ *with*

$$G(I) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}.$$

*Then*

$$lcm(I) \cong lcm(J).$$

# Results

**Lemma 12** *Let* $I =< x_1x_2, x_2x_3, \ldots, x_{n-1}x_n > \subset k[x_1, x_2, \ldots, x_n]$*. Then number of elements in the minimal set of generators for* $I^k$ *is given by*

$$\binom{k+n-3}{n-2}.$$

# Observations and further work

- Let $I =< x_1x_2, x_2x_3 >\subset k[x_1, x_2, x_3]$. Then number of elements in $\mathrm{lcm}(I^n)$ is given by
$$\frac{n(n+1)}{2}.$$
- Let $I =< x_1x_2, x_2x_3, x_3x_4 >\subset k[x_1, x_2, x_3, x_4]$. Then number of elements in $\mathrm{lcm}(I^n)$ is given by
$$\frac{n^2(n^2-1)}{12}.$$

# Observations and further work

- Let $I =< x_1x_2, x_2x_3, x_3x_4, x_4x_5 >\subset k[x_1, x_2, x_3, x_4, x_5]$. Then number of elements in $\mathrm{lcm}(I^n)$ is given by

$$\frac{(n+1)(n+2)(n+3)(n^3+6n^2+11n+12)}{72}.$$

- Let $I =< x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6 >\subset k[x_1, x_2, x_3, x_4, x_5, x_6]$. Then number of elements in $\mathrm{lcm}(I^n)$ is given by

$$\frac{((n+2)^6-(n+1)^6)-((n+2)^2-(n+1)^2)}{60}.$$

# THANK YOU