Castelnuovo-Mumford regularity of ladder determinantal ideals via Grothendieck polynomials

Colleen Robichaux
University of Illinois at Urbana-Champaign

joint work with Jenna Rajchgot and Anna Weigandt
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Kazhdan-Lusztig varieties of Woo–Yong '06

Let $\mathcal{F}l_n(\mathbb{C})$ be the set of complete flags

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n, \quad \text{where } \dim V_i = i.$$

We can identify $\mathcal{F}l_n(\mathbb{C})$ with $\mathrm{GL}_n(\mathbb{C})/B$, where $B \subset \mathrm{GL}_n(\mathbb{C})$ is the Borel subgroup.

B acts on $\mathcal{F}l_n(\mathbb{C})$ with finitely many orbits X_w° called **Schubert cells**. The **Schubert varieties** X_w are closures of these orbits. For the opposite Schubert cell Ω_v°:

Theorem [Kazhdan-Lusztig '79]

$$X_w \cap v\Omega_{id}^\circ \cong (X_w \cap \Omega_v^\circ) \times \mathbb{A}^{\ell(v)}$$

Of particular interest is the **Kazhdan-Lusztig variety**

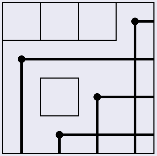
$$\mathcal{N}_{v,w} = X_w \cap \Omega_v^\circ.$$

Kazhdan-Lusztig varieties

Kazhdan-Lusztig variety $\mathcal{N}_{v,w}$ has defining ideal

$$I_{v,w} = \langle r_w(i,j) + 1 \text{ minors of } \mathbf{z}_{i \times j}(v) \rangle.$$

Example: $w = 4132$, $v = 4231$



$$\xrightarrow{r_w} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{\mathbf{z}(v)} \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$I_{v,w} = \langle z_{11},\ z_{12}, z_{13},\ z_{11} - z_{12}z_{21},\ -z_{12}z_{31},\ -z_{31} \rangle$$

Matrix Schubert varieties and determinantal varieties are all examples of KL varieties.

Minimal free resolution

Consider the coordinate ring S/I. The **minimal free resolution**

$$0 \rightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{l,j}} \rightarrow \cdots \rightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{0,j}} \rightarrow S/I \rightarrow 0.$$

The K-**polynomial** of S/I

$$\mathcal{K}(S/I; \mathbf{t}) := \sum_{j \in \mathbb{Z}, i \geq 0} (-1)^i \beta_{i,j} t^j.$$

The **Castelnuovo-Mumford regularity** of S/I

$$\operatorname{reg}(S/I) := \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

Proposition

For Cohen-Macaulay S/I

$$\operatorname{reg}(S/I) = \deg \mathcal{K}(S/I; \mathbf{t}) - \operatorname{codim}_S I.$$

Matrix Schubert varieties

Matrix Schubert varieties $\overline{X}_w$ are special cases of $\mathcal{N}_{v,w'}$.

Theorem

$$\operatorname{reg}(\mathbb{C}[\overline{X}_w]) = \deg(\mathfrak{G}_w(x_1, \ldots, x_n)) - \ell(w),$$

where $\mathfrak{G}_w(x_1, \ldots, x_n)$ is the Grothendieck polynomial and $\ell(w)$ is the Coxeter length of w.

Problem

Give an easily computable formula for $\deg(\mathfrak{G}_w(x_1, \ldots, x_n))$, where $w \in S_n$.

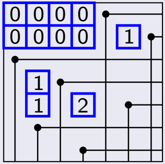
Finding the degree of $\mathfrak{G}_v$ vexillary

Theorem [Rajchgot-R.-Weigandt '22+]

Suppose $v \in S_n$ vexillary. Then

$$\deg(\mathfrak{G}_v) = \ell(v) + \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $v = 5713624$



$\rightsquigarrow$

0	0	0	0	1
0	0	0	0	
1	2			
1				

gives $\deg(\mathfrak{G}_v) = \ell(v) + (3+1) = 12 + 4 = 16$.

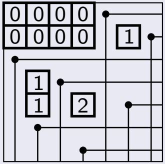
Finding the regularity of $\overline{X}_v$ vexillary

Theorem [Rajchgot-R.-Weigandt '22+]

Suppose $v \in S_n$ vexillary. Then

$$\operatorname{reg}(\mathbb{C}[\overline{X}_v]) = \deg(\mathfrak{G}_v) - \ell(v) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $v = 5713624$



$\rightsquigarrow$

0	0	0	0	1
0	0	0	0	
1	2			
1				

gives $\operatorname{reg}(\mathbb{C}[\overline{X}_v]) = \deg(\mathfrak{G}_v) - \ell(v) = 3 + 1 = 4.$

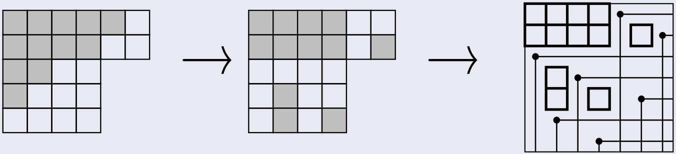
Computing CM-regularity of certain KL varieties

Theorem [Rajchgot-R.-Weigandt '22+]

For $u_\rho, w_\gamma \in S_n$ Grassmannian with descent k, $(u_\rho, w_\gamma) \mapsto v$ vexillary such that

$$\operatorname{reg}(\mathbb{C}[\mathcal{N}_{u_\rho,w_\gamma}]) = \operatorname{reg}(\mathbb{C}[\overline{X}_v]) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $u_{(6,6,4,4,4)}, w_{(5,4,2,1,0)} \mapsto v = 5713624$



gives $\operatorname{reg}(\mathbb{C}[\mathcal{N}_{u_\rho,w_\gamma}]) = \operatorname{reg}(\mathbb{C}[\overline{X}_v]) = 4$.

Application I: KLSS Conjecture

Fix $k \in [n]$. Let Y denote the space of $n \times n$ matrices of the form

$$\begin{bmatrix} A & I_k \\ I_{n-k} & 0 \end{bmatrix}, \text{ where } A \in M_{k\times(n-k)}(\mathbb{C}).$$

The map

$$\pi : GL_n(\mathbb{C}) \to Gr(k,n)$$

induces an isomorphism from Y onto an affine open subvariety U of $Gr(k,n)$. Let $Y_w := \pi|_Y^{-1}(X_w \cap U)$.

Kummini-Lakshmibai-Sastry-Seshadri were interested in the free resolutions of Y_w, and conjectured their CM-regularities when $w_1 = 1$ and w_{n-k-i} not 'too big'.

Application I: KLSS Conjecture

Conjecture [Kummini-Lakshmibai-Sastry-Seshadri '15]

For certain $w_\nu \in S_n$ Grassmannian with descent k,

$$\operatorname{reg}(\mathbb{C}[Y_{w_\nu}]) = \sum_{i=1}^{k-1} i(\nu_i - \nu_{i+1}).$$

But these Y_{w_ν} are just KL varieties!

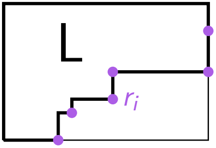
Corollary [Rajchgot-R.-Weigandt '22+]

For $w_\nu \in S_n$ as in KLSS and $u_\rho = (\mathrm{Id}_k + n - k) \times (\mathrm{Id}_{n-k})$

$$\operatorname{reg}(\mathbb{C}[Y_{w_\nu}]) = \operatorname{reg}(\mathbb{C}[\mathcal{N}_{u_\rho, w_\nu}]) = \operatorname{reg}(\mathbb{C}[\overline{X}_{w_\nu}]).$$

Application II: one-sided mixed ladder determinantal ideals

Consider a matrix $X = (x_{ij})$ of indeterminates. Let L denote the submatrix of X defined by choosing SE corners. $I(L)$ is the ideal generated by the NW r_i minors of L. This defines the one-sided mixed ladder determinantal variety $X(L)$.



Further, these are KL-varieties

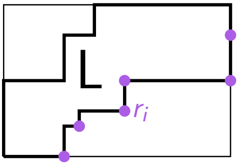
$$X(L) \cong \mathcal{N}_{u_\rho, w_\nu} \cong \overline{X}_v.$$

Corollary [Rajchgot-R.-Weigandt '22+]

$$\mathrm{reg}(\mathbb{C}[X(L)]) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}})$$

Future work: two-sided mixed ladder determinantal ideals

Consider a matrix $X = (x_{ij})$ of indeterminates. Let L denote the submatrix of X defined by choosing SE and NW corners. $I(L)$ is the ideal generated by the NW r_i minors of L. This defines the two-sided mixed ladder determinantal variety $\tilde{X}(L)$.



Further, for any such variety, we can construct $v, w \in S_n$ 321-avoiding such that

$$\tilde{X}(L) \cong \mathcal{N}_{v,w}.$$

Conclusions

- We can express $\operatorname{reg}(\mathbb{C}[\overline{X}_w])$ in terms of the degree of the K-polynomial and the codimension of I_w.
- Use that $\operatorname{reg}(\mathbb{C}[\overline{X}_w]) = \deg \mathfrak{G}_w - \ell(w)$.
- For v vexillary, we obtain an easily computable formula for $\deg \mathfrak{G}_v$, and thus for $\operatorname{reg}(\mathbb{C}[\overline{X}_v])$.
- By relating $\mathcal{N}_{u_\rho, w_\nu}$ to $\overline{X}_v$, we correct a conjecture of KLSS and obtain formulas for regularities of one-sided ladders.