OPEN QUESTIONS FROM THE PROBLEM SESSION

JANUARY 17, 2004

Questions raised by Tony Geramita

Related to the talk of Mike Roth:

Let $A = k[x_0, \ldots, x_n]$, $S = k[y_0, \ldots, y_n]$ be as in Tony's talks, so the x_i act as partial derivatives on S.

Let $B = A^{S_n}$ where S_n is the symmetric group acting by permutations on the basis $\langle x_i \rangle$ of A_1, i.e. B is the subring of invariants for this action.

If $J = \langle e_1, \ldots, e_n \rangle$ is the ideal of B generated by the elementary symmetric functions, then we know that $J = IA$, the ideal of A generated by J has inverse system generated by $H = \Pi_{i<j}(y_i - y_j)$. As Mike R. noted, H is not an invariant of the action of S_n on S, but rather an *alternant*, i.e. if $g \in S_n$ then $gH = sgn(g)H$.

A-submodules of S generated by alternants are inverse systems of ideals of A which are extensions of ideals of B, moreover, the quotients of A by such ideals are the sum of the regular representation of S_n (and such direct sums arise as Artinian quotients of A only in this way).

This gives rise to several questions:

Q1: Let $H_1, \ldots, H_t$ be alternants in S, let M be the A-submodule of S generated by the H_i, i.e. $M = \langle H_1, \ldots, H_t \rangle$.

1) If $M = I^{-1}$, describe I.

2) As we said above, $I = JA$, where J is an ideal of B, describe J.

3) Let H be the alternant described explicitly above. Suppose that $P = H^2$ and $M = \langle P \rangle$ be the A-submodule of S generated by P. If $M = I^{-1}$, find I.

Write A/I as a sum of S_n modules. From Roth's result we know that A/I is not the regular representation in this case (nor is it the direct sum of copies of the regular representation).

4) More generally, suppose that $F \in S$ is a homogeneous invariant of the S_n action. If $M = \langle F \rangle$ is the A-submodule of S generated by F give a way to describe the ideal I such that $M = I^{-1}$. Would it be easier to find the H-vector or Hilbert series of the Gorenstein ring A/I without actually finding I explicitly?

So we have from Nantel's talk: $\lambda \vdash n$.

$R = k[x_1, x_2, \ldots, x_n]$, $S = k[y_1, \ldots, y_n]$, $V_\lambda \subseteq S_{n(\lambda)}$. $M_\lambda = \langle V_\lambda \rangle$ from this we can obtain I_λ. $h(R/I_\lambda)$ is known. The open part of this problem is to find what is the minimal resolution of I_λ (naturally, in terms of λ). Nantel thinks that we can find the the minimal generators carefully choosing among the one he has given.

Punctured Partitions (collections of cells)

Suppose we began with a $\lambda \vdash n$ take the "shape" associated to the partition and remove some cells. Call the resulting shape D. $|D| = r$ (number of cells of D). A tableau T of shape D is an injective map $T\colon D \to \{1, 2, \ldots, r\}$. Given a Tableau T of shape D, let $\{(i_1, j), (i_2, j), \ldots, (i_k, j)\}$ be the cells of D in the column j. We consider the alternating polynomial

$$\Delta_T^{(j)} = \det \left[y_{T(i_s,j)}^{i_t} \right]_{1 \leq s,t \leq k}$$

Then we let

$$\Delta_T = \prod_j \Delta_T^{(j)}.$$

This is a polynomial of degree $n(D)$ in $S = k[y_1, \ldots, y_n]$. Let $V_D \subset S_{n(D)}$ be the subspace spanned by the polynomials $\{\Delta_T : T\colon D \to \{1, 2, \ldots, r\}\}$. The dimension of V_D can be shown to be the number of standard tableaux T of shape D. (T is standard if the entries of T are increasing in the rows and the columns of D.) Nantel does not know of any published proof of this, but it can be deduced adapting the proof one find in Sagan's book on representation of S_n. One see that the Garnir's relation still holds for this kind of modules and deduce that a basis is given by the standard tableaux.

Again let $R = k[x_1, \ldots, x_n]$ and assume R acts on S by partial differentiation. and let M_D be the R-submodule of S generated by V_D. We know the dimension of the space V_D (see above) but, the dimensions of the graded pieces of M_D are not known. Find those dimensions, or equivalently, if $M_D = I_D^{-1}$, where I_D is an ideal of R, then $B = R/I_D$ is a level algebra and we would like to know its h-vector.

One can, of course, take this further. Find I_D explicitly! Even more, find a minimal set of generators of I_D, extend that problem to finding the graded Betti numbers in a minimal free resolution of I_D.

The answers to these questions are known if D is the diagram of a partiiton (not the graded Betti numbers question, however). All is known (again, apart from the graded Betti numbers problem) in the case that D is obtained from a partition by removing a single cell (see J.-C. Aval, F. Bergeron and N. Bergeron, *Lattice Diagram polynomials in one set of variables*, Adv. Appl. Math. **28** (2002) 343–359. and the reference therein) and, from Mike Roth's talk, when D is a diagram which has all its cells on one line.

Questions raised by François Bergeron

$\mathcal{R} = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ you want to look at the same kind of problems but in the case where the symmetric group acts diagonally (i.e. $\sigma p(x_1, \ldots, x_n, y_1, \ldots, y_n) = p(x_{\sigma_1}, \ldots, x_{\sigma_n}, y_{\sigma_1}, \ldots, y_{\sigma_n})$)

$D \subseteq \mathbb{N} \times \mathbb{N}$ with $|D| = n$. Define the statistic $n(D) = \sum_{(a,b)\in D} a$.

$M_D(X;Y) = \langle \Delta_D(X;Y) \rangle$ (that is, $M_D(X;Y)$ will be $\mathcal{L}_{\partial_x,\partial_y}(\Delta_D(X;Y))$ the linear span of all derivatives of $\Delta_D(X;Y)$ of any degree in X and any degree in Y). $\Delta_D(X;Y) = det(x_i^a y_i^b)_{(a,b)\in D, 1\leq i \leq n}$.

Example:

$$\Delta_{(0,0),(1,0),(0,1)} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = x_2\, y_3 - x_3\, y_2 - x_1\, y_3 + x_1\, y_2 + y_1\, x_3 - y_1\, x_2$$

$$M_{(0,0),(0,1),(1,0)} = \mathcal{L}\{\Delta_D(X;Y), x_1 - x_2, x_2 - x_3, y_1 - y_2, y_2 - y_3, 1\}$$

For this example then the bigraded Hilbert series (the polynomial corresponding to the h-vector) is $qt + 2q + 2t + 1$.

1) Known: $D = \lambda$ (partition), everything representation theory is known, including a formula for the Frobenius series $\mathcal{F}_{M_\lambda}(q,t)$ which is given by the Macdonald polynomial indexed by a partition λ. This is the "n! conjecture" which is actually a theorem of M. Haiman, although nothing is explicit. We would like to have a basis.

We do not know the the corresponding ideal $I_D(X;Y)$.

2) If we take $D = \lambda/(a,b)$, the diagram for a partition less a single hole. We have very explicit conjectures for $\mathcal{F}_{M_D}(q,t)$.

Take a partition λ of size $n+1$ and put a small hole in it. The dimension is given by the number of cells in the shadow of the missing cell times $n!$.

3) $M_{\lambda,(i,j)} = \sum_{(a,b),(c,d)\in\lambda (i,j)\leq(a,b)(i,j)\leq(c,d)} M_{\lambda-\{(a,b),(c,d)\}}$ François also has a conjecture for $\mathcal{F}_{M_{\lambda,(i,j)}}(q,t)$

4) We know of instances for which $dim(M_D(X;Y))$ is not a multiple of $n!$.

Questions raised by Christophe Reutenauer

1) If we take G acting on $V = \langle x_1, x_2, \ldots, x_n \rangle$ as a regular representation. $R = k[x_1, x_2, \ldots, x_n]$. I is generated by the invariant polynomials without constant term.

Question: What can be said about R/I? dimension? dimensions of the graded pieces? algebraic properties of the ring?

An application of this is to take K a Galois extension of k, $k \subseteq K$ and the Galois group G. G acts on K as the regular representation. $V = K$.

2) Theorem of Macaulay characterizes Hilbert functions of commutative standard, finitely generated algebras by the condition that $a_{n+1} \leq a_n^{<n>}$. Is there a non-commutative version of this theorem? One part of this is to characterize the generating functions of "factorial languages." Take an

alphabet X of non-commuting variables and let X^* = the free monoid over X = set of non-commutative monomials (words). A language is a subset of X^*. A language is factorial if $\forall u, v, w \in X^*$, then $uvw \in L \Rightarrow v \in L$.

Example

$$L = \{1, x, y, xx, xyx, yx, xy\}$$

Question: Can one characterize the generating functions of factorial languages?