The Squish Map and the SL_2 Double Dimer Model

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Combinatorial Algebra meets Algebraic Combinatorics
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Plane partitions

A **plane partition** is an infinite matrix of nonnegative integers which are weakly decreasing in both rows and columns (all of which are zero sufficiently far from the origin).

5	5	5	3	1
5	4	1	1	1
5	2	1	1	1
3	2	1		
3	1	1		
1				

Plane partitions - counts

Definition

(MacMahon 1915) The generating function for boxed $x \times y \times z$ plane partitions is

$$\prod_{i=1}^{x} \prod_{j=1}^{y} \prod_{k=1}^{z} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}},$$

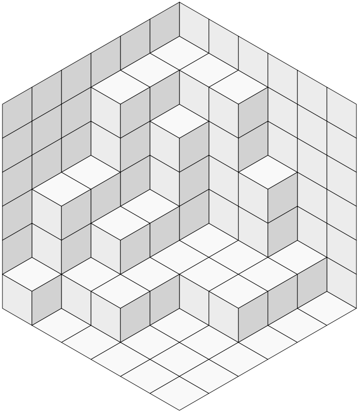
and for infinite plane partitions is

$$M(1, q) = \prod_{i \geq 1} \left(\frac{1}{1 - q^i} \right)^i.$$

Folklore bijection

We can interpret a plane partition as a stack of unit cubes in the corner of a room.

5	5	5	3	1
5	4	1	1	1
5	2	1	1	1
3	2	1		
3	1	1		
1				



Folklore bijection

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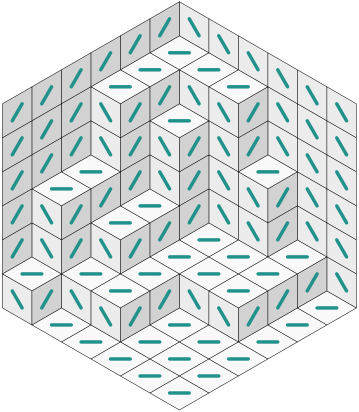
1				



Folklore bijection

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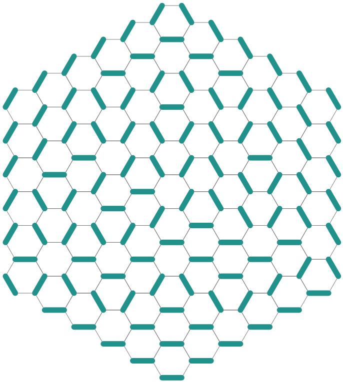
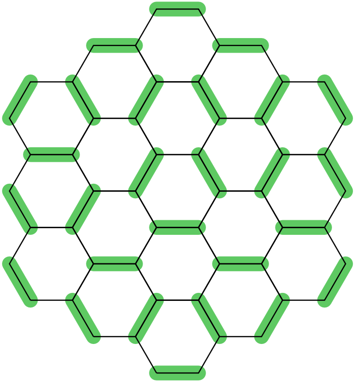
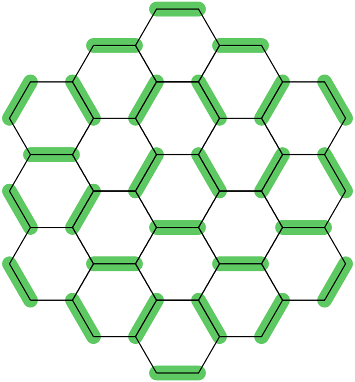


The single dimer model



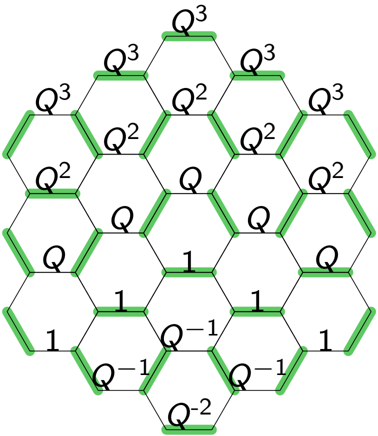
(Kasteleyn 1961) A **single dimer configuration** is a perfect matching of a bipartite graph, which for us is the honeycomb grid.
(Or, a subset of graph edges such that every vertex has degree one.)

The single dimer model



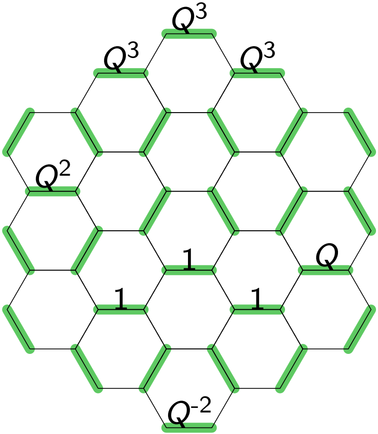
(Kasteleyn 1961) A **single dimer configuration** is a perfect matching of a bipartite graph, which for us is the honeycomb grid.
(Or, a subset of graph edges such that every vertex has degree one.)

So the finite MacMahon generating function also counts single dimer configurations on the $a \times b \times c$ hexagon grid.



(Kasteleyn, 1961) The **weight** of a graph is an assignment $\nu : E \to \mathbb{R}_{\geq 0}$ of real numbers onto each edge of the graph, where E is the set of edges in a graph G.

A monochromatic weighting - horizontal edges get powers of Q, while all other edges get weight 1.

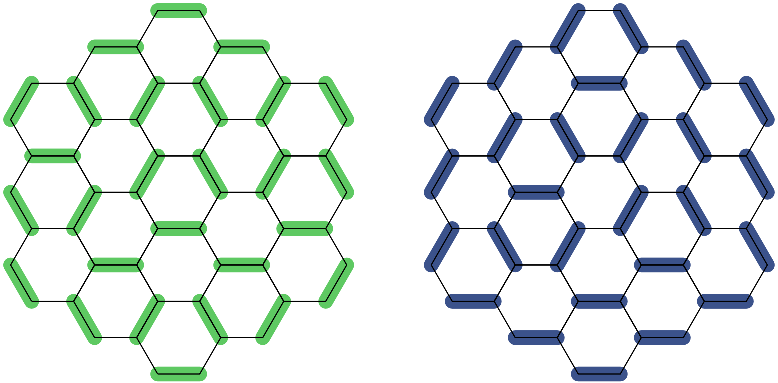


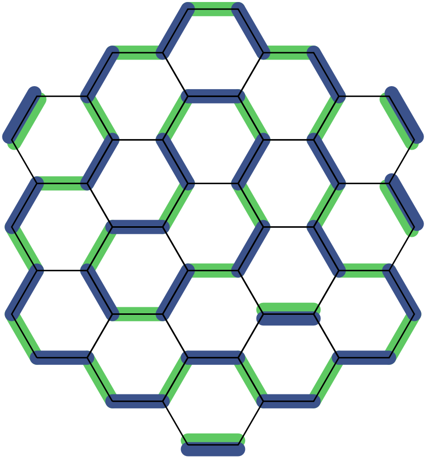
(Kasteleyn, 1961) The **weight** of a graph is an assignment $\nu : E \to \mathbb{R}_{\geq 0}$ of real numbers onto each edge of the graph, where E is the set of edges in a graph G.
The **weight** of a dimer configuration D is

$$w(D) = \prod_{e \in D} w(e).$$

A monochromatic weighting - horizontal edges get powers of Q, while all other edges get weight 1. The weight of the above dimer configuration is Q^{10}.

The single dimer model





(Kenyon, Wilson 2011) A **double dimer configuration** is a subset of graph edges such that every vertex has degree two (or, an overlaying of two single dimer configurations)

The Squish Map I

(Young 2009) The **squish map** Sq sends a single dimer configuration (a perfect matching on the hexagon lattice) to a double dimer configuration (a subset of graph edges where every vertex has degree two – every edge is part of a closed loop or is doubled).

The Squish Map I

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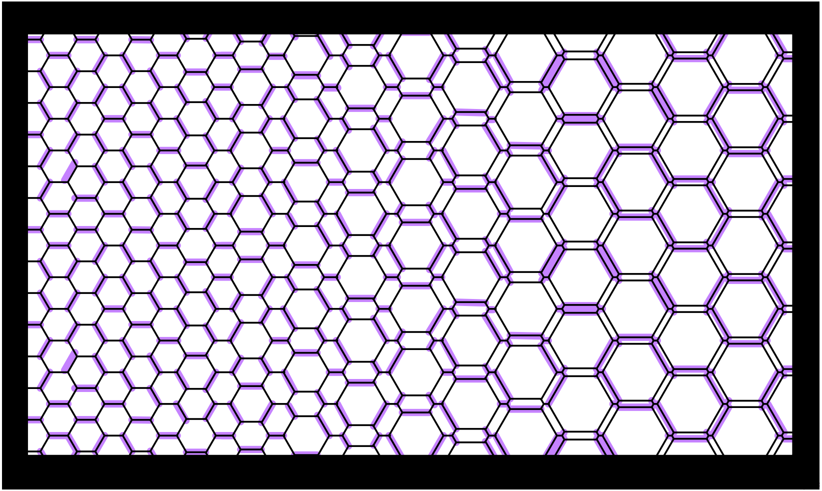
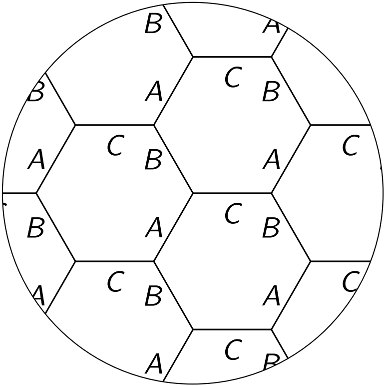


The Squish Map I

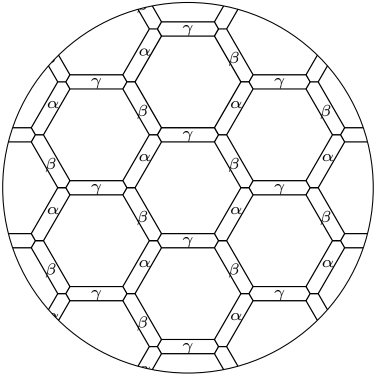
mathematicleigh.com/squish

(Kenyon 2014) Let $G = (V, E)$ be a bipartite graph with a *scalar weight* $w : E \to \mathbb{C}$ as well as an SL_2 *connection*: a map $\Gamma : E \to SL_2(\mathbb{C})$. Then the contribution of a double dimer configuration DD is defined to be

$$\left(\prod_{e\in m} w(e)\right) \times \prod_{\text{closed loops } L} \mathrm{Tr}\left(\prod_{e\in L} \Gamma(e)\right).$$



Weights A, B, and C on
the single dimer model



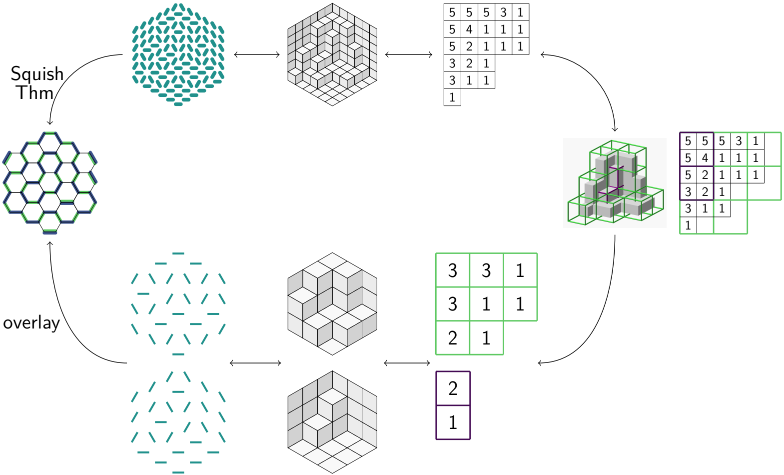
The α, β, and γ connection
on the squished hexagon lattice

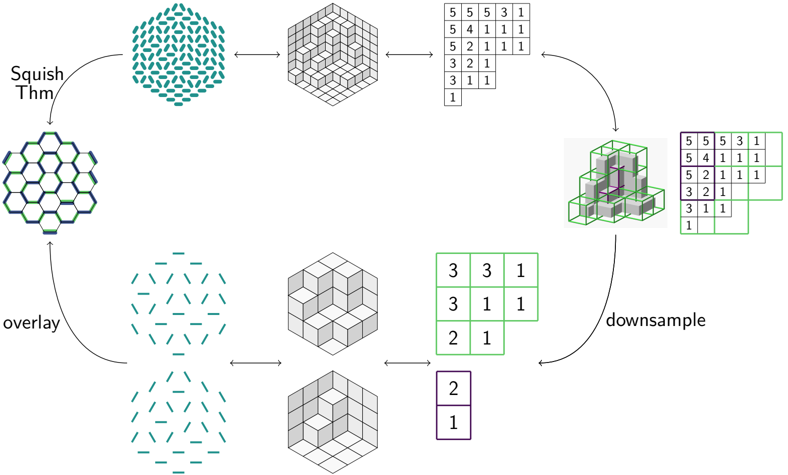
Define the matrices α, β, and γ as follows, with $a, b, c \in \mathbb{C}$:

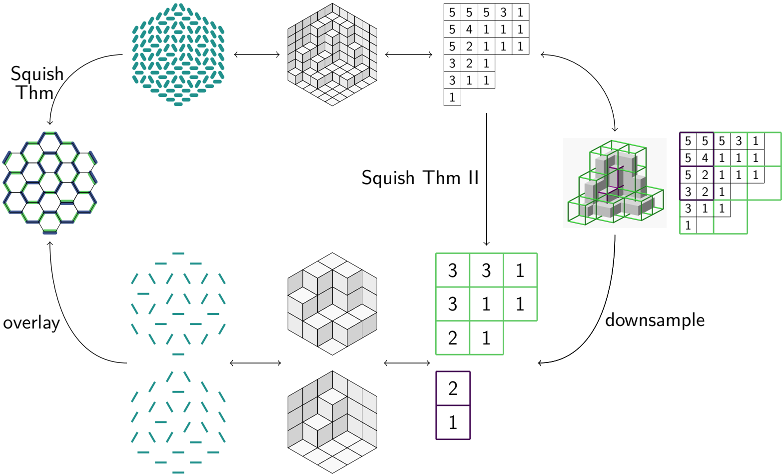
$$\alpha := \begin{bmatrix} ia & 0 \\ 0 & \frac{1}{ia} \end{bmatrix}, \qquad \beta := \begin{bmatrix} ib & -ib - \frac{i}{b} \\ 0 & \frac{1}{ib} \end{bmatrix}, \qquad \gamma := \begin{bmatrix} \frac{1}{ic} & 0 \\ -ic - \frac{i}{c} & ic \end{bmatrix}.$$

Theorem

(F., Young 2024)
Using a particular weight function on the single dimer model and a connection given by placing α on every northeast/southwest edge, β on every northwest/southeast edge, and γ on every horizontal (east/west) edge, then the squish map is measure preserving.


Squish Thm
overlay
5 5 5 3 1
5 4 1 1 1
5 2 1 1 1
3 2 1
3 1 1
1
5 5 5 3 1
5 4 1 1 1
5 2 1 1 1
3 2 1
3 1 1
1
3 3 1
3 1 1
2 1
2
1


Squish Thm
overlay
downsample
5 5 5 3 1
5 4 1 1 1
5 2 1 1 1
3 2 1
3 1 1
1
3 3 1
3 1 1
2 1
2
1


Squish Thm
5 5 5 3 1
5 4 1 1 1
5 2 1 1 1
3 2 1
3 1 1
1
Squish Thm II
downsample
overlay
3 3 1
3 1 1
2 1
2
1

Theorem (The Squish Map II)

Consider a plane partition P. In each 2×2 section of P, let

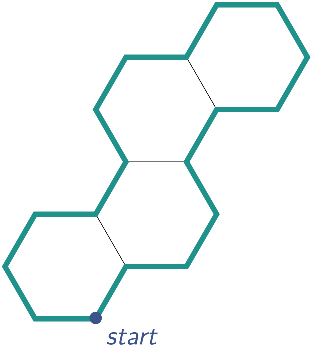
$$\pi_{min}\left(\begin{array}{|c|c|}\hline m & n \\ \hline o & p \\ \hline\end{array}\right) = \left\lfloor \frac{\min(m,n,o,p)}{2} \right\rfloor, \text{ and}$$

$$\pi_{max}\left(\begin{array}{|c|c|}\hline m & n \\ \hline o & p \\ \hline\end{array}\right) = \left\lceil \frac{\max(m,n,o,p)}{2} \right\rceil.$$

These are the minimal and maximal plane partitions that, when overlayed, give us a particular double dimer configuration $\mathcal{D}$.

Example

edge 'direction'



$$\gamma^{-1}(\beta^{-1}\alpha^{-1}\gamma\alpha^{-1})^2\gamma(\beta\alpha\gamma^{-1}\alpha)^2$$

$$\gamma^{-1}(\beta^{-1}\alpha^{-1}\gamma\alpha^{-1})^2\gamma(\beta\alpha\gamma^{-1}\alpha)^2 = \begin{bmatrix} R & S \\ T & U \end{bmatrix} \text{ where}$$

$$\gamma^{-1}(\beta^{-1}\alpha^{-1}\gamma\alpha^{-1})^2\gamma(\beta\alpha\gamma^{-1}\alpha)^2 = \begin{bmatrix} R & S \\ T & U \end{bmatrix} \text{ where}$$

$R = (a^{-6}b^{-4}c^{-4})(a^{14}b^8c^{10} + a^{14}b^8c^8 + a^{14}b^6c^{10} + 3a^{12}b^8c^{10} + a^{14}b^6c^8 + 7a^{12}b^8c^8 + 5a^{12}b^6c^{10} + 3a^{10}b^8c^{10} + 4a^{12}b^8c^6 + 12a^{12}b^6c^8 + 12a^{10}b^8c^8 + 2a^{12}b^4c^{10} + 7a^{10}b^6c^{10} + a^8b^8c^{10} + 7a^{12}b^6c^6 + 15a^{10}b^8c^6 + 5a^{12}b^4c^8 + 30a^{10}b^6c^8 + 7a^8b^8c^8 + 5a^{10}b^4c^{10} + 3a^8b^6c^{10} + 6a^{10}b^8c^4 + 3a^{12}b^4c^6 + 38a^{10}b^6c^6 + 15a^8b^8c^6 + 23a^{10}b^4c^8 + 25a^8b^6c^8 + a^6b^8c^8 + a^{10}b^2c^{10} + 3a^8b^4c^{10} + 15a^{10}b^6c^4 + 13a^8b^8c^4 + 30a^{10}b^4c^6 + 56a^8b^6c^6 + 4a^6b^8c^6 + 5a^{10}b^2c^8 + 28a^8b^4c^8 + 6a^6b^6c^8 + a^8b^2c^{10} + 4a^8b^8c^2 + 12a^{10}b^4c^4 + 47a^8b^6c^4 + 6a^6b^8c^4 + 7a^{10}b^2c^6 + 66a^8b^4c^6 + 27a^6b^6c^6 + 11a^8b^2c^8 + 10a^6b^4c^8 + 13a^8b^6c^2 + 4a^6b^8c^2 + 3a^{10}b^2c^4 + 56a^8b^4c^4 + 40a^6b^6c^4 + 28a^8b^2c^6 + 48a^6b^4c^6 + 3a^4b^6c^6 + a^8c^8 + 6a^6b^2c^8 + a^6b^8 + 15a^8b^4c^2 + 23a^6b^6c^2 + 25a^8b^2c^4 + 73a^6b^4c^4 + 9a^4b^6c^4 + 3a^8c^6 + 31a^6b^2c^6 + 10a^4b^4c^6 + a^6c^8 + 4a^6b^6 + 7a^8b^2c^2 + 40a^6b^4c^2 + 9a^4b^6c^2 + 3a^8c^4 + 48a^6b^2c^4 + 31a^4b^4c^4 + 6a^6c^6 + 10a^4b^2c^6 + 6a^6b^4 + 3a^4b^6 + a^8c^2 + 27a^6b^2c^2 + 30a^4b^4c^2 + 10a^6c^4 + 32a^4b^2c^4 + 3a^2b^4c^4 + 3a^4c^6 + 4a^6b^2 + 9a^4b^4 + 6a^6c^2 + 31a^4b^2c^2 + 6a^2b^4c^2 + 10a^4c^4 + 6a^2b^2c^4 + a^6 + 9a^4b^2 + 3a^2b^4 + 10a^4c^2 + 12a^2b^2c^2 + 3a^2c^4 + 3a^4 + 6a^2b^2 + 6a^2c^2 + b^2c^2 + 3a^2 + b^2 + c^2 + 1)$

$S = (a^6b^4c^6)(a^{14}b^8c^{12} + a^{14}b^8c^{10} + a^{14}b^6c^{12} + 3a^{12}b^8c^{12} + 2a^{14}b^6c^{10} + 7a^{12}b^8c^{10} + 5a^{12}b^6c^{12} + 3a^{10}b^8c^{12} + a^{14}b^6c^8 + 4a^{12}b^8c^8 + 15a^{12}b^6c^{10} + 12a^{10}b^8c^{10} + 2a^{12}b^4c^{12} + 7a^{10}b^6c^{12} + a^8b^8c^{12} + 14a^{12}b^6c^8 + 15a^{10}b^8c^8 + 7a^{12}b^4c^{10} + 33a^{10}b^6c^{10} + 7a^8b^8c^{10} + 5a^{10}b^4c^{12} + 3a^8b^6c^{12} + 4a^{12}b^6c^6 + 6a^{10}b^8c^6 + 8a^{12}b^4c^8 + 51a^{10}b^6c^8 + 15a^8b^8c^8 + 27a^{10}b^4c^{10} + 26a^8b^6c^{10} + a^6b^8c^{10} + a^{10}b^2c^{12} + 3a^8b^4c^{12} + 3a^{12}b^4c^6 + 31a^{10}b^6c^6 + 13a^8b^8c^6 + 48a^{10}b^4c^8 + 65a^8b^6c^8 + 4a^6b^8c^8 + 6a^{10}b^2c^{10} + 30a^8b^4c^{10} + 6a^6b^6c^{10} + a^8b^2c^{12} + 6a^{10}b^6c^4 + 4a^8b^8c^4 + 35a^{10}b^4c^6 + 67a^8b^6c^6 + 6a^6b^8c^6 + 12a^{10}b^2c^8 + 85a^8b^4c^8 + 29a^6b^6c^8 + 12a^8b^2c^{10} + 10a^6b^4c^{10} + 9a^{10}b^4c^4 + 29a^8b^6c^4 + 4a^6b^8c^4 + 10a^{10}b^2c^6 + 100a^8b^4c^6 + 49a^6b^6c^6 + 38a^8b^2c^8 + 54a^6b^4c^8 + 3a^4b^6c^8 + a^8c^{10} + 6a^6b^2c^{10} + 4a^8b^6c^2 + a^6b^8c^2 + 3a^{10}b^2c^4 + 51a^8b^4c^4 + 36a^6b^6c^4 + 50a^8b^2c^6 + 102a^6b^4c^6 + 10a^4b^6c^6 + 4a^8c^8 + 36a^6b^2c^8 + 10a^4b^4c^8 + a^6c^{10} + 9a^8b^4c^2 + 11a^6b^6c^2 + 29a^8b^2c^4 + 84a^6b^4c^4 + 12a^4b^6c^4 + 6a^8c^6 + 74a^6b^2c^6 + 37a^4b^4c^6 + 7a^6c^8 + 10a^4b^2c^8 + a^6b^6 + 6a^8b^2c^2 + 29a^6b^4c^2 + 6a^4b^6c^2 + 4a^8c^4 + 67a^6b^2c^4 + 49a^4b^4c^4 + 16a^6c^6 + 40a^4b^2c^6 + 3a^2b^4c^6 + 3a^4c^8 + 3a^6b^4 + a^4b^6 + a^8c^2 + 26a^6b^2c^2 + 27a^4b^4c^2 + 16a^6c^4 + 57a^4b^2c^4 + 8a^2b^4c^4 + 13a^4c^6 + 6a^2b^2c^6 + 3a^6b^2 + 5a^4b^4 + 7a^6c^2 + 34a^4b^2c^2 + 7a^2b^4c^2 + 20a^4c^4 + 17a^2b^2c^4 + 3a^2c^6 + a^6 + 7a^4b^2 + 2a^2b^4 + 13a^4c^2 + 16a^2b^2c^2 + 9a^2c^4 + b^2c^4 + 3a^4 + 5a^2b^2 + 9a^2c^2 + 2b^2c^2 + c^4 + 3a^2 + b^2 + 2c^2 + 1)$

$T = (a^8b^4c^4)(a^{12}b^8c^8 + a^{12}b^8c^6 + 2a^{12}b^6c^8 + 2a^{10}b^8c^8 + 2a^{12}b^6c^6 + 5a^{10}b^8c^6 + a^{12}b^4c^8 + 5a^{10}b^6c^8 + a^8b^8c^8 + 3a^{10}b^8c^4 + a^{12}b^4c^6 + 13a^{10}b^6c^6 + 5a^8b^8c^6 + 4a^{10}b^4c^8 + 3a^8b^6c^8 + 8a^{10}b^6c^4 + 7a^8b^8c^4 + 11a^{10}b^4c^6 + 18a^8b^6c^6 + a^6b^8c^6 + a^{10}b^2c^8 + 3a^8b^4c^8 + 3a^8b^8c^2 + 7a^{10}b^4c^4 + 25a^8b^6c^4 + 3a^6b^8c^4 + 3a^{10}b^2c^6 + 21a^8b^4c^6 + 6a^6b^6c^6 + a^8b^2c^8 + 10a^8b^6c^2 + 3a^6b^8c^2 + 2a^{10}b^2c^4 + 31a^8b^4c^4 + 19a^6b^6c^4 + 9a^8b^2c^6 + 10a^6b^4c^6 + a^6b^8 + 12a^8b^4c^2 + 17a^6b^6c^2 + 15a^8b^2c^4 + 34a^6b^4c^4 + 3a^4b^6c^4 + a^8c^6 + 6a^6b^2c^6 + 4a^6b^6 + 6a^8b^2c^2 + 30a^6b^4c^2 + 6a^4b^6c^2 + 2a^8c^4 + 23a^6b^2c^4 + 10a^4b^4c^4 + a^6c^6 + 6a^6b^4 + 3a^4b^6 + a^8c^2 + 21a^6b^2c^2 + 20a^4b^4c^2 + 5a^6c^4 + 10a^4b^2c^4 + 4a^6b^2 + 9a^4b^4 + 5a^6c^2 + 21a^4b^2c^2 + 3a^2b^4c^2 + 3a^4c^4 + a^6 + 9a^4b^2 + 3a^2b^4 + 7a^4c^2 + 6a^2b^2c^2 + 3a^4 + 6a^2b^2 + 3a^2c^2 + 3a^2 + b^2 + 1$

$U = (a^8b^4c^6)(a^{12}b^8c^{10} + a^{12}b^8c^8 + 2a^{12}b^6c^{10} + 2a^{10}b^8c^{10} + 3a^{12}b^6c^8 + 5a^{10}b^8c^8 + a^{12}b^4c^{10} + 5a^{10}b^6c^{10} +$

$$\mathrm{tr}(\gamma^{-1}(\beta^{-1}\alpha^{-1}\gamma\alpha^{-1})^2\gamma(\beta\alpha\gamma^{-1}\alpha)^2) =$$

$$\mathrm{tr}(\gamma^{-1}(\beta^{-1}\alpha^{-1}\gamma\alpha^{-1})^2\gamma(\beta\alpha\gamma^{-1}\alpha)^2) =$$

$$\begin{aligned}
&a^{-8}\cdot c^{-6}\cdot b^{-4}\cdot(a^{16}b^8c^{12}+a^{16}b^8c^{10}+a^{16}b^6c^{12}+3a^{14}b^8c^{12}+a^{16}b^6c^{10}+7a^{14}b^8c^{10}+5a^{14}b^6c^{12}+3a^{12}b^8c^{12}+\\
&4a^{14}b^8c^8+12a^{14}b^6c^{10}+13a^{12}b^8c^{10}+2a^{14}b^4c^{12}+7a^{12}b^6c^{12}+a^{10}b^8c^{12}+7a^{14}b^6c^8+16a^{12}b^8c^8+5a^{14}b^4c^{10}+\\
&32a^{12}b^6c^{10}+9a^{10}b^8c^{10}+5a^{12}b^4c^{12}+3a^{10}b^6c^{12}+6a^{12}b^8c^6+3a^{14}b^4c^8+41a^{12}b^6c^8+20a^{10}b^8c^8+24a^{12}b^4c^{10}+\\
&30a^{10}b^6c^{10}+2a^8b^8c^{10}+a^{12}b^2c^{12}+3a^{10}b^4c^{12}+16a^{12}b^6c^6+16a^{10}b^8c^6+32a^{12}b^4c^8+71a^{10}b^6c^8+9a^8b^8c^8+\\
&5a^{12}b^2c^{10}+32a^{10}b^4c^{10}+9a^8b^6c^{10}+a^{10}b^2c^{12}+4a^{10}b^8c^4+13a^{12}b^4c^6+60a^{10}b^6c^6+13a^8b^8c^6+7a^{12}b^2c^8+\\
&80a^{10}b^4c^8+46a^8b^6c^8+a^6b^8c^8+12a^{10}b^2c^{10}+13a^8b^4c^{10}+16a^{10}b^6c^4+7a^8b^8c^4+3a^{12}b^2c^6+71a^{10}b^4c^6+\\
&71a^8b^6c^6+3a^6b^8c^6+32a^{10}b^2c^8+71a^8b^4c^8+9a^6b^6c^8+a^{10}c^{10}+7a^8b^2c^{10}+a^8b^8c^2+20a^{10}b^4c^4+41a^8b^6c^4+\\
&3a^6b^8c^4+30a^{10}b^2c^6+118a^8b^4c^6+30a^6b^6c^6+3a^{10}c^8+41a^8b^2c^8+20a^6b^4c^8+a^8c^{10}+7a^8b^6c^2+a^6b^8c^2+\\
&9a^{10}b^2c^4+71a^8b^4c^4+32a^6b^6c^4+3a^{10}c^6+71a^8b^2c^6+71a^6b^4c^6+3a^4b^6c^6+7a^8c^8+16a^6b^2c^8+13a^8b^4c^2+\\
&12a^6b^6c^2+a^{10}c^4+46a^8b^2c^4+80a^6b^4c^4+7a^4b^6c^4+13a^8c^6+60a^6b^2c^6+13a^4b^4c^6+4a^6c^8+a^6b^6+9a^8b^2c^2+\\
&32a^6b^4c^2+5a^4b^6c^2+9a^8c^4+71a^6b^2c^4+32a^4b^4c^4+16a^6c^6+16a^4b^2c^6+3a^6b^4+a^4b^6+2a^8c^2+30a^6b^2c^2+\\
&24a^4b^4c^2+20a^6c^4+41a^4b^2c^4+3a^2b^4c^4+6a^4c^6+3a^6b^2+5a^4b^4+9a^6c^2+32a^4b^2c^2+5a^2b^4c^2+16a^4c^4+\\
&7a^2b^2c^4+a^6+7a^4b^2+2a^2b^4+13a^4c^2+12a^2b^2c^2+4a^2c^4+3a^4+5a^2b^2+7a^2c^2+b^2c^2+3a^2+b^2+c^2+1)
\end{aligned}$$

Definition

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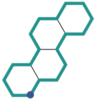

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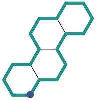

Theorem (F. 2024)

If $n = 1$ or $n = 2$, then the trace of L gives the number of single dimer configurations that squish to that loop.

When $a = b = c = 1$, then

$$\gamma^{-1}(\beta^{-1}\alpha^{-1}\gamma\alpha^{-1})^2\gamma(\beta\alpha\gamma^{-1}\alpha)^2 = \begin{bmatrix} 1393 & 576 \\ 2208 & 913 \end{bmatrix}.$$

So there are 2306 single dimer configurations that squish to this particular closed loop.

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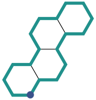
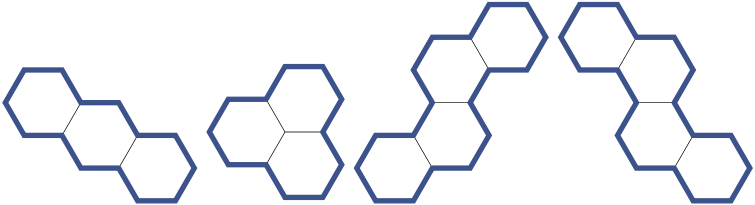


Theorem (F. 2025)

Given a finite, simply-connected region R of the hexagon grid with boundary L, then the trace of L is $\pm I$ (the 2×2 identity matrix) if there exists a signed tiling of R by the stone, bone, and snake tiles.

Tilings



bone stone snake snake

Definition

(Conway, Lagarias 1990) A **signed tiling** of a region R is a collection of tiles, each with a weight of ± 1, covering R in such a way that the total contribution at each hexagon inside of R is 1, and outside of R is zero.

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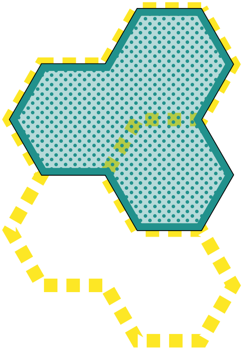

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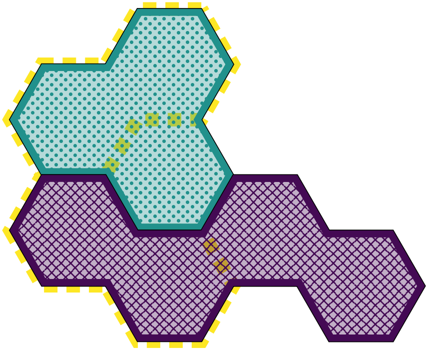

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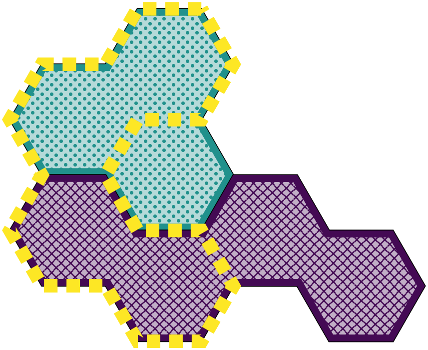

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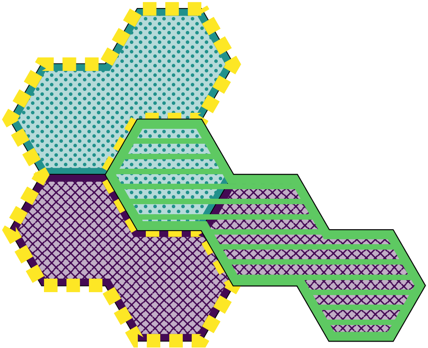

Lemma

When specializing to the 3rd or 6th roots of unity case, removing a bone or a snake from a region R does not change the product of the path around R, and removing a stone negates the product.

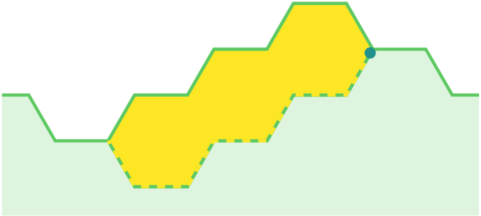

Figure: The path along ∂R, then all the way around a bone, then back out along the bone to ∂R again

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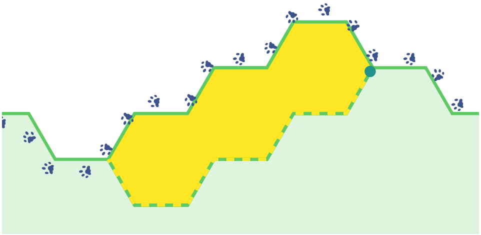

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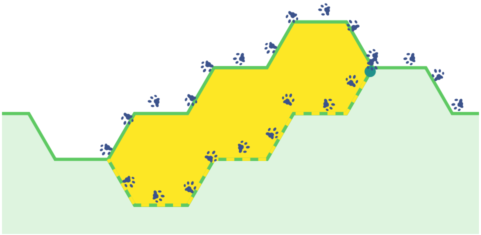

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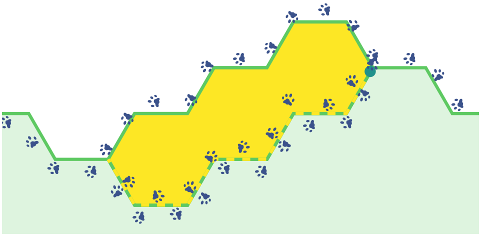

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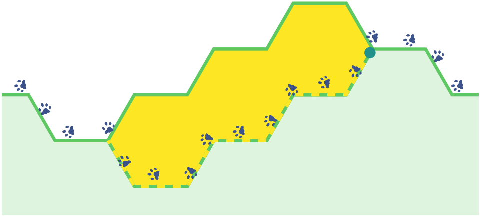

Figure: The path along ∂R, then all the way around a bone, then back out along the bone to ∂R again

Definition

Let $\llbracket \partial R \rrbracket$ be the **tile boundary**: all cyclic permutations of ∂R, as well as all rotations by 120°, and each of these elements' inverses.

Conjecture

Let I be the 2×2 identity matrix. Let $X := \beta\alpha$, $Y := \alpha^{-1}\gamma$, and $Z := \gamma^{-1}\beta^{-1}$, $x := X^{-1}$, $y := Y^{-1}$, and $z := Z^{-1}$. Then the subgroup generated by X, Y, and Z is a subgroup of $SL_2(\mathbb{C})$, and can be written

$$\mathcal{G} = \langle X, Y, Z \mid \llbracket XXX \rrbracket = \llbracket yXyX \rrbracket = \llbracket yzYX \rrbracket = \llbracket YzzX \rrbracket = \llbracket yZxYzX \rrbracket = -I \rangle \, .$$

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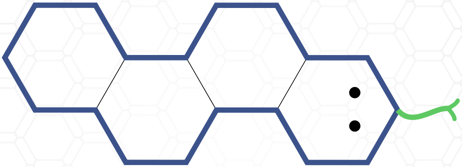
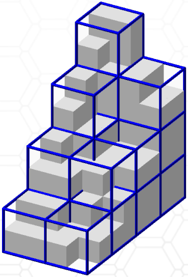
Conjecture

$\mathcal{G}$ is a representation of the Conway-Lagarias tile homotopy group.

Thank you!





Corollary (To the main theorem)

The partition function for the SL_2 double dimer model with 2-periodic weights q, r, s, and t, is

$$Z_Q = M(1,Q)^4 \frac{\widetilde{M}(rs,Q)\widetilde{M}(st,Q)\widetilde{M}(tr,Q)}{\widetilde{M}(-r,Q)\widetilde{M}(-s,Q)\widetilde{M}(-t,Q)\widetilde{M}(-rst,Q)}.$$

where $Q := qrst$,
$\widetilde{M}(x,y) := M(x,y)M(x^{-1},y)$, and
$M(a,q) = \prod_{i\geq 1}\left(\frac{1}{1-aq^i}\right)^i$ is the weighted MacMahon generating function.

Proof (sketch) of main theorem.

The matrices α, β, and γ can be written using transfer matrices (Kramers, Wannier 1941) instead of an $SL_2(\mathbb{C})$ connection. $\square$

Proof (sketch) of main theorem.

Consider a closed loop in the hexagon lattice. Pick a starting edge and walk around the loop counterclockwise. At each vertex, place matrices $L := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ or $R := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ at each left or right turn, respectively. At each northeast edge place matrix A, northwest edges B, and horizontal edges C, where $A := \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$, with B and C defined similarly. □

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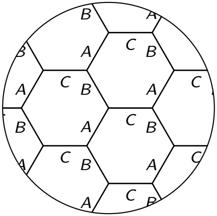
Let $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $i = \sqrt{-1}$.

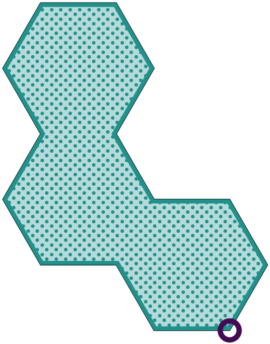
Then α, β, and γ can be written as follows:



$$\alpha := iAJ \qquad \beta := iRBLJ \qquad \gamma := -iLCRJ.$$

$\square$

~~Proof~~ Example.



□

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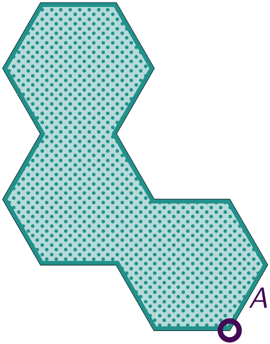

A

□

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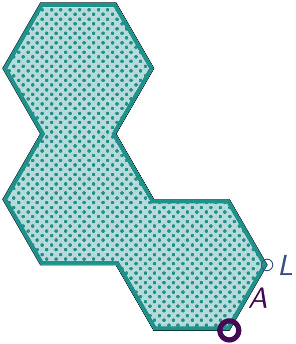

LA

□

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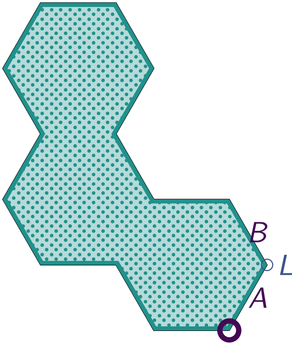
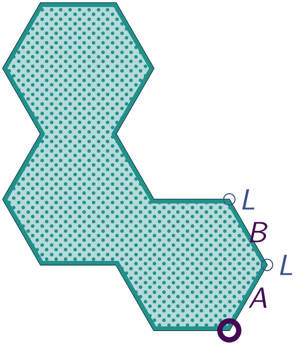


BLA

□

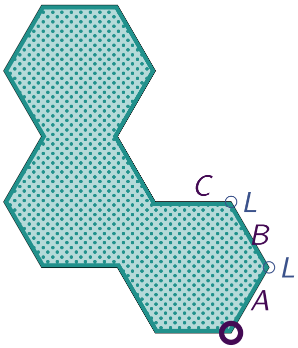
~~Proof~~ Example.



$LBLA$

□

~~Proof~~ Example.



$CLBLA$

□

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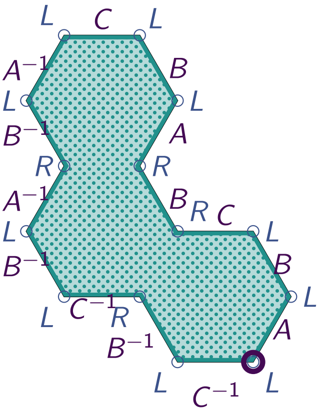

$$LC^{-1}LB^{-1}RC^{-1}LB^{-1}LA^{-1}RB^{-1}LA^{-1}LCLBLARBRCLBLA$$

□

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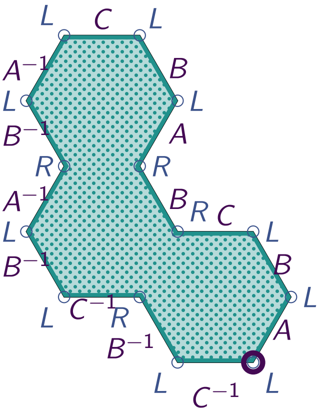

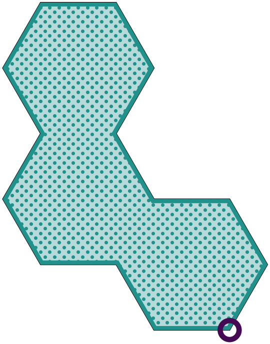

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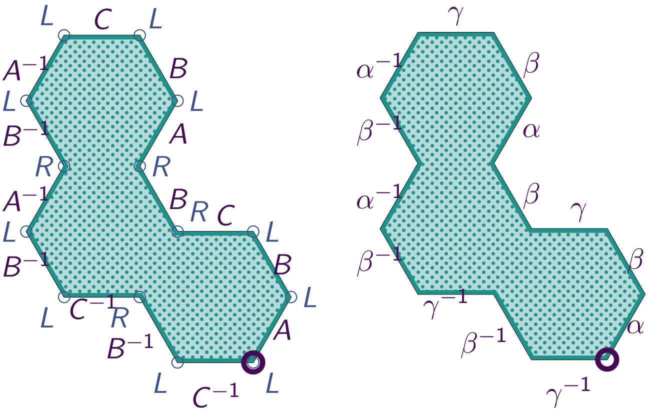

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$$\gamma^{-1}\beta^{-1}\gamma^{-1}\beta^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}\gamma\beta\alpha\beta\gamma\beta\alpha$$

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Then use the fact that $\alpha := iAJ$, $\beta := iRBLJ$, $\gamma := -iLCRJ$ to make substitutions. Pull out all i and $-i$ terms to the front, and then note that J commutes with A, B, and C, but not L or R, and that $J^2 = I$, $RJR = L$, $LJL = R$.

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We show this by calculating the twelve possible turns on the hexagon lattice, as follows:

Left Turns	Right Turns
$\beta\alpha = -RBLA$	$\alpha\beta = -JARBLJ$
$\gamma\beta = LCLBLJ$	$\beta\gamma = -RBRCRJ$
$\alpha^{-1}\gamma = -JA^{-1}LCRJ$	$\gamma\alpha^{-1} = -LCRA^{-1}$
$\beta^{-1}\alpha^{-1} = -RB^{-1}LA^{-1}$	$\alpha^{-1}\beta^{-1} = -JA^{-1}RB^{-1}LJ$
$\gamma^{-1}\beta^{-1} = LC^{-1}LB^{-1}LJ$	$\beta^{-1}\gamma^{-1} = RB^{-1}RC^{-1}RJ$
$\alpha\gamma^{-1} = -JALC^{-1}RJ$	$\gamma^{-1}\alpha = -LC^{-1}RA$