Permutohedral & Multipermutohedral Chow rings

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Permutohedral Chow ring:

- Boolean matroid
- Chow ring of the Boolean matroid
- Group action
- Hilbert series

Multipermutohedral Chow ring:

- Definition
- Multipermutohedral Chow ring
- Hilbert series
- Group action
- Hilbert series

Boolean matroids

We think of a **matroid** $M = (E, \mathscr{L})$ on ground set $E \cong [n]$ defined by its lattice of flats $\mathscr{L} \subseteq 2^E$.

- A Boolean matroid has $\mathscr{L} = 2^E$ and thus the set of flats, when ordered by inclusion, forms a Boolean lattice B_n.

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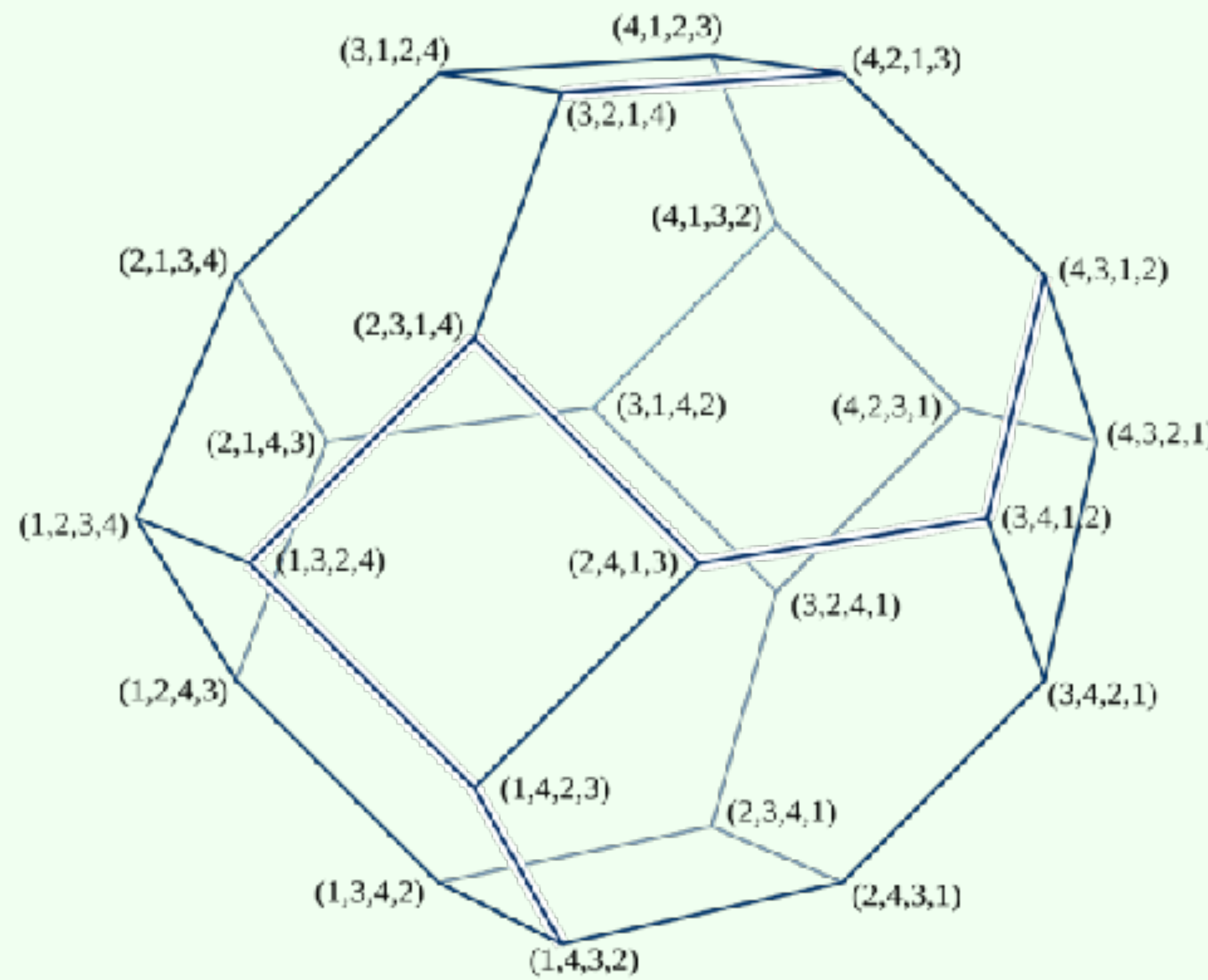
The **permutohedron** of order n is a polytope whose vertices correspond to $\sigma \in \mathfrak{S}_n$ and edges correspond to transpositions.

- The normal fan Σ of the permutohedron is the type-A Coxeter fan
- The associated toric variety X_Σ is called the permutohedral variety
- the cohomology ring of the permuhedral variety is given by the Chow ring of the Boolean matroid



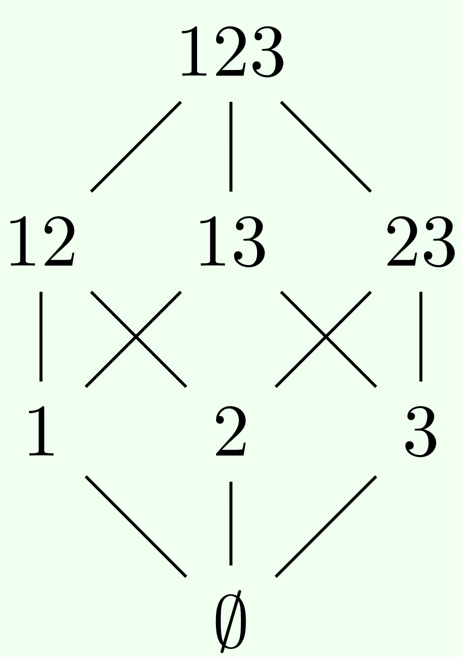
Boolean Chow ring

The Chow ring of a Boolean matroid B_n is

$$A(n) = \frac{\mathbb{Z}[x_S : S \in B_n \backslash \varnothing]}{I + J}$$

$$I = \left(x_S x_T \colon S, T \text{ incomparable in } B_n\right)$$

$$J = \left(\sum_{S \ni a} x_S \colon a = 1, \ldots, n\right)$$



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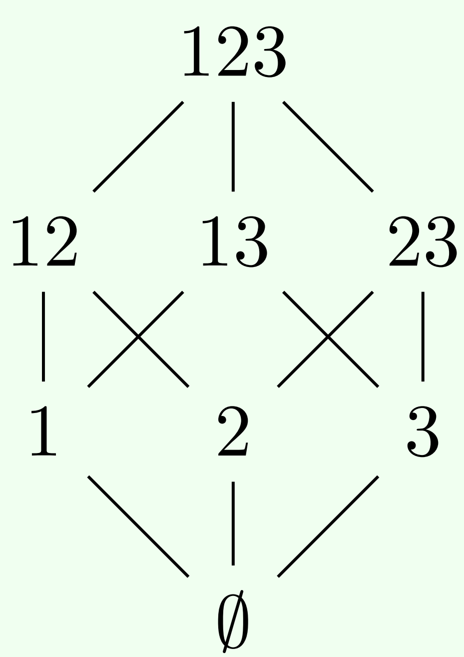

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Theorem [Feichtner–Yuzvinsky '04]

The ideal $I + J$ of the ring $A(n)$ has a monic Gröbner basis.

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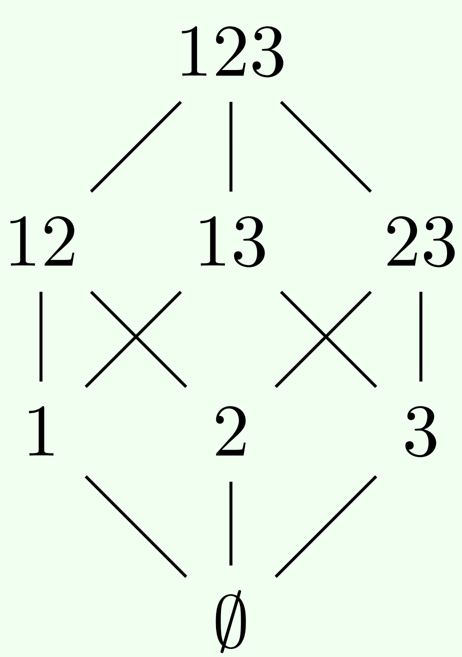

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Theorem [Feichtner–Yuzvinsky '04]

The ideal $I + J$ of the ring $A(n)$ has a monic Gröbner basis.

Corollary [Feichtner–Yuzvinsky '04]

The Boolean Chow ring is free as a $\mathbb{Z}$-module, with $\mathbb{Z}$-basis given by:

$$\prod_{i=1}^{\ell} x_{S_i}^{f_i} \colon \varnothing = S_0 \subset S_1 \subset \ldots \subset S_\ell,$$

$$f_i < |S_i| - |S_{i-1}|$$

We call these Feichtner-Yuzvinski (or FY) monomials.

Degree k FY monomials, FY^k, form a basis for A^k.

FY-monomials:

$$\prod_{i=1}^{\ell} x_{S_i}^{f_i} : \varnothing = S_0 \subset S_1 \subset \ldots \subset S_\ell$$

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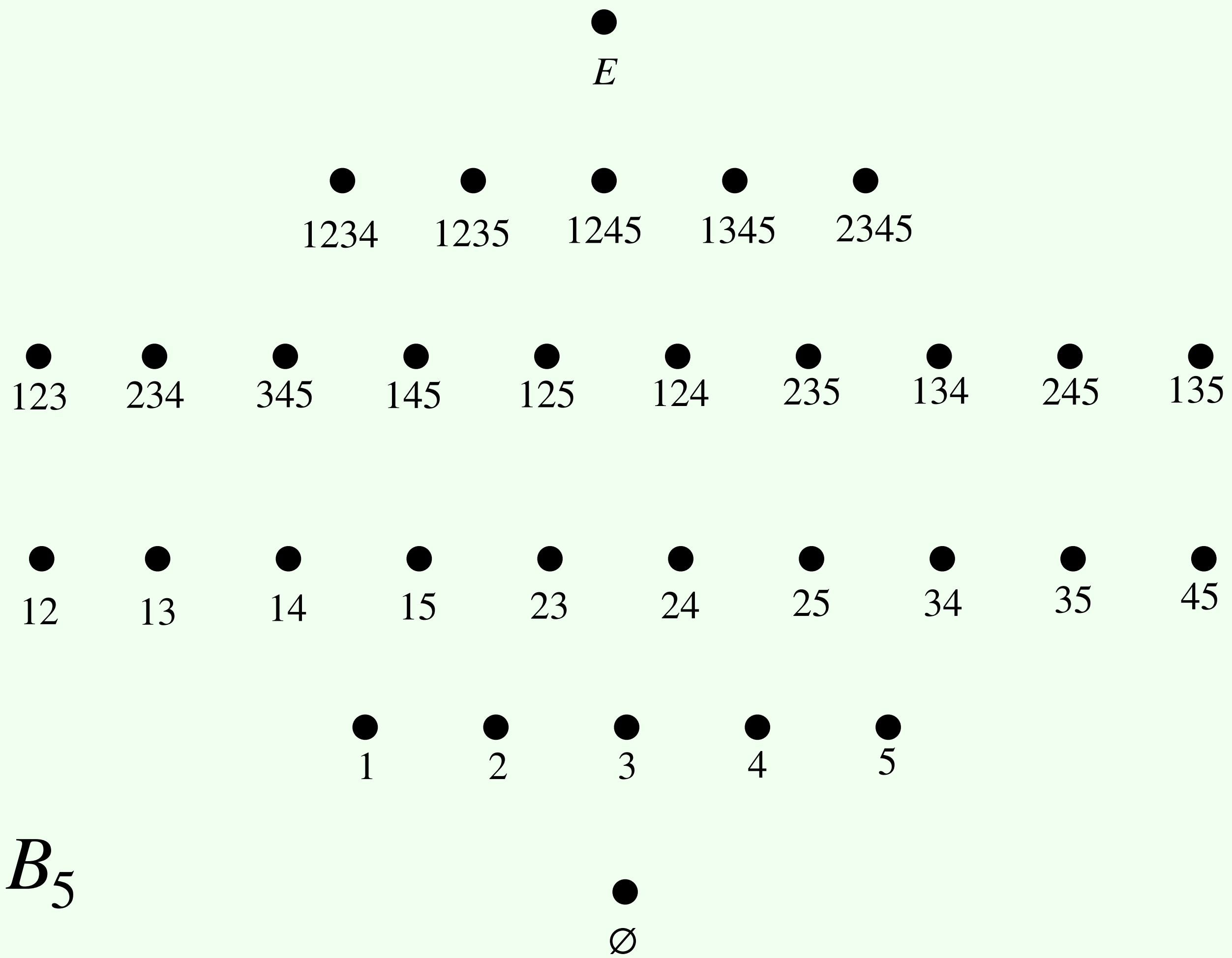

B_5

A^4		
A^3		
A^2		
A^1		
A^0		

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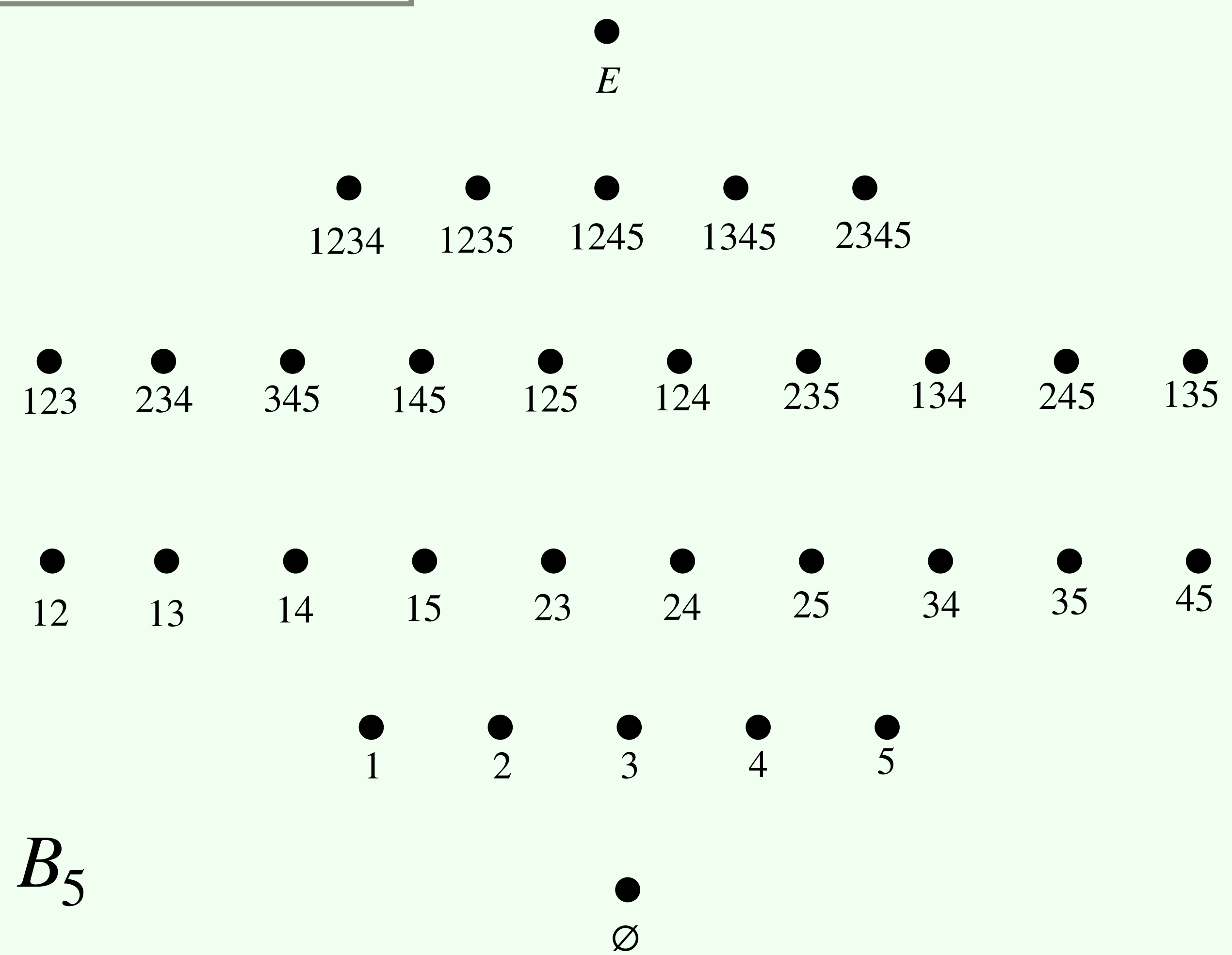

A^4		
A^3		
A^2		
A^1	x_{ij}, x_{ijk}, x_{ijkl}, x_E	
A^0	1	

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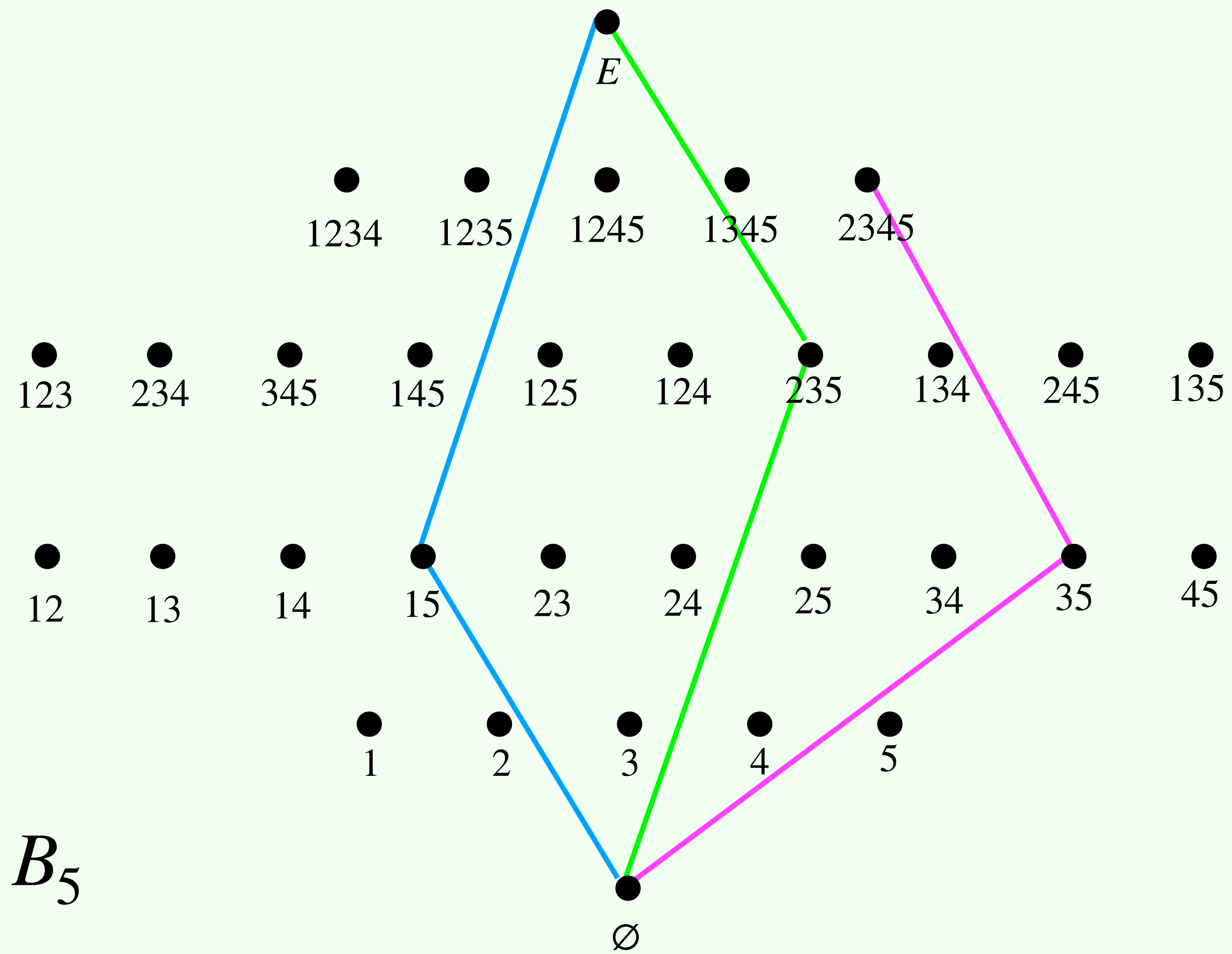

A^4		
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A^2	$x_{ijk}^2,\ x_{ijkl}^2,\ x_E^2$ $x_{ij}x_E,\ x_{ijk}x_E,\ x_{ij}x_{ijkl}$	
A^1	$x_{ij},\ x_{ijk},\ x_{ijkl},\ x_E$	
A^0	1	

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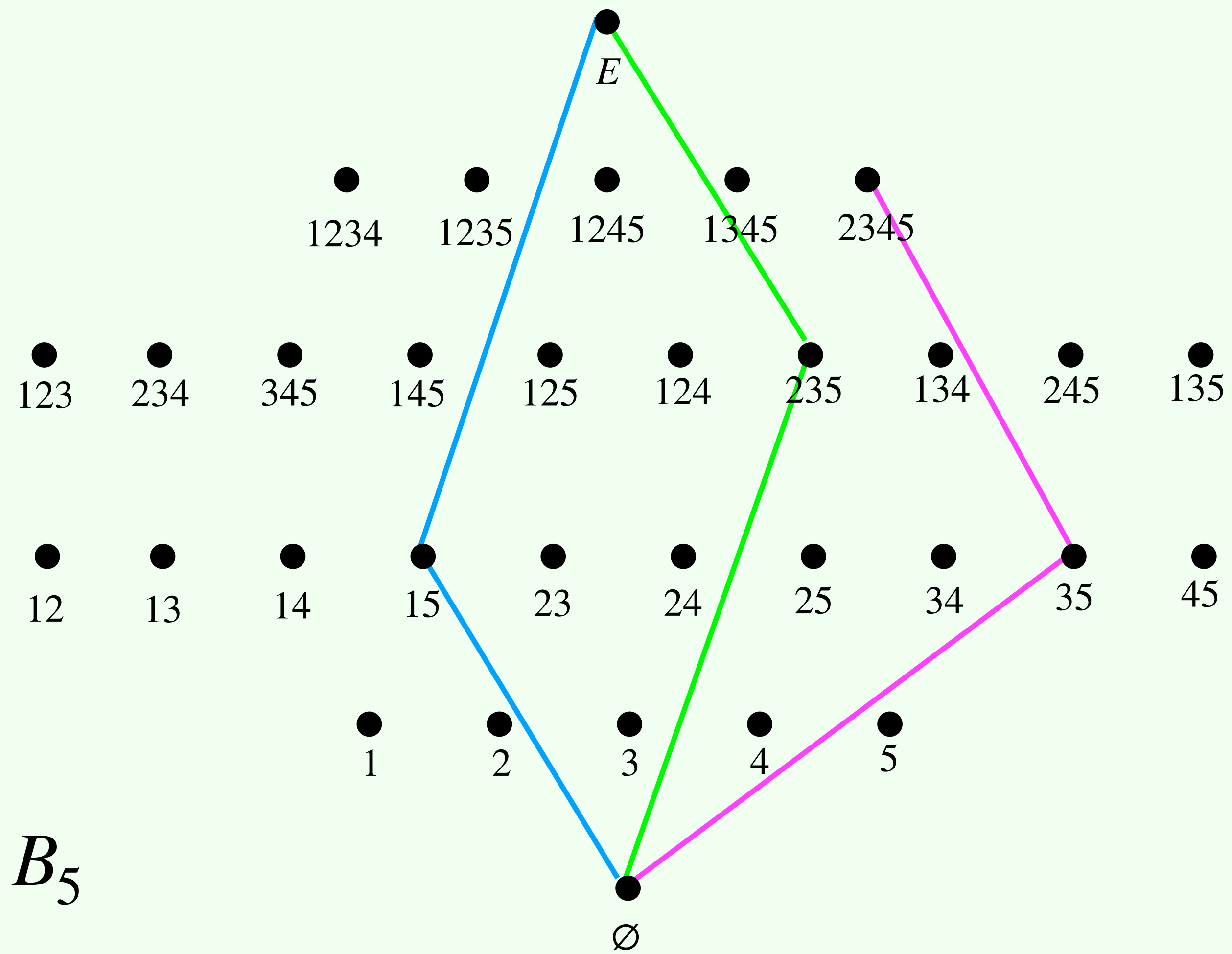

A^4	x_E^4	
A^3	$x_{ij}x_E^2$, $x_{ijk}^2 x_E$, x_{ijkl}^3, x_E^3	
A^2	x_{ijk}^2, x_{ijkl}^2, x_E^2 $x_{ij}x_E$, $x_{ijk}x_E$, $x_{ij}x_{ijkl}$	
A^1	x_{ij}, x_{ijk}, x_{ijkl}, x_E	
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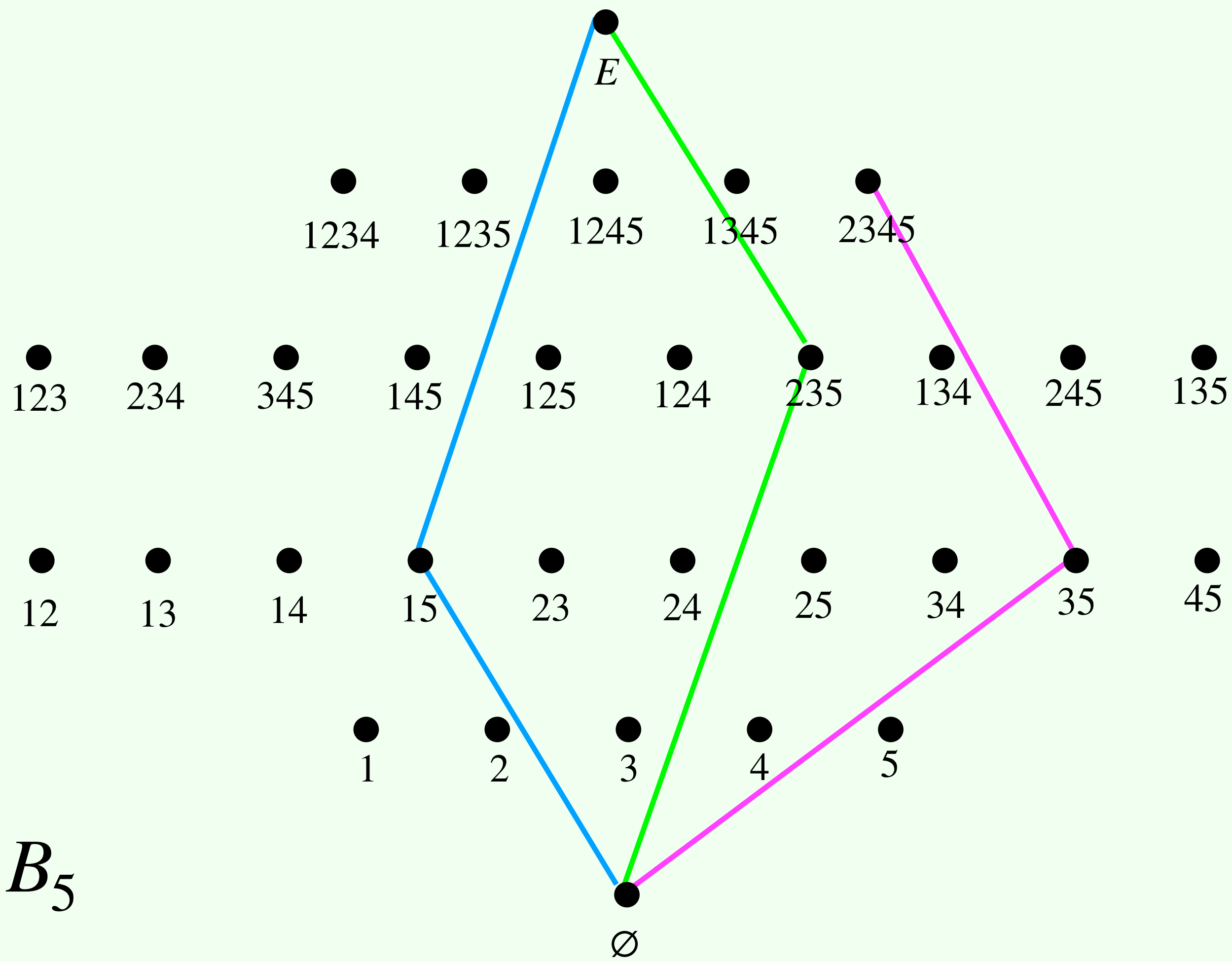

B_5

A^4	x_E^4 1	1
A^3	$x_{ij}x_E^2$, 10 $x_{ijk}^2x_E$, 10 x_{ijkl}^3, 5 x_E^3 1	26
A^2	x_{ijk}^2, 10 x_{ijkl}^2, 5 x_E^2 1 $x_{ij}x_E$, 10 $x_{ijk}x_E$, 10 $x_{ij}x_{ijkl}$ 30	66
A^1	x_{ij}, 10 x_{ijk}, 10 x_{ijkl}, 5 x_E 1	26
A^0	1 1	1

Group Action

The Chow ring carries an action of $\mathfrak{S}_n$ where $\sigma \in \mathfrak{S}_n$ acts on $\mathbb{Z}[x_S : S \in B_n]$ by extending $\sigma \curvearrowright B_n$ to send $\sigma \cdot x_S \to x_{\sigma \cdot S}$

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- every σ preserves cardinality and inclusion relations:
  - it follows that it preserves the ideals $I + J$, so $A(n)$ is an $\mathfrak{S}_n$ representation
  - even further, it only permutes the FY basis, so $A(n)$ is an $\mathfrak{S}_n$ permutation representation

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In general, for $x_{S_1}^{f_1} \cdots x_{S_\ell}^{f_\ell} \in \mathrm{FY}^k$

$$\sigma \cdot x_{S_1}^{f_1} \cdots x_{S_\ell}^{f_\ell} = x_{\sigma \cdot S_1}^{f_1} \cdots x_{\sigma \cdot S_\ell}^{f_\ell} \in \mathrm{FY}^k$$

Thus, each A^k is a permutation representation itself.

Hilbert and Froebenius series

Fact [Ancient]

The Hilbert series of $A(n)$ is computed by

$$\sum_{n\geq 0} \mathrm{Hilb}(A(n);q)x^n = \frac{(1-q)\,e^q}{e^{qx}-xe^q}.$$

- Hsin-Chieh Liao provides a geometric interpretation of these!
- With Robbie Angarone and Vic Reiner, we showed that the sequence of permutation representations $(A^0, A^1, \ldots, A^{n-1})$ is unimodal and palindromic

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Theorem [Stembridge '92]

The Frobenius series is computed by

$$\sum_{n\geq 0} \chi(A(n);q)x^n = \frac{(1-q)H(q)}{H(qx)-xH(q)},$$

where χ is the Frobenius characteristic and $H(q) = \sum_{k\geq 0} h_k(x)q^n$ where h_k means the complete homogeneous symmetric polynomial of degree k.

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Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \mathbb{Z}^n_{\geq 2}$.
Define color classes $E_i \simeq [\pi_i]$ and $E := E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$.

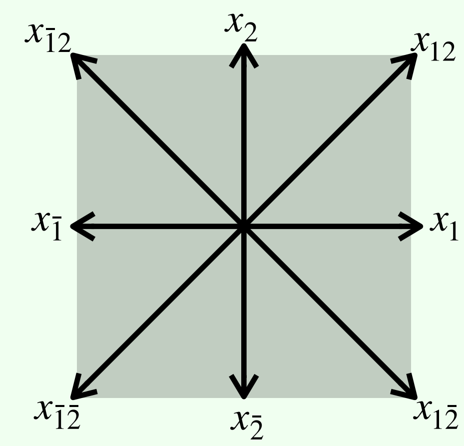
This construction defines a Boolean multimatroid, first defined by Bouchet in 1987.

It was recently used by Clader–Damiolini–Eur–Huang–Li to study moduli space parametrizing pinwheel curves via multipermutohedral fans.

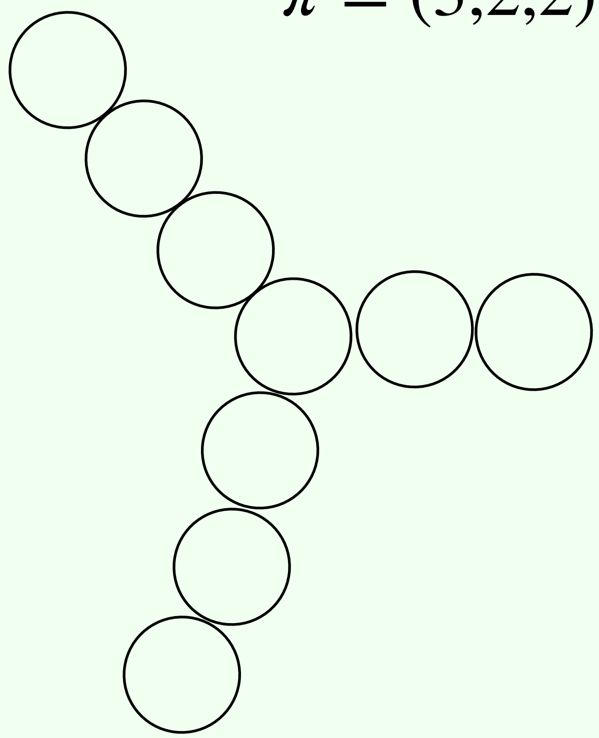
If $\pi = (a, a)$, this defines a delta-matroid where $E = [a] \sqcup [\bar{a}]$.

If $\pi = (2,\ldots,2)$, the multipermutohedral fan is the type B Coxeter fan.

$\pi = (2,2)$



$\pi = (3,2,2)$



π-colored sets

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \mathbb{Z}^n_{\geq 2}$.
Define color classes $E_i \simeq [\pi_i]$ and $E := E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$.

A set $S \subseteq E$ is π-**colored** if $|S \cap E_i| \leq 1$ for all $i \in [n]$.

- Let $\mathcal{S}_\pi = \{S\ \pi\text{-colored}\}$. Ordered by containment, this is a meet-semilattice.
- Given a π-colored set S, define $\mathring{S} = \{i \in [n] : |S \cap E_i| = \varnothing\}$.

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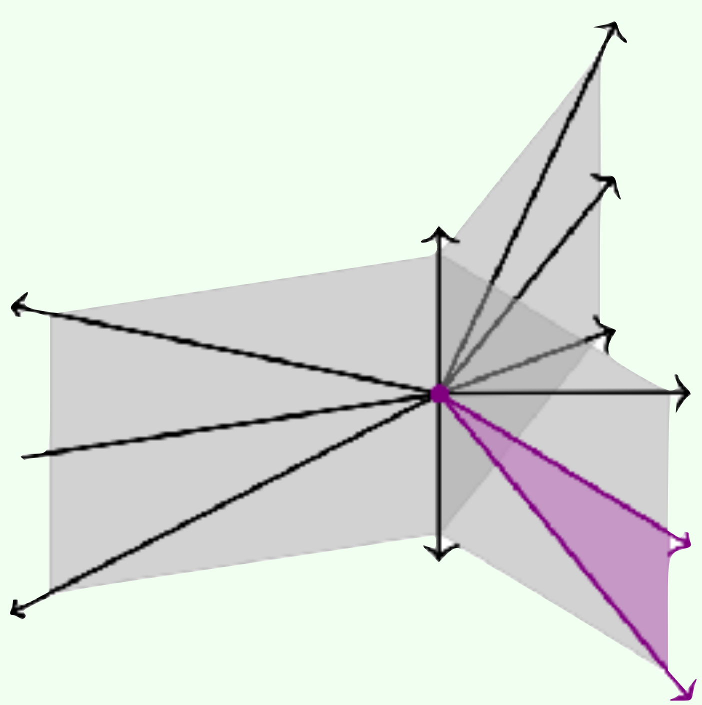
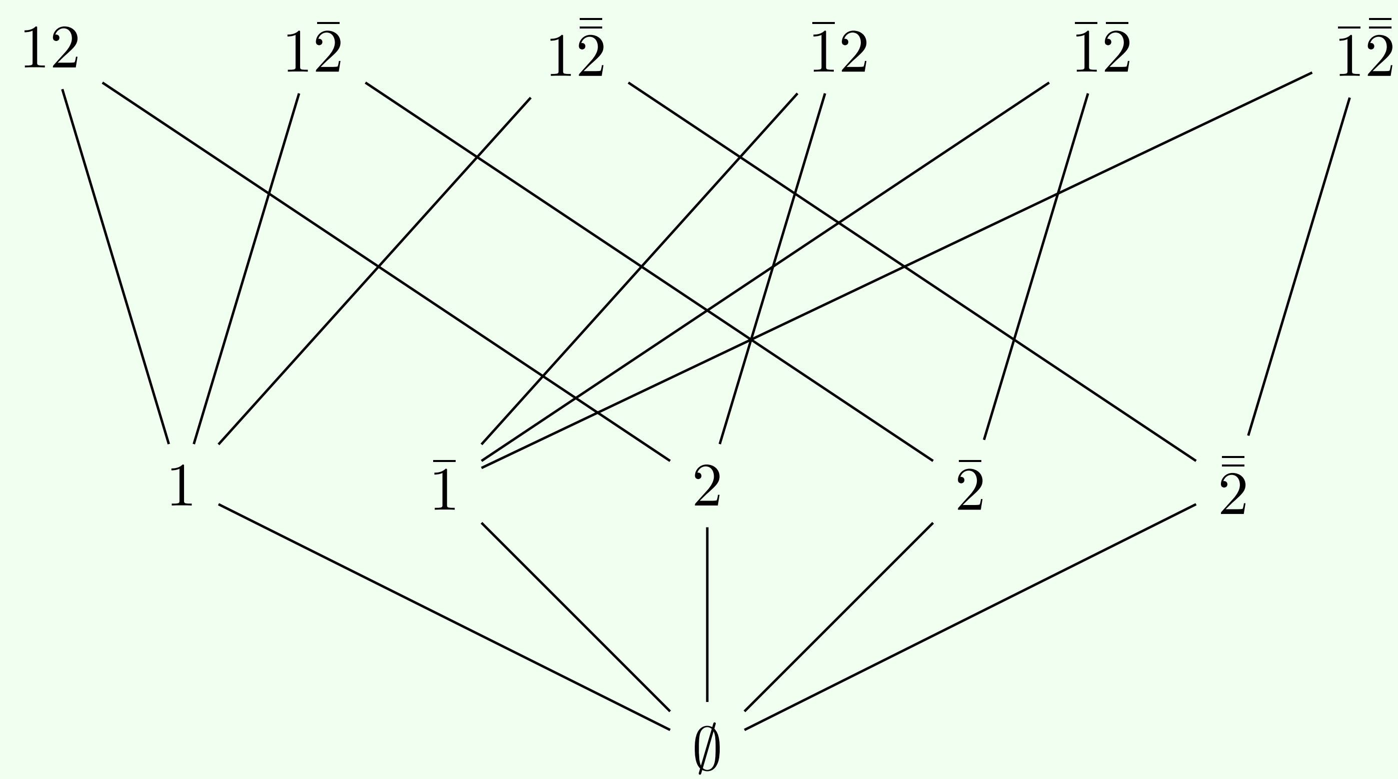
Given $\pi = (2,3)$, we get the ground set $E = \{1, \bar{1}\} \sqcup \{2, \bar{2}, \bar{\bar{2}}\}$.





Multipermutohedral Chow ring

The Chow ring defined by
$\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is

$$A(\pi) = \frac{\mathbb{Z}[x_S \colon \varnothing \neq S \in \mathcal{S}_\pi][y_i \colon i \in [n]]}{I + J}$$

$$I = \left(x_S x_T \colon S, T \text{ incomparable in } \mathcal{S}_\pi\right)$$

$$J = \left(y_i + \sum_{S \ni a_i} x_S \colon a_i \in E_i, i \in [n]\right)$$

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Theorem [N. '26+]

The ideal $I + J$ of the ring $A(\pi)$ has a monic Gröbner basis.

Corollary [N. '26+]

The multipermutohedral Chow ring is free as a $\mathbb{Z}$-module, with $\mathbb{Z}$-basis given by:

$$\prod_{i=1}^{\ell} x_{S_i}^{f_i} \prod_{j \in Z} y_j \colon \varnothing = S_0 \subset S_1 \subset \ldots \subset S_\ell, \quad f_i < |S_i| - |S_{i-1}|, \quad Z \subseteq \mathring{S}_\ell \,.$$

We call these multipermutohedral Feichtner-Yuzvinski (or FY_π) monomials.

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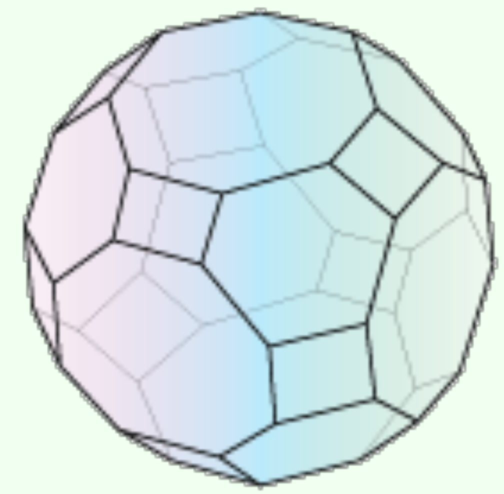

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1 $\quad \bar{1}$ $\quad 2$ $\quad \bar{2}$ $\quad 3$ $\quad \bar{3}$

$\emptyset$

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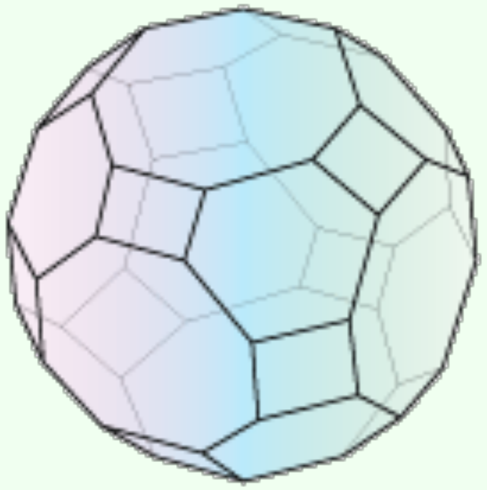

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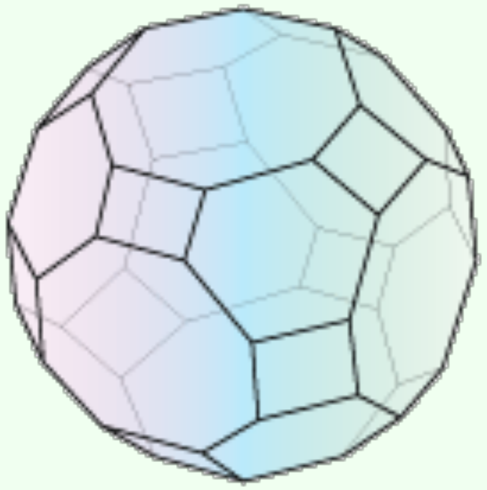

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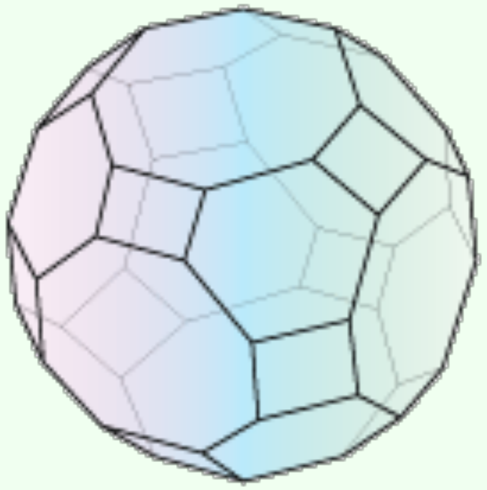

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$\emptyset$

$\pi = (2,2,2)$

A^3	$y_1y_2y_3$ **1**	1
A^2	$x_{ij}y_k$, **12** x_{ijk}^2, **8** y_iy_j **3**	23
A^1	x_{ij}, **12** x_{ijk}, **8** y_i **3**	23
A^0	1 **1**	1

Hilbert series

Theorem [N. '26+]

For $|\pi| = n$, the Hilbert series of $A(\pi)$ is computed by

$$\sum_{k=0}^{n} e_k(\pi_1, \ldots, \pi_n)(1+q)^{n-k}\left(\text{coeff. of } x^k/k! \text{ in } \frac{1-q}{e^{qx} - xe^q}\right)$$

where e_k is the elementary symmetric polynomial.

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$\mathrm{Hilb}(A((\pi_1, \pi_2, \pi_3)); q) =$

	$k=0$	$k=1$	$k=2$	$k=3$
A^3	1	0	0	0
A^2	$3q^2$	0	$e_2(\pi_1, \pi_2, \pi_3)q^2$	$e_3(\pi_1, \pi_2, \pi_3)q^2$
A^1	$3q$	0	$e_2(\pi_1, \pi_2, \pi_3)q$	$e_3(\pi_1, \pi_2, \pi_3)q$
A^0	1	0	0	0

Hilbert series

Theorem [N. '26+]

For $|\pi| = n$, the Hilbert series of $A(\pi)$ is computed by

$$\sum_{k=0}^{n} e_k(\pi_1, \ldots, \pi_n)(1+q)^{n-k}\left(\text{coeff. of } x^k/k! \text{ in } \frac{1-q}{e^{qx} - xe^{q}}\right)$$

where e_k is the elementary symmetric polynomial.

$\mathrm{Hilb}(A((\pi_1, \pi_2, \pi_3)); q) =$

	$k=0$	$k=1$	$k=2$	$k=3$
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A^2	$3q^2$	0	$(\pi_1\pi_2 + \pi_1\pi_3 + \pi_2\pi_3)q^2$	$(\pi_1\pi_2\pi_3)q^2$
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A^2	$23q^2$
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where e_k is the elementary symmetric polynomial.

Define $F(x, u; q)$ to be the generating function for the coefficient of $e_k(\pi_1, \ldots, \pi_n)$ in $\mathrm{Hilb}(A(\pi); q)$, with x tracking the value of k and u tracking $n - k$.

This function is computed by

$$F(x, u; q) = \frac{1-q}{(e^{qx} - qe^{x})(1 - u(1+q))}$$

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Corollary [N. '26+]

Each coefficient of $e_k(\pi)$ in $\mathrm{Hilb}(A(\pi); q)$ is itself palindromic centered at the power of $q^{n/2}$. Equivalently, we show

$$F(x, u; q) = F(qx, qu; 1/q)\,.$$

Group action

Let π define the π-symmetry group

$$G_\pi := \mathfrak{S}_{m_2}[\mathfrak{S}_2] \times \mathfrak{S}_{m_3}[\mathfrak{S}_3] \times \cdots \qquad \text{where } m_k = \#\{\pi_i = k\}\,.$$

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$$H_\pi = \mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_3$$

Each $\sigma \in G_\pi$ preserves cardinality and inclusion, so it presrves $A(\pi)$ and only permutes the monomial basis FY_π and each graded subset FY_π^k.

Equivariant Hilbert Series

For $\pi = (p, \ldots, p) := (p^n)$, the π-symmetry group is $G_\pi = \mathfrak{S}_n[\mathfrak{S}_p]$.

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- Let ω_n be the character of the action of the coset $\mathbb{C}\left[\left(\mathfrak{S}_n[\mathfrak{S}_p]\right)/\mathfrak{S}_n\right]$ which permutes π-colored sets of size n and write $\Omega(x) = \sum_{n \geq 0} \omega_n x^n$.

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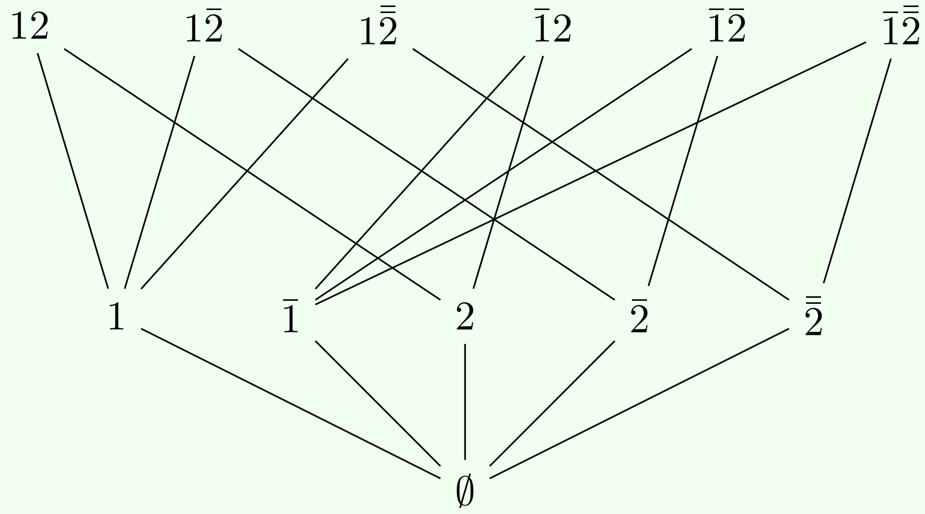
Theorem [N. '26+]

The graded equivariant Hilbert series of $A_{(p^n)}$ under the action of G_π is given by

$$\sum_{n=0}^{\infty} \chi(A_{(p^n)}; q)x^n = \frac{(1-q)\, T(qx)\, T(x)}{\Omega(pqx) - q\Omega(px)}.$$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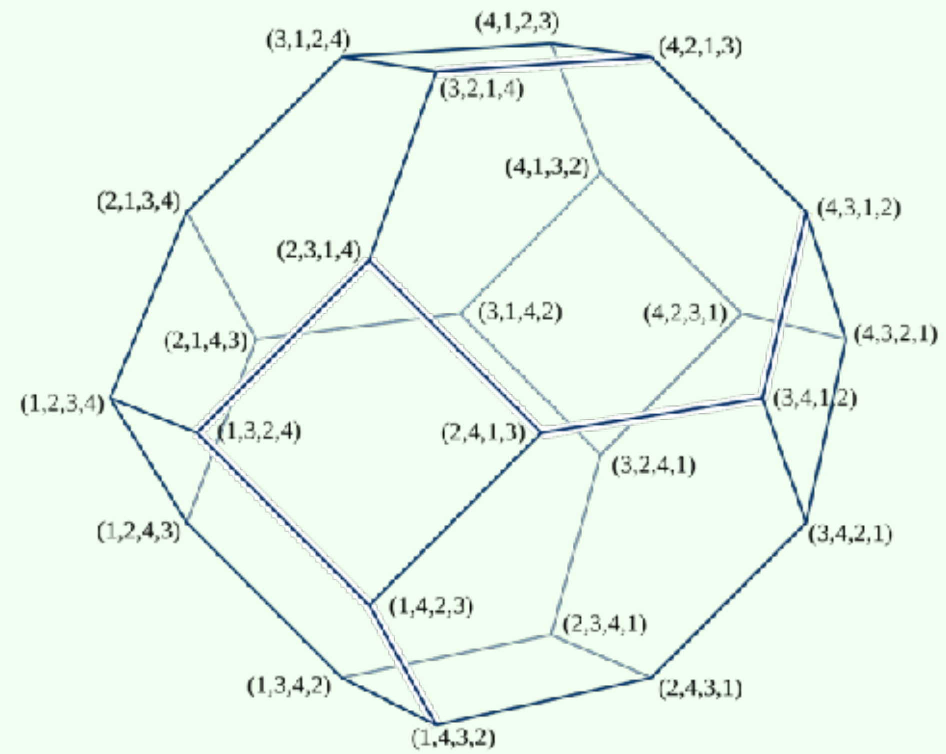
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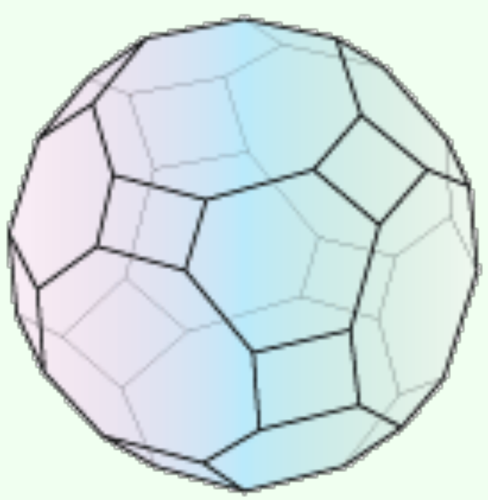

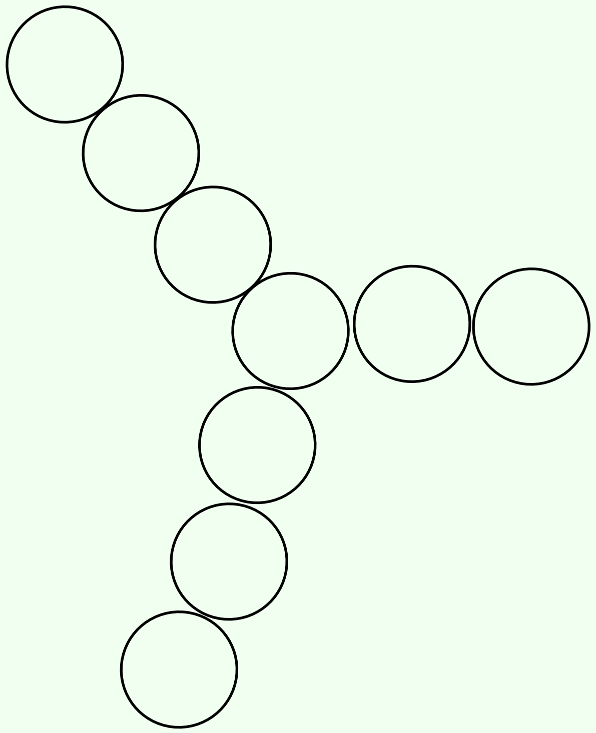

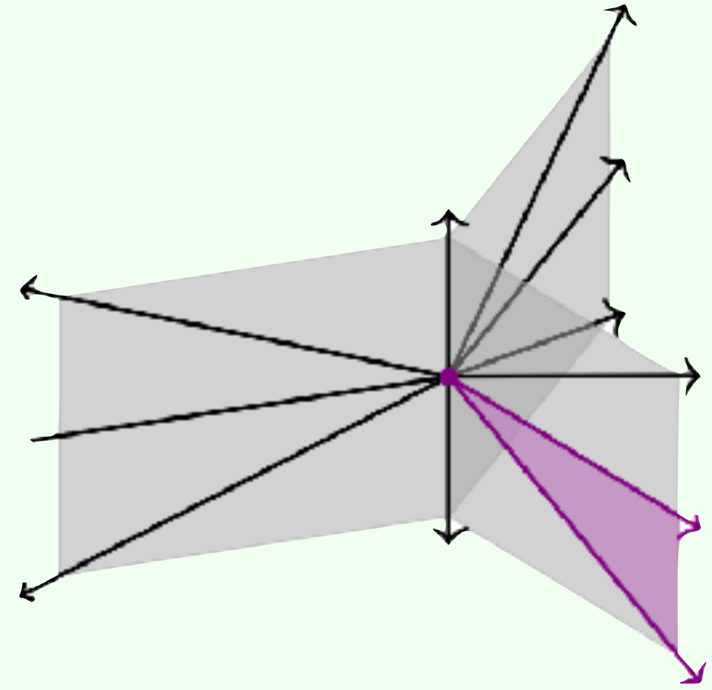

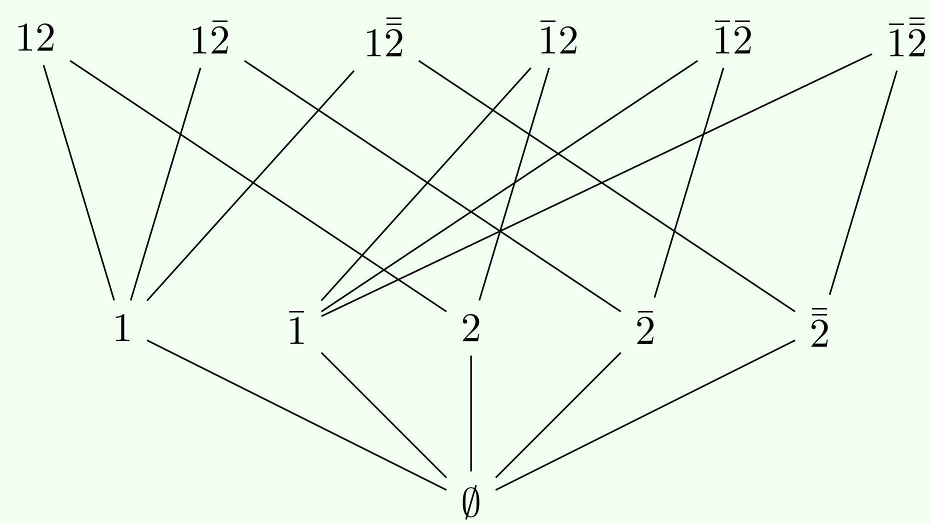
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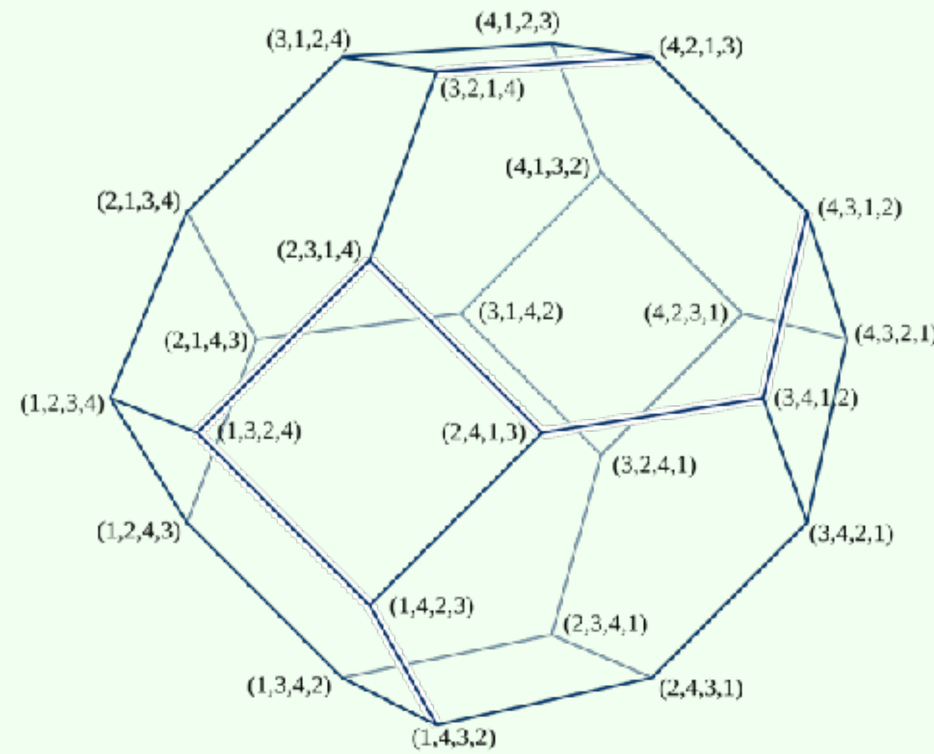
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THANK YOU