Graded Ehrhart Theory of Unimodular Zonotopes

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CAAC

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Ehrhart Theory

Let P be a lattice polytope.

Q How do the numbers

$i_P(m) = \#$ lattice points of mP

$\hat{i}_P(m) = \#$ interior lattice points of mP

behave as m grows?

The answers connect beautiful combinatorics, algebra & geometry.

For unimodular zonotopes, the answers are matroidal.
[Stanley '91], [Beck-Johemko-McCullough '19]

Graded Ehrhart Theory

For each number m, graded Ehrhart theory constructs

$$i_P(m;q),\ \tilde{i}_P(m;q) \in \mathbb{Z}[q]$$

such that, when $q \to 1$,

$$i_P(m;1) = i_P(m) \quad \text{and} \quad \tilde{i}_P(m;1) = \tilde{i}_P(m).$$

Q How do the polynomials $i_P(m;q)$ and $\tilde{i}_P(m;q)$ behave as m grows?

Graded Ehrhart Theory (of Unimodular Zonotopes)

> **Q** How do the polynomials $i_P(m;q)$ and $\tilde{i}_P(m;q)$ behave as m grows?

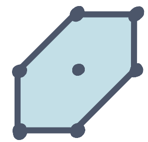
Our work answers this for unimodular zonotopes.

In our answer, we find lots of interesting combinatorics, algebra & geometry.

The answers are matroidal!

Unimodular Zonotopes

Def A unimodular zonotope Z is the projection of a unit cube via $A: \mathbb{R}^n \longrightarrow \mathbb{R}^d$, a totally unimodular matrix.

Ex $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $Z =$  $M = U_{2,3}$

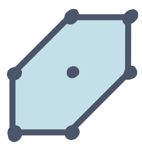
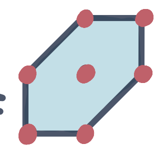
The matrix A determines a matroid M that knows a lot about Z. Let $T_M(x,y)$ be the Tutte poly. of M.

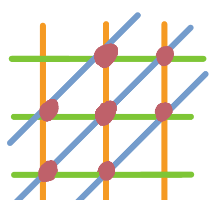
Thrm [Stanley '91]

$$i_Z(m) = m^d \, T_M\left(\frac{m+1}{m}, 1\right) \qquad \tilde{i}_Z(m) = m^d \, T_M\left(\frac{m-1}{m}, 1\right).$$

Orbit Harmonics

Def Given $X \subset \mathbb{R}^d$ finite, let $\text{Orb}(X) = \mathbb{R}[x_1, \ldots, x_d] / \text{gr} I_X$
where $\text{gr} I_X = \langle f_k \mid f := f_k + f_{k-1} + \ldots + f_0 \text{ vanishes on } X \rangle$
Orb(X) is a graded ring w/ total dimension equal to $\#X$.

Ex $Z =$  $X = Z \cap \mathbb{Z}^2 =$ 



$x(x-1)(x-2)$ vanishes so $x^3 \in \text{gr} I_X$
$y(y-1)(y-2)$ vanishes so $y^3 \in \text{gr} I_X$
$(x-y-1)(x-y)(x-y+1)$ vanishes so $y^2x - x^2y \in \text{gr} I_X$

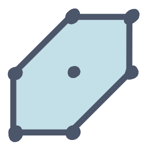
In fact, $\text{gr} I_X = \langle x^3, y^3, y^2x - x^2y \rangle$.

Orbit Harmonics of Unimodular Zonotopes

Def For $R = \bigoplus_{m \geq 0} R_m$, let $\mathrm{Hilb}(R;q) = \sum_{m \geq 0} \dim(R_m) q^m$.

Def Let, $i_P(m;q) = \mathrm{Hilb}(\mathrm{Orb}(mP \cap \mathbb{Z}^d);q)$

$\tilde{i}_P(m;q) = \mathrm{Hilb}(\mathrm{Orb}(\mathrm{int}(mP) \cap \mathbb{Z}^d);q)$.

Ex $Z =$ 

$i_Z(1;q) = 1 + 2q + 3q^2 + q^3$

$\tilde{i}_Z(1;q) = 1$

$i_Z(2;q) = 1 + 2q + 3q^2 + 4q^3 + 5q^4 + 3q^5 + q^6$

$\tilde{i}_Z(2;q) = 1 + 2q + 3q^2 + q^3$

q-Ehrhart Polynomials

Let $[m]_q = 1 + q + \ldots + q^{m-1}$.

Thrm [Crowley-P.]

$$i_Z(m;q) = q^{m(n-d)} [m]_q^d T_M\left(\frac{[m+1]_q}{[m]_q}, q^{-m}\right)$$

$$\tilde{i}_Z(m;q) = q^{m(n-d)} [m]_q^d T_M\left(\frac{[m-1]_q}{[m]_q}, q^{-m}\right)$$

Stanley:

$$i_Z(m) = m^d T_M\left(\frac{m+1}{m}, 1\right)$$

$$\tilde{i}_Z(m) = m^d T_M\left(\frac{m-1}{m}, 1\right)$$

q-Ehrhart Polynomials

Let $[m]_q = 1+q+\dots+q^{m-1}$.

Thrm [Crowley-P.]

$$i_Z(m;q) = q^{m(n-d)}\,[m]_q^d\,T_n\!\left(\frac{[m+1]_q}{[m]_q},\, q^{-m}\right)$$

$$\tilde{i}_Z(m;q) = q^{m(n-d)}\,[m]_q^d\,T_n\!\left(\frac{[m-1]_q}{[m]_q},\, q^{-m}\right)$$

$\Rightarrow \exists$ a "q-Ehrhart poly." $f_Z \in \mathbb{Q}(q)[t]$ s.t. $f_Z([m]_q) = i_Z(m;q)$.

f_Z is "quantum integer-valued" [Harman-Hopkins '16].

q-Ehrhart Series

Let $E_Z(t,q)=\sum_{m\geq 0} i_Z(m;q)t^m$ and $\widetilde{E}_Z(t,q)=\sum_{m\geq 0} \widetilde{i}_Z(m;q)t^m$.

Thrm [Crowley-P.]

$E_Z(t,q)$, $\widetilde{E}_Z(t,q)$ are rational. They have the form

$$E_Z(t,q)=\frac{N_Z(t,q)}{(1-t)(1-tq)\cdot\ldots\cdot(1-tq^n)} \qquad \widetilde{E}_Z(t,q)=\frac{\widetilde{N}_Z(t,q)}{(1-t)(1-tq)\cdot\ldots\cdot(1-tq^n)}$$

where $N_Z, \widetilde{N}_Z \in \mathbb{Z}[t,q]$. They are related by

$$q^d\,\widetilde{E}_Z(t,q)=(-1)^{d+1}E_Z(t^{-1},q^{-1}).$$

Algebra & Geometry

[Reiner–Rhoades '24] construct a bigraded ring $\mathcal{H}_P$ and ideal $\widetilde{\mathcal{H}_P} \subseteq \mathcal{H}_P$ such that

$$\mathrm{Hilb}(\mathcal{H}_P; t,q) = E_P(t,q) \qquad \mathrm{Hilb}(\widetilde{\mathcal{H}_P}; t,q) = \widetilde{E_P}(t,q)$$

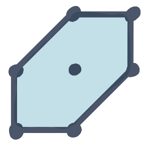
For unimodular zonotopes Z, these have geometric origin!

Thrm [Crowley – P.]

$\mathcal{H}_Z$ is the coordinate ring of the matroid Schubert variety Y_Z under the Segre embedding $Y_Z \hookrightarrow (\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^{2^n-1}$.

Algebra & Geometry

Thrm [Crowley-P.] $\mathcal{H}_Z$ has an explicit presentation!

Ex $Z =$ 

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\mathcal{H}_Z \cong \frac{\mathbb{R}[x_S : S \subseteq \{1,2,3\}]}{\langle x_1 + x_2 - x_3 \rangle + \langle x_S x_T - x_{S\cup T} x_{S \cap T} : S, T \subseteq \{1,2,3\} \rangle}$$

$\deg(x_S) = (1, |S|)$

↑ circuit-like relations ↑ Segre equations

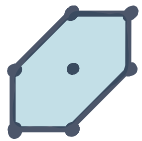
With work of [Ardila-Boocher '16], [Brion '03], [Crowley-Proudfoot '25],

Thrm [Crowley-P.]

$\mathcal{H}_Z$ is Cohen-Macaulay. After a degree shift by q-degree d, $\widehat{\mathcal{H}}_Z$ is the canonical module.

h^*-Vectors & Gorenstein-ness

The numerator of $E_Z(t,q)$ is quite mysterious...

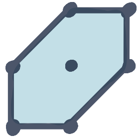
$Z =$ 

$$E_Z(t,q) = \frac{-q^4t^3 - (q^3+2q^2)t^2 + (2q^2+q)t + 1}{(1-t)(1-tq)(1-tq^2)(1-tq^3)}$$

Thrm [Crowley–P.]

$\mathcal{H}_Z$ is Gorenstein iff M is Boolean or a direct sum of circuits. In this case, the numerator of $E_Z(t,q)$ is "quantum palindromic".

Thank You!

$Z =$ 

$$f_Z(t) = 1\begin{bmatrix} t \\ 0 \end{bmatrix} + (q^3+2q)\begin{bmatrix} t \\ 1 \end{bmatrix} + (q^6+3q^5+4q^4-2q^2)\begin{bmatrix} t \\ 2 \end{bmatrix} + (q^9+2q^8+q^7-q^6-2q^5-q^4)\begin{bmatrix} t \\ 3 \end{bmatrix} \in \mathbb{Q}(q)[t].$$

$M = U_{2,3}$

$$E_Z(t,q) = \frac{-q^4t^3-(q^3-2q^2)t^2+(2q^2+q)t+1}{(1-t)(1-tq)(1-tq^2)(1-tq^3)}$$

$$E_Z(t,1) = \frac{-t^3-3t^2+3t+1}{(1-t)^4} = \frac{t^2+4t+1}{(1-t)^3}$$

$$\widetilde{E}_Z(t,q) = \frac{-q^4t^4-(q^3-2q^2)t^3+(2q^2+q)t^2+t}{(1-t)(1-tq)(1-tq^2)(1-tq^3)}$$