Positivity of Schubert coefficients

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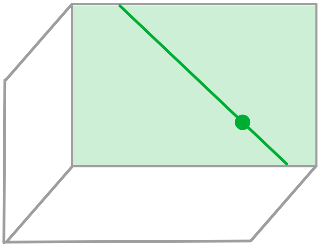
Flag variety

Let $\mathcal{F}l_n(\mathbb{C})$, the **complete flag variety**, be the set of complete flags $V_\bullet$:

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n, \quad \text{where } \dim V_i = i.$$



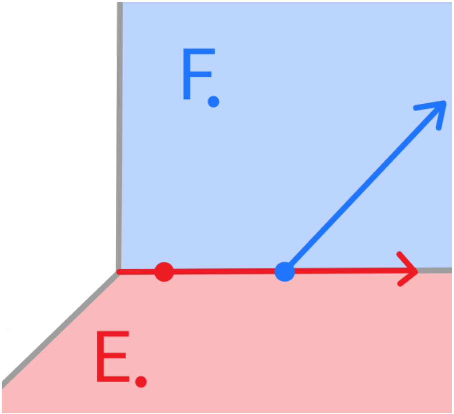
Schubert varieties

$$X_w(E_\bullet) = \{F_\bullet \in \mathcal{F}l_n(\mathbb{C}) \mid \dim(E_i \cap F_j) \geq \mathrm{rank}(w)_{ij}, \text{ for } i,j \in [n]\}$$

Example (Schubert variety, $w = 3142$)

We check $F_\bullet \in X_w(E_\bullet)$ using

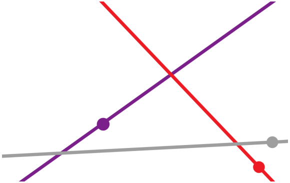
$$\mathrm{rank}(w) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$



Schubert Intersections

Question

In general, how many points are in the 0-dimensional intersection of a given triple of Schubert varieties?



$$\#(X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_w(G_\bullet)) = ?$$

Schubert Calculus

Schubert classes $[X_w]$ give a $\mathbb{Z}$-basis for $H^*(\mathcal{F}l_n(\mathbb{C}))$. Then

$$[X_u] \smile [X_v] = \sum_w c_{u,v,w}[X_{w_\circ w}], \text{ where}$$

$$c_{u,v,w} = \#(X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_w(G_\bullet)) \in \mathbb{Z}_{\geq 0}$$

for generic $E_\bullet, F_\bullet, G_\bullet \in \mathcal{F}l_n(\mathbb{C})$.

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This gives a rigorous framework to compute Schubert intersections but does not allow one to "predict" the number of solutions.

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Alternatively, we can use the *Schubert polynomials* $\mathfrak{S}_w$ of Lascoux and Schützenberger ('82) to compute $c_{u,v,w}$:

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_n} c_{u,v,w} \mathfrak{S}_w.$$

Schubert Problem

Central Problems

1. Determine a manifestly positive combinatorial rule for $c_{u,v,w}$.
2. Determine a rule to reliably predict when $c_{u,v,w} \overset{?}{>} 0$.

Complexity Background

Decision Problems

- P: decide 'yes' or 'no' in poly time
  - 2-coloring
  - primality [Agrawal–Kayal–Saxena, '02]
  - planarity [Demoucron–Malgrange–Pertuiset, '64]

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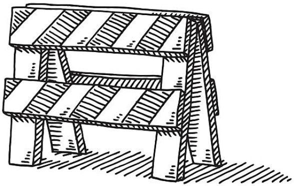
  - primality [Agrawal–Kayal–Saxena, ’02]
  - planarity [Demoucron–Malgrange–Pertuiset, ’64]
- NP: check evidence of ‘yes’ in poly time
  - 3-coloring
  - graph isomorphism
  - knottedness [Hass–Lagarias–Pippenger ’99]

Initial Obstructions

Obstructions to "easy" arguments:

- How to ensure sufficient genericity of an intersection?
- Points in Schubert intersections could be VERY small or VERY large



CR: FrankRamspott

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Question

How efficiently can we predict $[c_{u,v,w} >^? 0]$?

Upper bound: signed combinatorial rules imply $[c_{u,v,w} >^? 0]$ is in PSPACE.

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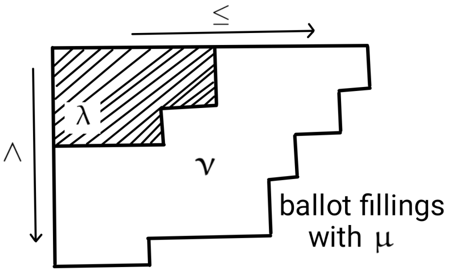
These LR coefficients count particular Young tableaux called LR tableaux:



Special Case: Grassmannian

$c^{\nu}_{\lambda,\mu} > 0$

($\Leftrightarrow$) $\exists$ LR tableaux for (λ, μ, ν)

($\Leftrightarrow$) $\exists$ integral point in a polytope $\mathcal{P}(\lambda, \mu, \nu)$

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Saturation Theorem (Knutson–Tao '99)

$c_{\lambda,\mu}^{\nu} \neq 0$ if and only if $c_{N\lambda,N\mu}^{N\nu} \neq 0$ for any $N > 1$

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Thm [DeLoera–McAllister '06; Mulmuley–Narayanan–Sohoni '12]

$[c_{\lambda,\mu}^{\nu} \stackrel{?}{=} 0] \in \mathrm{P}.$

What structure can we use?

There is no known positive combinatorial rule for $[c_{u,v,w} >^? 0]$. Additionally:

Theorem (Pak–R. '26)

For an infinite family of $u, v, w \in S_n$,

$$c_{u,v,w} = 1 \quad \textit{but} \quad c_{N\star u,N\star v,N\star w} = 0 \textit{ for all } N > 1.$$

What do we know?

Necessary OR sufficient conditions for vanishing

1. Knutson '01 (descent cycling)
2. Purbhoo '06 (root games)
3. Belkale–Kumar '06 (Levi movability)
4. Billey–Vakil '08 (permutation arrays)
5. St. Dizier–Yong '22
6. + all cases where the Schubert problem is resolved

Recall: Signed combinatorial rules imply $[c_{u,v,w} >^{?} 0]$ is in PSPACE.

Initial Results

Theorem (Pak–R. '25)

$[c_{u,v,w} >^? 0] \in \mathsf{AM} \cap \mathsf{coAM}$, *assuming* GRH.

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$[c_{u,v,w} >^? 0] \in \mathsf{AM} \cap \mathsf{coAM}$, *assuming* GRH.

Here GRH is the Generalized Riemann Hypothesis and AM is the Arthur–Merlin class of decision problems with interactive proofs with two rounds.

PC: arthurslegends

Zooming Out: Algebraic Framework

Suppose $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$ for $i \in [m]$ are 'small' (poly size in n). Take the system

$$S = \left\{ f_i = 0 \right\}_{i \in [m]}.$$

We formulate the decision problem, called HN:

Definition: HN (Hilbert Nullstellensatz)

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The weak Nullstellensatz + Hermann (1926) imply HN is in EXPSPACE (Mayr–Meyer '82).

Effective Nullstellensatz (Brownawell '87, Kollár '88, Jelonek '05, +) imply HN is in PSPACE.

Argument Outline

Theorem (Koiran '96)

HN *is in* AM, *assuming* GRH.

Theorem (Pak–R. '25)

$[c_{u,v,w} >^? 0] \in \mathrm{AM} \cap \mathrm{coAM}$, *assuming* GRH.

1. We parameterize equations for both Schubert vanishing and positivity using Purbhoo '06.
2. We use a parameterized version of HN studied by Ait El Manssour–Balaji–Nosan–Shirmohammadi–Worrel '24 to simulate genericity.

Main Results

We obtain the following (with NO additional assumptions):

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Corollary (Pak–R. '25)

$[c_{u,v,w} >^? 0] \in \mathrm{NP}$

Main Ingredients - Part 1

Lemma (Purbhoo '06)

Let $u, v, w \in S_n$. *Then for* $A, B, C \in \mathrm{N} \subset \mathrm{GL}_n(\mathbb{C})$ *generic:*

$$c_{u,v,w} > 0 \iff A \cdot R_u + B \cdot R_v + C \cdot R_{w_\circ w} = A \cdot R_u \oplus B \cdot R_v \oplus C \cdot R_{w_\circ w}.$$

Here $R_\rho := \mathfrak{n} \cap (\rho \mathrm{B}_- \rho^{-1})$, *where* $\mathfrak{n}$ *is the Lie algebra of* N.

This lemma arises by translating conditions for transverse intersections into Lie algebraic statements via the Killing form.

Main Ingredients - Part 2

Lemma (Schwarz–Zippel)

Let $Q \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be non-zero with degree $d \geq 0$ over $\mathbb{C}$. Take $S \subset \mathbb{C}$ be a finite set. Then:

$$\mathbf{P}\big[Q(c_1, c_2, \ldots, c_n) = 0\big] \;\leq\; \frac{d}{|S|},$$

over random, independent and uniform choices $c_1, c_2, \ldots, c_n \in S$.

Lovász ('79) uses the Schwarz–Zippel to give an RP algorithm for deciding perfect matchings in a bipartite graph.

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3. Determine when $\det(M_{u,v,w}) \equiv 0$. This is an instance of PIT (polynomial identity testing).
4. Test a random evaluation $\overrightarrow{\det(M_{u,v,w})} =^? 0$.
5. Use Schwartz–Zippel to determine the likelihood of a faulty test.

Main Results

Theorem (Pak–R. '25)

Schubert positivity can be decided in probabilistic poly-time for all classical G and arbitrary k-fold intersections, with one-sided error.

Corollary (Pak–R. '25)

$[c_{u,v,w} >^? 0] \in \mathrm{NP}$

Our NP certificates are random bits where $\overrightarrow{\det(M_{u,v,w})} \neq 0$.

Corollary (Pak–R. '25)

$[c_{u,v,w} >^? 0] \in \mathrm{P}$, *assuming derandomization (i.e.,* $\mathrm{BPP} = \mathrm{P}$*).*

Overview

To summarize:

1. Schubert vanishing asks if a given intersection of Schubert varieties is empty.
2. We prove the general case of the vanishing problem lies in $\mathsf{AM} \cap \mathsf{coAM}$, assuming GRH.
3. Later we prove the general case of the vanishing problem can be decided in probabilistic poly-time for all classical G and arbitrary k-fold intersections, with one-sided error.

Thank you!

Koiran's improvement via GRH

Theorem (Koiran '96)

HN *is in* AM, *assuming* GRH.

Koiran uses that quantities A and B differ enough in size, where:

- If S is unsatisfiable over $\mathbb{C}$, there are **at most** A primes p such that S is satisfiable over $\mathbb{F}_p$.
- If S is satisfiable over $\mathbb{C}$, there are **at least** B primes p such that S is satisfiable over $\mathbb{F}_p$.
  - Ensuring the existence of small p is given by the effective Chebotarev Density Theorem (Lagarias–Odlyzko, '77)