On the second symbolic power of extremal ideals

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Fact: $I^{(1)} = I$.

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Partially partial answer: We found the formula for $\mathcal{E}_q^{(2)}$.

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A partial ordering on $\mathbb{Z}^n_{\geq 0}$: $\mathbf{a} \leq \mathbf{b}$ if $\mathbf{x}^{\mathbf{a}} \mid \mathbf{x}^{\mathbf{b}}$.

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Then $\mathcal{E}_q = (\epsilon_1, \ldots, \epsilon_q)$ is the *q-extremal ideal*.

Example: $\mathcal{E}_3 = (\epsilon_1, \epsilon_2, \epsilon_3)$ where

$$\epsilon_1 = y_1 y_{12} y_{13} y_{123}, \quad \epsilon_2 = y_2 y_{12} y_{23} y_{123}, \quad \epsilon_3 = y_3 y_{13} y_{23} y_{123}.$$

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are associated to vectors in $\mathbb{Z}^{2^q-1}$, whose standard basis is $\{\mathbf{e}_A \mid \emptyset \neq A \subseteq [q]\}$.

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Lemma

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2. *If* $\mathbf{y}^{\mathbf{b}}$ *is a minimal generator of* $\mathcal{E}_q^{(2)}$, *then* $b_A \leq 2$ *for all* A.

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$$\begin{array}{ccc} & b_1 + b_{23} \geq 2, & b_{12} + b_{13} \geq 2 \\ b_1 + b_2 + b_3 \geq 2, & b_2 + b_{13} \geq 2, & b_{12} + b_{23} \geq 2 \\ b_{123} \geq 2, & b_3 + b_{12} \geq 2, & b_{13} + b_{23} \geq 2 \end{array}$$

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Minimal solutions:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_{12} \\ b_{13} \\ b_{23} \\ b_{123} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

and permutations.

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and permutations. There are **seven** minimal solutions here.

A guessing game

Fill in the ???:

$$\text{There are} \quad \begin{matrix} 1 \\ 3 \\ 7 \\ 15 \\ ??? \end{matrix} \quad \text{minimal solutions if} \quad \begin{matrix} q = 1 \\ q = 2 \\ q = 3 \\ q = 4 \\ q = 5 \end{matrix}$$

A guessing game

Fill in the ???:

	1		$q = 1$
	3		$q = 2$
There are	7	minimal solutions if	$q = 3$
	15		$q = 4$
	31		$q = 5$

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Educated guess: There are $2^q - 1$ minimal solutions to the system $\sum_{A \in \mathfrak{C}} b_A \geq 2$ where $\mathfrak{C}$ is any minimal cover of $[q]$!!!!!

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Theorem (CDFHMŞ '26)

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$$\mathbf{1} + \sum_{X\subseteq A\subseteq [q]} \mathbf{e}_A - \sum_{\emptyset\neq A\subseteq \overline{X}} \mathbf{e}_A$$

for some subset $\emptyset \neq X \subseteq [q]$.

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$$\mathbf{1} + \sum_{\{1\}\subseteq A\subseteq[q]} \mathbf{e}_A - \sum_{\emptyset\neq A\subseteq\{2,3\}} \mathbf{e}_A = \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\0\\0\\1\\1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\\0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\0\\2\\2\\0\\2\end{pmatrix}.$$

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Let I be a squarefree monomial ideal with $q > 0$ generators. Then

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for any integer k.

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The Betti numbers $\beta_k(I)$ measure how far the module I is from being free. A module M is **free** if $\beta_k(M) = 0$ for any $k > 0$.

Higher symbolic powers

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(c) $\sum_{A \in \mathfrak{C}} b_A \geq r$ *for every **minimal** cover $\mathfrak{C}$ of $[q]$.*

Problem: For a fixed $r \geq 2$. Find all minimal solutions to the system $\sum_{A \in \mathfrak{C}} b_A \geq r$ where $\mathfrak{C}$ is any minimal cover of $[q]$

Another guessing game

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Consider the system $\sum_{A \in \mathfrak{C}} b_A \geq 3$ where $\mathfrak{C}$ is any minimal cover of $[q]$.

$$\text{There are} \quad \begin{matrix} 1 \\ 4 \\ 13 \\ 44 \\ \cdots \\ \cdots \end{matrix} \quad \text{minimal solutions if} \quad \begin{matrix} q = 1 \\ q = 2 \\ q = 3 \\ q = 4 \\ q = 5 \\ q = 6. \end{matrix}$$

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This sequence is **not** on OEIS.

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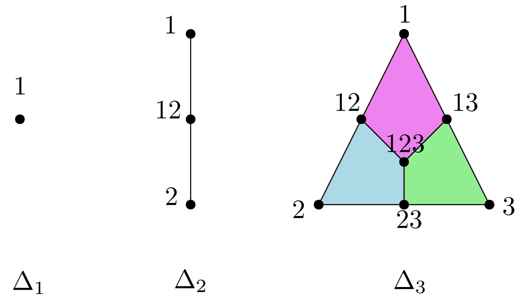

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THE END

THANK YOU!!!