q,t-Fub-Catalan numbers

for finite reflection groups

Christian Stump, UQAM

Jan. 24, 2010
Overview

Finite reflection groups

$q, t$-Fuß-Catalan numbers for real reflection groups

Algebraic Combinatorics – the extended Shi arrangement

Combinatorial Algebra – rational Cherednik algebras

$q, t$-Fuß-Catalan numbers for complex reflection groups
Finite real reflection groups
Finite real reflection groups

Let $V$ be a finite-dimensional real vector space.

- A (finite) real reflection group

$$W = \langle t_1, \ldots, t_{\ell} \rangle \subseteq O(V)$$

is a finite group generated by reflections.
The following list of root systems determine (up to isomorphisms) the irreducible finite real reflection groups:

- $A_{n-1}$ (symmetric group),
- $B_n$ (group of signed permutations),
- $D_n$ (group of even-signed permutations),
- $I_2(k)$ (dihedral group of order $2k$) and
- $H_3, H_4, F_4, E_6, E_7, E_8$ (exceptional groups).
The most classical example of a reflection group is the symmetric group $S_n$ of all permutations of $n$ letters.

$$123 = (), \quad 132 = (23), \quad 213 = (12), \quad 231 = (123), \quad 312 = (132), \quad 321 = (13).$$

This group can be seen as the reflection group of type $A_{n-1}$:

- $transposition \overset{\sim}{\longleftrightarrow} reflection$
  $$(i, j) \quad = \quad e_i \leftrightarrow e_j$$
- $simple \ transposition \overset{\sim}{\longleftrightarrow} simple \ reflection$
  $$(i, i + 1) \quad = \quad e_i \leftrightarrow e_{i+1}.$$
Finite complex

Reflections

Reflection groups

Square groups
Finite complex reflection groups

Let $V$ be a finite-dimensional complex vector space.

- A complex reflection $s \in U(V)$
  
  i. has finite order and
  
  ii. its fixed-point space has codimension 1.

- A (finite) complex reflection group

  $$W = \langle t_1, \ldots, t_\ell \rangle \subseteq O(V)$$

  is a finite group generated by complex reflections.

Irreducible complex reflection groups are determined by the following types:

- $G(m, p, n)$ with $p|m$ of order $m^n n!/p$,

- $G_4 - G_{37}$ 34 exceptional types.

  (Shephard–Todd, Chevalley, 1950’s)
The cyclic group

It acts on \( \mathbb{C} \) by multiplication of a **primitive root of unity** \( \zeta \).
In real reflection groups, any reflection $t$ has order two and there is a 1:1-correspondence

\[
\{ \text{reflections} \} \leftrightarrow \{ \text{reflecting hyperplanes} \}
\]

\[ t \leftrightarrow H_\alpha. \]

In complex reflection groups, any reflection $t$ has order $k \geq 2$ and there is a correspondence

\[
\{ \text{reflections} \} \sim \leftrightarrow \{ \text{reflecting hyperplanes} \}
\]

\[ t, t^2, \ldots, t^{k-1} \leftrightarrow H_\alpha. \]
For a permutation $\sigma \in S_n$, define a diagonal action on $\mathbb{C}[x, y] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ by $\sigma(x_i) := x_{\sigma(i)}, \sigma(y_i) := y_{\sigma(i)}$. E.g.,

$$231(2x_1x_2y_2^2y_3) = 2x_2x_3y_3^2y_1.$$ 

A polynomial $f \in \mathbb{C}[x, y]$ is called

- **invariant** if $\sigma(f) = f$,
- **alternating** if $\sigma(f) = \text{sgn}(\sigma) f$.

Example

$x_1y_2 + x_2y_1$ is invariant, $x_1y_2 - x_2y_1$ is alternating.
Let $W$ be a real reflection group acting on $V$. The **contragredient action** of $W$ on $V^* = \text{Hom}(V, \mathbb{C})$ is given by

$$\omega(\rho) := \rho \circ \omega^{-1}.$$ 

This gives an action of $W$ on the symmetric algebra $S(V^*) = \mathbb{C}[x]$ and 'doubling up' this action gives a **diagonal action** on $\mathbb{C}[x, y] := \mathbb{C}[V \oplus V]$.

A polynomial $f \in \mathbb{C}[x, y]$ is called

- **invariant** if $\omega(f) = f$,
- **alternating** if $\omega(f) = \text{det}(\omega) f$. 

q(t - Fulp - Catalan numbers)
Let $W$ be a reflection group now acting on $\mathbb{C}[x, y]$ and let

$$A := \langle \text{alternating polynomials} \rangle \subseteq \mathbb{C}[x, y].$$

Define the $W$-module $M^{(m)}(W)$ to be minimal generating space of the ideal $A^m$,

$$M^{(m)}(W) := A^m/\langle x, y \rangle A^m \cong \mathbb{C}B,$$

where $B$ is any homogeneous minimal generating set for $A^m$.

$M^{(m)}(W)$ sits inside a larger $W$-module $DR^{(m)}(W)$ as its isotropic component,

$$M^{(m)}(W) \cong e_{\det}(DR^{(m)}(W)).$$
Definition

For any real reflection group $W$, define $q, t$-Fuß-Catalan numbers to be the bigraded Hilbert series of $M^{(m)}(W)$,

$$\text{Cat}^{(m)}(W; q, t) := \mathcal{H}(M^{(m)}(W); q, t) = \sum_{f \in \mathcal{B}} q^{\deg_x(f)} t^{\deg_y(f)}.$$

- $\text{Cat}^{(m)}(W; q, t)$ is a symmetric polynomial in $q$ and $t$,
- it reduces in type $A_{n-1}$ to the classical $q, t$-Fuß-Catalan numbers,

$$\text{Cat}^{(m)}(S_n; q, t) = \text{Cat}_n^{(m)}(q, t)$$

introduced by Haiman in the 1990’s.
Example: \( \text{Cat}^{(1)}(S_3; q, t) \)

For \( W = S_3 \), one can show that

\[
\mathcal{M}^{(1)}(S_3) = \mathbb{C} \left\{ \Delta\{(1,0),(2,0)\}, \Delta\{(1,0),(1,1)\}, \Delta\{(0,1),(1,1)\}, \right. \\
\left. \Delta\{(0,1),(0,2)\}, \Delta\{(1,0),(0,1)\} \right\},
\]

where

\[
\Delta\{(i_1,j_1),(i_2,j_2)\}(x, y) := \det \begin{pmatrix}
1 & 1 & 1 \\
1^{i_1} y_1^{j_1} & 1^{i_1} y_1^{j_1} & 1^{i_1} y_1^{j_1} \\
1^{i_2} y_2^{j_2} & 1^{i_2} y_2^{j_2} & 1^{i_2} y_2^{j_2} \\
1^{i_1} y_1^{j_1} & 1^{i_1} y_1^{j_1} & 1^{i_1} y_1^{j_1}
\end{pmatrix}
\]

is the \textbf{generalized Vandermonde determinant}. This gives

\[
\text{Cat}^{(1)}(S_3; q, t) = \mathcal{H}(\mathcal{M}^{(1)}(S_3); q, t) \\
= q^3 + q^2 t + qt^2 + t^3 + qt.
\]
A conjectured formula for the dimension of $M^{(m)}(W)$

Computations of the **dimensions** of $M^{(m)}(W)$ were the first motivation for further investigations:

**Conjecture**

Let $W$ be a real reflection group. Then

$$
\text{Cat}^{(m)}(W; 1, 1) = \prod_{i=1}^{\ell} \frac{d_i + mh}{d_i},
$$

where

- $\ell$ is the **rank** of $W$,
- $h$ is the **Coxeter number** and
- $d_1, \ldots, d_\ell$ are its **degrees**.
Fuß-Catalan numbers

- These numbers, called **Fuß-Catalan numbers**, count several combinatorial objects, e.g.,
  - positive regions in the generalized Shi arrangement (Athanasiadis, Postnikov),
  - \textit{m-divisible non-crossing partitions} (Armstrong, Bessis, Reiner),
  - facets in the generalized Cluster complex (Fomin, Reading, Zelevinsky).

- They reduce for \( m = 1 \) to the well-known Catalan numbers associated to real reflection groups:

\[
\begin{array}{c|c|c}
  \text{ } & A_{n-1} & B_n \\
  \hline
  \frac{1}{n+1} \binom{2n}{n} & \binom{2n}{n} & \binom{2n}{n} - \binom{2(n-1)}{n-1} \\
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
  \hline
  I_2(k) & H_3 & H_4 & F_4 & E_6 & E_7 & E_8 \\
  \hline
  k + 2 & 32 & 280 & 105 & 833 & 4160 & 25080 \\
  \hline
\end{array}
\]
The classical $q$, $t$-Fuss-Catalan numbers

In type $A$, $\text{Cat}^{(m)}(S_n; q, t)$ occurred within the past 15 years in various fields of mathematics:

- Hilbert series of space of diagonal coinvariants (Haiman),
- complicated rational function in the context of modified Macdonald polynomials (Garsia, Haiman),
- Hilbert series of some cohomology module in the theory of Hilbert schemes of points in the plane (Haiman),
- they have a conjectured combinatorial interpretation in terms of two statistics on partitions fitting inside the partition 
  \[ \mu := ((n-1)m, \ldots, 2m, m), \]

\[ \text{Cat}^{(m)}_n(q, t) = \sum_{\lambda \subseteq \mu} q^{\text{area}(\lambda)} t^{\text{bounce}(\lambda)}. \]

- Proved for $m = 1$ (Garsia, Haglund) and for $t = 1$ (Haiman).
Combinatorics

The extended Shafi arrangement
The extended Shi arrangement

Let $\mathcal{W}$ be a crystallographic reflection group.

$\text{Shi}^{(m)}(\mathcal{W})$ is defined to be the collection of (translates of the reflecting) hyperplanes in $V$ given by

$$\{H_k^\alpha : \alpha \in \Phi^+, -m < k \leq m\},$$

where $H_k^\alpha = \{x \in V : (x, \alpha) = k\}$.

- A region of $\text{Shi}^{(m)}(\mathcal{W})$ is a connected component of its complement.

Remark

**Coxeter arrangement $\subseteq$ extended Shi arrangement.**
The extended Shi arrangement

Let $W$ be a crystallographic reflection group.

**Theorem (Yoshinaga)**

The number of regions in $\text{Shi}^{(m)}(W)$ is given by

$$(mh + 1)^\ell.$$ 

**Theorem (Athanasiadis)**

The number of positive regions in $\text{Shi}^{(m)}(W)$ – regions which lie in the fundamental chamber of the associated Coxeter arrangement – is given by

$$\prod_{i=1}^{\ell} \frac{d_i + mh}{d_i}.$$
Example: $\text{Shi}^{(1)}(A_2)$

$|\{\text{regions}\}| = 16 = (1 \cdot 3 + 1)^2,

|\{\text{positive regions}\}| = 5 = \frac{5 \cdot 6}{2 \cdot 3}.$
Example: $\text{Shi}^{(1)}(A_2)$ and $\text{Cat}^{(1)}(A_2; q)$

$$\text{Cat}^{(1)}(A_2; q) = \sum q^{\text{coh}(R)} = 1 + 2q + q^2 + q^3.$$
**Conjecture**

Let $W$ be a crystallographic reflection group. Then

$$\text{Cat}^{(m)}(W; q, 1) = \sum q^{\text{coh}(R)},$$

where the sum ranges over all regions of $\text{Shi}^{(m)}(\Phi)$ which lie in the fundamental chamber of the associated Coxeter arrangement and where $\text{coh}$ denotes the coheight statistic.

- The conjecture is known to be true for type $A$,
- was validated by computations for several types.
**Specialization** \( t = q^{-1} \).

**Conjecture**

Let \( W \) be a reflection group. Then

\[
\text{Cat}^{(m)}(W; q, q^{-1}) = q^{-mN} \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q},
\]

where

- \( N \) is the number of reflections in \( W \),
- \( [n]_q := q^{n-1} + \ldots + q + 1 \) is the usual \( q \)-analogue of \( n \).
The dihedral groups

**Theorem**

Let $W$ be the dihedral group of type $l_2(k)$. Then

$$\text{Cat}^{(m)}(W; q, t) = \sum_{j=0}^{m} q^{m-j} t^{m-j} [jk + 1]_{q,t},$$

where

$$[n]_{q,t} := q^{n-1} + q^{n-2} t + \ldots + qt^{n-2} + t^{n-1}.$$

**Theorem**

All shown conjectures hold for the dihedral groups.
Rational Cherednik algebras
Let $W$ be a real reflection group. The **rational Cherednik algebra**

$$H_c = H_c(W)$$

is an **associative algebra** generated by

$$V, V^*, W,$$

subject to **defining relations** depending on a **rational parameter** $c$, such that

- the polynomial rings $\mathbb{C}[V], \mathbb{C}[V^*]$ and
- the group algebra $\mathbb{C}W$

are subalgebras of $H_c$. 
A simple $H_c$-module

For $c = m + \frac{1}{\hbar}$ there exists a unique simple $H_c$-module $L$ which carries a natural filtration.

**Theorem (Berest, Etingof, Ginzburg)**

*Let $W$ be a real reflection group. Then*

$$
\mathcal{H}(\text{gr}(L); q) = q^{-mn}[mh + 1]^\ell_q,
$$

$$
\mathcal{H}(e(\text{gr}(L)); q) = q^{-mn} \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q},
$$

*where $\text{gr}(L)$ is the associated graded module of $L$ and where $e(\text{gr}(L))$ is its trivial component.*
The connection between $L$ and the space of generalized diagonal coinvariants

**Theorem**

Let $W$ be a real reflection group and let

$$DR^{(m)}(W) = A^{m-1}/I A^{m-1}$$

be the **generalized diagonal coinvariants** graded by degree in $x$ minus degree in $y$. Then there exists a natural surjection of graded modules,

$$DR^{(m)}(W) \otimes \det \twoheadrightarrow \text{gr}(L),$$

where $\det$ denotes the **determinant representation**.

**Remark**

This theorem generalizes a theorem by Gordon, who proved the $m = 1$ case, following mainly his approach.
The connection between $L$ and the space of generalized diagonal coinvariants

Conjecture (Haiman)
The $W$-stable kernel of the surjection in the previous theorem does not contain a copy of the trivial representation.

Corollary
*If the previous conjecture holds, then*

\[ M^{(m)}(W) \cong e(\text{gr}(L)) \]

*as graded modules. In particular,*

\[ \text{Cat}(W; q, q^{-1}) = q^{-mN} \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q}. \]
Finite complex groups
The cyclic group

The cyclic group $C_k = \langle \zeta \rangle$ would act on $\mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}]$ by

$$
\zeta(x^a y^b) = \zeta^a \cdot x^a \cdot \zeta^b \cdot y^b = \zeta^{a+b} \cdot x^a y^b.
$$

This would give

$$
\mathbb{C}[x, y]^{C_k} = \text{span}\left\{x^a y^b : a + b \equiv 0 \mod k\right\},
$$

$$
\mathbb{C}[x, y]^{\text{det}} = \text{span}\left\{x^a y^b : a + b \equiv 1 \mod k\right\}
$$

$$
= x\mathbb{C}[x, y]^{C_k} + y\mathbb{C}[x, y]^{C_k},
$$

$$
\mathbb{C}[x, y]^{\text{det}^{-1}} = \text{span}\left\{x^a y^b : a + b \equiv k - 1 \mod k\right\}
$$

$$
= \sum_{i+j = k-1} x^i y^j \mathbb{C}[x, y]^{C_k}.
$$
The cyclic group

\[
\mathbb{C}[x, y]^{\text{det}} = x\mathbb{C}[x, y]^{C_k} + y\mathbb{C}[x, y]^{C_k},
\]
\[
\mathbb{C}[x, y]^{\text{det}^{-1}} = \sum_{i+j=k-1} x^i y^j \mathbb{C}[x, y]^{C_k}.
\]

We would have two possible choices to define \(q, t\)-Catalan numbers for the cyclic group \(C_k\):

\[
\text{Cat}^{(1)}(C_k; q, t) = q + t \quad \text{or} \quad \text{Cat}^{(1)}(C_k; q, t) = q^{k-1} + q^{k-2}t + \ldots + qt^{k-2} + t^{k-1}.
\]

- Both choices would be in contradiction to the previously shown conjectures!
Let $W$ be a real reflection group acting on $V$. The **contragredient action** of $W$ on $V^* = \text{Hom}(V, \mathbb{C})$ is given by

$$\omega(\rho) := \rho \circ \omega^{-1}.\$$

This gives an action of $W$ on the symmetric algebra $S(V^*) = \mathbb{C}[x]$ and ‘doubling up’ this action gives a **diagonal action** on

$$\mathbb{C}[x, y] := \mathbb{C}[V \oplus V].$$

A polynomial $f \in \mathbb{C}[x, y]$ is called

- **invariant** if $\omega(f) = f$ for all $\omega \in W$,
- **alternating** if $\omega(f) = \det(\omega) f$ for all $\omega \in W$. 

**Alternating polynomials** — Generalization
Let $W$ be a complex reflection group acting on $V$. The **contragredient action** of $W$ on $V^* = \text{Hom}(V, \mathbb{C})$ is given by

$$\omega(\rho) := \rho \circ \omega^{-1}.$$  

This gives an action of $W$ on the symmetric algebra $S(V^*) = \mathbb{C}[x]$ and 'doubling up' this action gives a **diagonal action** on

$$\mathbb{C}[x, y] := \mathbb{C}[V \oplus V^*].$$

A polynomial $f \in \mathbb{C}[x, y]$ is called

- **invariant** if $\omega(f) = f$ for all $\omega \in W$,
- **alternating** if $\omega(f) = \det(\omega) f$ for all $\omega \in W$. 

The cyclic group $C_k = \langle \xi \rangle$ would act on $\mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}]$ by

\[
\xi(x^a y^b) = \xi^a \cdot x^a \xi^b \cdot y^b = \xi^{a+b} \cdot x^a y^b.
\]
The cyclic group $C_k = \langle \zeta \rangle$ acts on $C[x, y] := C[C \oplus C^*]$ by

$$\zeta(x^a y^b) = \zeta^a \cdot x^a \cdot \zeta^{-b} \cdot y^b = \zeta^{a-b} \cdot x^a y^b.$$
The cyclic group \( C_k = \langle \zeta \rangle \) acts on \( \mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}^*] \) by

\[
\zeta(x^a y^b) = \zeta^a \cdot x^a \cdot \zeta^{-b} \cdot y^b = \zeta^{a-b} \cdot x^a y^b.
\]

This gives

\[
\mathbb{C}[x, y]^{C_k} = \text{span} \{ x^a y^b : a \equiv b \mod k \},
\]

\[
\mathbb{C}[x, y]^{\text{det}} = \text{span} \{ x^a y^b : a + 1 \equiv b \mod k \}
\]

\[
= x\mathbb{C}[x, y]^{C_k} + y^{k-1}\mathbb{C}[x, y]^{C_k},
\]

\[
\mathbb{C}[x, y]^{\text{det}^{-1}} = \text{span} \{ x^a y^b : a \equiv b + 1 \mod k \}
\]

\[
= x^{k-1}\mathbb{C}[x, y]^{C_k} + y\mathbb{C}[x, y]^{C_k}.
\]
The cyclic group

The cyclic group \( C_k = \langle \zeta \rangle \) acts on \( \mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}^*] \) by

\[
\zeta(x^a y^b) = \zeta^a \cdot x^a \cdot \zeta^{-b} \cdot y^b = \zeta^{a-b} \cdot x^a y^b.
\]

\[
\mathbb{C}[x, y]^{\text{det}} = x\mathbb{C}[x, y]^{C_k} + y^{k-1}\mathbb{C}[x, y]^{C_k},
\]

\[
\mathbb{C}[x, y]^{\text{det}^{-1}} = x^{k-1}\mathbb{C}[x, y]^{C_k} + y\mathbb{C}[x, y]^{C_k}.
\]

Now, we have (beside interchanging the roles of \( q \) and \( t \)) only the following choice:

\[
\text{Cat}^{(1)}(C_k; q, t) := q + t^{k-1}.
\]
The cyclic group

The cyclic group $C_k = \langle \zeta \rangle$ acts on $\mathbb{C}[x, y] := \mathbb{C}[\mathbb{C} \oplus \mathbb{C}^*]$ by

$$\zeta(x^a y^b) = \zeta^a \cdot x^a \zeta^{-b} \cdot y^b = \zeta^{a-b} \cdot x^a y^b.$$ 

Now, we have (beside interchanging the roles of $x$ and $y$) only the following choice:

$$\text{Cat}^{(1)}(C_k; q, t) := q + t^{k-1}.$$ 

The $q$-power equals the number of reflecting hyperplanes $N^* = 1$ and the $t$-power equal the number of reflections $N = k - 1$. We have

$$q^N \text{Cat}^{(1)}(C_k; q, q^{-1}) = q^{N^*} \text{Cat}^{(1)}(C_k; q^{-1}, q) = 1 + q^k = [2k]_q/[k]_q.$$
**Conjecture**

Let $W$ be a well-generated complex reflection group. Then

$$q^{mN} \text{Cat}^{(m)}(W; q, q^{-1}) = q^{mN^*} \text{Cat}^{(m)}(W; q^{-1}, q)$$

$$= \ell \prod_{i=1}^{\ell} \frac{[d_i + mh]_q}{[d_i]_q},$$

where

- $N$ is the number of reflections in $W$ and where
- $N^*$ is the number of reflecting hyperplanes.