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The Facet Ideal of a Simplicial Complex

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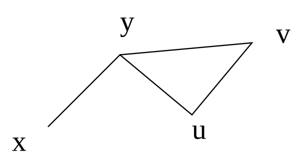
Abstract. To a simplicial complex, we associate a square-free monomial ideal in the polynomial ring generated by its vertex set over a field. We study algebraic properties of this ideal via combinatorial properties of the simplicial complex. By generalizing the notion of a tree from graphs to simplicial complexes, we show that ideals associated to trees satisfy sliding depth condition, and therefore have normal and Cohen-Macaulay Rees rings. We also discuss connections with the theory of Stanley-Reisner rings.

1. Introduction

Given a graph on n vertices, Villarreal ([Vi1]) defined the *edge ideal* associated to that graph in a polynomial ring in n variables (each variable representing a vertex of the graph) to be the ideal generated by monomials xy, where the corresponding vertices to x and y are connected by an edge. For example, the ideal $I = (xy, yu, yv, uv)$ corresponds to the following graph:



Later in [SVV], Simis, Vasconcelos and Villarreal used this construction along with properties of graphs to show that edge ideals of trees satisfy sliding depth condition. Among other things, this implies that the Rees ring of the edge ideal of a tree is normal and Cohen-Macaulay.

Our goal here is generalize this construction to simplicial complexes. We define the notion of tree for simplicial complexes, and show that ideals corresponding to trees satisfy sliding depth (Theorem 1) and therefore have normal and Cohen-Macaulay Rees rings (Corollary 4). We also show that if the ideal of the tree is Cohen-Macaulay to begin with, it is strongly Cohen-Macaulay (Corollary 3), meaning that all Koszul homology modules

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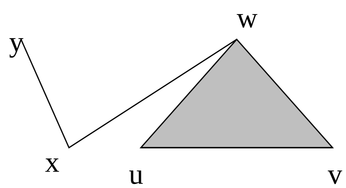
of generators of that ideal are Cohen-Macaulay. Consequently we recover a rather large class of normal square-free monomial ideals with sliding depth condition.

Traditionally, given a simplicial complex Δ one would associate to it the so-called *Stanley Reisner* ideal, that is, the ideal generated by monomials corresponding to *non-faces* of this complex (here again we are assigning to each vertex of the complex one variable of a polynomial ring generated by the vertices of the complex). For example, for the simplicial complex Δ below:



the Stanley-Reisner ideal is

$$\mathcal{N}(\Delta) = (xu, xv, yu, yv, yw).$$

Our approach in this paper is to assign to the same simplicial complex Δ the ideal generated by its *facets*:

$$\mathcal{F}(\Delta) = (uvw, xw, xy).$$

2. Basic Setup

We first fix some notation and terminology.

Definition 1 (simplicial complex, facet and more). *A* simplicial complex *Δ over a set of vertices $V = \{v_1, \ldots, v_n\}$ is a collection of subsets of V, with the property that $\{v_i\} \in \Delta$ for all i, and if $F \in \Delta$ then all subsets of F are also in Δ (including the empty set). An element of Δ is called a* face *of Δ, and the* dimension *of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called* vertices *and* edges*, respectively, and* $\dim \emptyset = -1$.

The maximal faces of Δ under inclusion are called facets *of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets; in other words*

$$\dim \Delta = \max\{\dim F \mid F \in \Delta\}.$$

We denote the simplicial complex Δ with facets $F_1, \ldots, F_q$ by

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

and we call $\{F_1, \ldots, F_q\}$ the facet set *of Δ.*

A simplicial complex with only one facet is called a simplex.

Definition 2 (subcomplex). *By a* subcomplex *of a simplicial complex Δ, in this paper, we mean a simplicial complex whose facet set is a subset of the facet set of Δ.*

Definition 3 (facet ideal, non-face ideal). *Let Δ be a simplicial complex over n vertices labeled $v_1, \ldots, v_n$. Let k be a field, $x_1, \ldots, x_n$ be indeterminates, and R be the polynomial ring $k[x_1, \ldots, x_n]$.*
(a) We define $\mathcal{F}(\Delta)$ to be the ideal of R generated by square-free monomials $x_{i_1} \ldots x_{i_s}$, where $\{v_{i_1}, \ldots, v_{i_s}\}$ is a facet of Δ. We call $\mathcal{F}(\Delta)$ the facet ideal *of Δ.*
(b) We define $\mathcal{N}(\Delta)$ to be the ideal of R generated by square-free monomials $x_{i_1} \ldots x_{i_s}$, where $\{v_{i_1}, \ldots, v_{i_s}\}$ is not a face of Δ. We call $\mathcal{N}(\Delta)$ the non-face ideal *or the* Stanley-Reisner ideal *of Δ.*

We refer the reader to [BH] for an extensive coverage of the theory of Stanley-Reisner ideals.

Definition 4 (facet complex, non-face complex). *Let $I = (M_1, \ldots, M_q)$ be an ideal in a polynomial ring $k[x_1, \ldots, x_n]$, where $M_1, \ldots, M_q$ are square-free monomials in $x_1, \ldots, x_n$ that form a minimal set of generators for I, and k is a field.*
(a) We define $\delta_{\mathcal{F}}(I)$ to be the simplicial complex over a set of vertices $v_1, \ldots, v_n$ with facets $F_1, \ldots, F_q$, where for each i, $F_i = \{v_j \mid x_j | M_i,\ 1 \leq j \leq n\}$. We call $\delta_{\mathcal{F}}(I)$ the facet complex *of I.*
(b) We define $\delta_{\mathcal{N}}(I)$ to be the simplicial complex over a set of vertices $v_1, \ldots, v_n$, where $\{v_{i_1}, \ldots, v_{i_s}\}$ is a face of $\delta_{\mathcal{N}}(I)$ if and only if $x_{i_1} \ldots x_{i_s} \notin I$. We call $\delta_{\mathcal{N}}(I)$ the non-face complex *or the* Stanley-Reisner complex *of I.*

Remark 1. It is worth observing that given a simplicial complex Δ, if

$$\mathcal{F}(\Delta) = (M_1, \ldots, M_q) \subseteq k[x_1, \ldots, x_n],$$

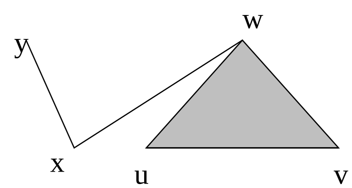
then $\mathcal{N}(\Delta)$ is generated by square free monomials that do not divide any of $M_1, \ldots, M_q$.

We now generalize some notions from graph theory to simplicial complexes.

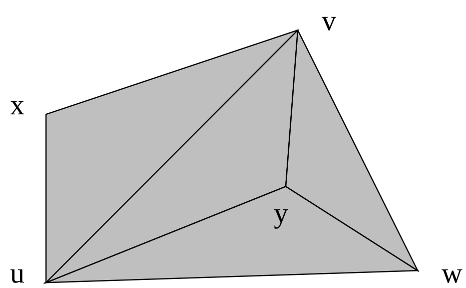
Definition 5 (minimal vertex cover, vertex covering number). *Let Δ be a simplicial complex with vertex set V and facets $F_1, \ldots, F_q$. A* vertex cover *for Δ is a subset A of V, with the property that for every facet F_i there is a vertex $v \in A$ such that $v \in F_i$. A* minimal vertex cover *of Δ is a subset A of V such that A is a vertex cover, and no proper subset of A is a vertex cover for Δ. The smallest cardinality of a minimal vertex cover of Δ is called the* vertex covering number *of Δ.*

A simplicial complex may have several minimal vertex covers.

Example 1. Let Δ be the simplicial complex below.



Here $\mathcal{N}(\Delta) = (xu, xv, yu, yv, yw)$, $\mathcal{F}(\Delta) = (uvw, xw, xy)$ and $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$ is the following simplicial complex:



Also, $\{x, w\}$, $\{y, w\}$, $\{x, v\}$, $\{x, u\}$ are all the minimal vertex covers for Δ.

Proposition 1. *Let Δ be a simplicial complex over n vertices. Consider the ideal $I = \mathcal{F}(\Delta)$ in the polynomial ring $k[x_1, \ldots, x_n]$ over a field k. Then an ideal $p = (x_{i_1}, \ldots, x_{i_s})$ of R is a minimal prime of I if and only if $\{x_{i_1}, \ldots, x_{i_s}\}$ is a minimal vertex cover for Δ.*

Proof. Suppose that $I = (M_1, \ldots, M_q)$. The minimal primes of I are generated by subsets of $\{x_1, \ldots, x_n\}$ (see Proposition 5.1.3 of [Vi2]). By definition $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover for Δ if and only if for each generator M_j of I, $x_{i_t} | M_j$ for some $1 \leq t \leq s$. It follows that $I \subseteq p = (x_{i_1}, \ldots, x_{i_s})$ if and only if $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover for Δ. The assertion now follows. $\square$

Definition 6 (unmixed). *A simplicial complex Δ is* unmixed *if all of its minimal vertex covers have the same cardinality.*

For instance, the simplicial complex in the previous example is unmixed.

Definition 7 (pure). *A simplicial complex Δ is* pure *if all of its facets have the same dimension. Equivalently, this means that if $\mathcal{F}(\Delta)$ is generated by $M_1, \ldots, M_q$, all the M_i are the product of the same number of variables.*

Corollary 1. *Let I be a square-free monomial ideal in the polynomial ring $k[x_1, \ldots, x_n]$. Then $\delta_{\mathcal{F}}(I)$ is unmixed if and only if $\delta_{\mathcal{N}}(I)$ is pure.*

Proof. By Proposition 1 $\delta_{\mathcal{F}}(I)$ is unmixed if and only if all minimal primes of I have the same number of minimal generators, say that number is s. By the proof of Theorem 5.1.4 of [BH], $(x_{i_1}, \ldots, x_{i_s})$ is a minimal prime of I if and only if $\{v_1, \ldots, v_n\} \setminus \{v_{i_1}, \ldots, v_{i_s}\}$ is a facet of $\delta_{\mathcal{N}}(I)$, which is equivalent to $\delta_{\mathcal{N}}(I)$ being pure. □

Corollary 2 (A Cohen-Macaulay simplicial complex is unmixed). *Suppose that Δ is a simplicial complex with vertex set $x_1, \ldots, x_n$ such that $k[x_1, \ldots, x_n]/\mathcal{F}(\Delta)$ is Cohen-Macaulay. Then Δ is unmixed.*

Proof. If $k[x_1, \ldots, x_n]/\mathcal{F}(\Delta)$ is Cohen-Macaulay, then $\delta_{\mathcal{N}}(\mathcal{F}(\Delta))$ is pure ([BH] Corollary 5.1.5). Corollary 1 then implies that $\Delta = \delta_{\mathcal{F}}(\mathcal{F}(\Delta))$ is unmixed. □

Remark 2. It is worth observing that for a square-free monomial ideal I, there is a natural way to construct $\delta_{\mathcal{N}}(I)$ and $\delta_{\mathcal{F}}(I)$ from each other using the structure of the minimal primes of I. To do this, consider the vertex set $V = \{x_1, \ldots, x_n\}$ consisting of all variables that divide a monomial in the generating set of I. The following correspondence holds:

$$F = \text{facet of } \delta_{\mathcal{N}}(I) \longleftrightarrow V \setminus F = \text{minimal vertex cover of } \delta_{\mathcal{F}}(I)$$

Also,

$$I = \bigcap p$$

where the intersection is taken over all prime ideals p of $k[x_1, \ldots, x_n]$ that are generated by a minimal vertex cover of $\delta_{\mathcal{F}}(I)$ (or equivalently, primes p that are generated by $V \setminus F$, where F is a facet of $\delta_{\mathcal{N}}(I)$; see [BH] Theorem 5.1.4).

Regarding the dimension and codimension of I, note that by Theorem 5.1.4 of [BH] and the discussion above, setting $R = k[x_1, \ldots, x_n]$ as above, we have

$$\dim R/I = \dim \delta_{\mathcal{N}}(I) + 1 = |V| - \text{ vertex covering number of } \delta_{\mathcal{F}}(I)$$

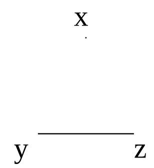
and

$$\text{height } I \;=\; \text{vertex covering number of } \delta_{\mathcal{F}}(I).$$

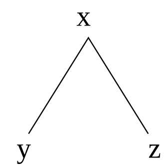
We illustrate all this through an example.

Example 2. For $I = (xy, xz)$,

$\underline{\delta_{\mathcal{N}}(I)}$



$\underline{\delta_{\mathcal{F}}(I)}$



facets of $\delta_{\mathcal{N}}(I)$	minimal vertex covers of $\delta_{\mathcal{F}}(I)$
$\{x\}$	$\{y,z\}$
$\{y,z\}$	$\{x\}$

Note that $I=(x)\cap(y,z)$, and

$$\dim k[x,y,z]/(xy,xz)=2$$

as asserted in Remark 2 above.

3. Simplicial complexes that are trees

We now explore a new notion of *tree* on simplicial complexes. This definition generalizes trees in graph theory, and turns out to behave well under localization and removal of a facet as described below.

To motivate the definition, recall that a connected graph is a tree if it has no cycles. An equivalent definition states that a connected graph is a tree if every subgraph has a *leaf*, where a leaf is a vertex that belongs to only one edge of the graph. This latter description is the one that we adapt, with a slight change in the definition of a leaf, to the class of simplicial complexes.

Definition 8 (leaf, universal set). *Suppose that Δ is a simplicial complex. A facet F of Δ is called a* leaf *if either F is the only facet of Δ, or there exists a facet G in Δ, $G\neq F$, such that*

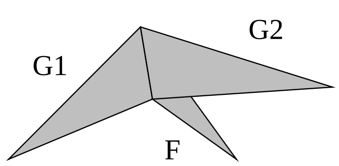
$$F\cap F'\subseteq F\cap G$$

for every facet $F'\in\Delta$, $F'\neq F$.

We denote the set of facets G in Δ with this property by $\mathcal{U}_\Delta(F)$ and call it the universal set *of F in Δ.*

Note that the facet G in Definition 8 is not necessarily unique:

Example 3. In the following simplicial complex Δ, F is a leaf and both G_1 and G_2 are in $\mathcal{U}_\Delta(F)$.



Remark 3. In order to be able to quickly identify a leaf in a simplicial complex, it is important to notice that a leaf must have a vertex that does not belong to any other facet of that simplicial complex. This follows easily from Definition 8: otherwise, a leaf F will be contained in the members of its universal set, which contradicts the fact that a leaf is a facet.

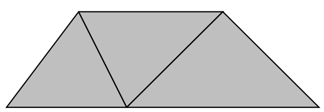
Definition 9 (tree). *Suppose that Δ is a connected simplicial complex. We say that Δ is a* tree *if every nonempty subcomplex of Δ (including Δ itself) has a leaf.*

Equivalently, Δ is a tree if every nonempty connected *subcomplex of Δ has a leaf.*

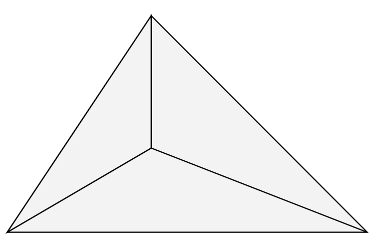
Recall that by a subcomplex of Δ we mean a simplicial complex whose facet set is a subset of the facet set of Δ.

Definition 10 (forest). *A simplicial complex Δ with the property that every connected component of Δ is a tree is called a* forest. *In other words, a forest is a simplicial complex with the property that every nonempty subcomplex has a leaf.*

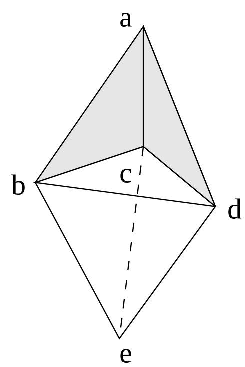
Example 4. The simplicial complex below is a tree.



Example 5. The simplicial complex below is not a tree because every vertex is shared by at least two facets (see Remark 3 above).



Example 6. The simplicial complex below with facets $F_1 = \{a, b, c\}$, $F_2 = \{a, c, d\}$ and $F_3 = \{b, c, d, e\}$ is not a tree because the only candidate for a leaf is the facet F_3, but neither one of $F_1 \cap F_3$ or $F_2 \cap F_3$ is contained in the other.



Notice that in the case that Δ is a graph, Definition 9 agrees with the definition of a tree in graph theory, with the difference that now the term "leaf" refers to an edge, rather than a vertex.

A notion that will be crucial in the following discussion is "removing a facet" from a simplicial complex. We therefore record a definition for this construction.

Definition 11 (facet removal). *Suppose $\Delta = \langle F_1, \ldots, F_q \rangle$ is a simplicial complex and $\mathcal{F}(\Delta) = (M_1, \ldots, M_q)$ is its facet ideal in $R = k[x_1, \ldots, x_n]$. The simplicial complex obtained by* removing the facet F_i *from Δ is the simplicial complex*

$$\langle F_1, \ldots, \hat{F_i}, \ldots, F_q \rangle.$$

An important property of a tree that we will use later is that the localization of a tree is a forest. Below, by abuse of notation, we use $x_1, \ldots, x_n$ to denote both the vertices of a simplicial complex and the variables of the polynomial ring corresponding to that complex.

Lemma 1 (Localization of a tree is a forest). *Let Δ be a tree with vertices $x_1, \ldots, x_n$, and let $I = \mathcal{F}(\Delta)$ be the facet ideal of Δ in the polynomial ring $R = k[x_1, \ldots, x_n]$ where k is a field. Then for any prime ideal p of R, $\delta_{\mathcal{F}}(I_p)$ is a forest.*

Proof. The first step is to show that it is enough to prove this for prime ideals of R generated by a subset of $\{x_1, \ldots, x_n\}$. To see this, assume that p is a prime ideal of R and that p' is the prime ideal of R generated by all $x_i \in \{x_1, \ldots, x_n\}$ such that $x_i \in p$ (recall that the minimal primes of I are generated by subsets of $\{x_1, \ldots, x_n\}$). So $p' \subseteq p$. If $I = (M_1, \ldots, M_q)$, then

$$I_{p'} = ({M_1}', \ldots, {M_q}')$$

where for each i, ${M_i}'$ is the image of M_i in $I_{p'}$. In other words, ${M_i}'$ is obtained by dividing M_i by the product of all the x_j such that $x_j | M_i$ and $x_j \notin p'$. But $x_j \notin p'$ implies that $x_j \notin p$, and so it follows that ${M_i}' \in I_p$.

Therefore $I_{p'} \subseteq I_p$. On the other hand since $p' \subseteq p$, $I_p \subseteq I_{p'}$, which implies that $I_{p'} = I_p$ (the equality and inclusions of the ideals here mean equality and inclusion of their generating sets).

We now prove the theorem for $p = (x_{i_1}, \ldots, x_{i_r})$. The main point to notice here is that the simplicial complex corresponding to I_p is obtained by removing all vertices except for $x_{i_1}, \ldots, x_{i_r}$ from Δ. So the proof reduces to showing that if we remove a vertex from Δ, the resulting simplicial complex is a forest.

Let $\Delta = \langle F_1, \ldots, F_q \rangle$, and suppose that we remove the vertex x_1. Setting

$$F_i{}' = F_i \setminus \{x_1\}$$

we would like to show that

$$\Delta' = \langle F_1', \ldots, F_q' \rangle$$

is a forest. Notice that some of the F_i' appearing in Δ' may be redundant. We need to show that every nonempty subcomplex of Δ' has a leaf.

Let $\Delta_1' = \langle F_{i_1}', \ldots, F_{i_s}' \rangle$ be a subcomplex of Δ' where $F_{i_1}', \ldots, F_{i_s}'$ are distinct facets. Consider the corresponding subcomplex $\Delta_1 = \langle F_{i_1}, \ldots, F_{i_s} \rangle$ of Δ, which has a leaf, say F_{i_1}. So there exists $G \in \Delta_1$ such that

$$F_{i_1} \cap F \subseteq F_{i_1} \cap G$$

for every facet $F \in \langle F_{i_2}, \ldots, F_{i_s} \rangle$. Now since each of the F_{i_j}' is a nonempty facet of Δ_1' the same statement holds in Δ_1', that is,

$$F_{i_1}' \cap F' \subseteq F_{i_1}' \cap G'$$

for every facet $F' \in \Delta_1' \setminus \{F_{i_1}'\}$. This implies that F_{i_1}' is a leaf for Δ_1'. $\square$

4. Cohen-Macaulay properties of simplicial complexes

The main result of this section states that the facet ideal of a tree (see Definition 9) has sliding depth in the polynomial ring generated by its set of vertices over a field k. A special case of this result for the case that Δ is a graph was proved in [SVV] (Theorem 1.3). We first define the notion of sliding depth. Then we carry out a similar argument to the one in [SVV] to show that the facet ideal of a tree has sliding depth.

Definition 12 (sliding depth). *Let I be an ideal in a ring R of dimension n. Let $\mathbb{K}$ be the Koszul complex on the set $\mathbf{a} = \{a_1, \ldots, a_q\}$ of generators of I, and denote by $\mathrm{H}_i(\mathbb{K})$ the homology modules of $\mathbb{K}$. I satisfies* sliding depth *if*

$$\text{depth } \mathrm{H}_i(\mathbb{K}) \geq n - q + i, \qquad \text{for all } i.$$

Definition 13 (Cohen-Macaulay simplicial complex). *A simplicial complex Δ over a set of vertices labeled $x_1, \ldots, x_n$ is Cohen-Macaulay, if the quotient ring $k[x_1, \ldots, x_n]/\mathcal{F}(\Delta)$ is Cohen-Macaulay.*

Below, we assume that the polynomial ring $k[x_1, \ldots, x_n]$ is localized at the maximal ideal $(x_1, \ldots, x_n)$. By abuse of notation, we use $x_1, \ldots, x_n$ to denote both the vertices of a simplicial complex and the variables of the polynomial ring corresponding to that complex.

Theorem 1 (Main Theorem: trees have sliding depth). *If a simplicial complex Δ on a set of vertices $\{x_1, \ldots, x_n\}$ is a tree and k is a field, then $\mathcal{F}(\Delta)$ has sliding depth in the polynomial ring $R = k[x_1, \ldots, x_n]$.*

Proof of Main Theorem. Suppose that Δ is a tree with facets $F_1, \ldots, F_q$ and vertex set $x_1, \ldots, x_n$, and that $\mathcal{F}(\Delta) = (M_1, \ldots, M_q)$, where each M_i is a square-free monomial in the variables $x_1, \ldots, x_n$. We want to show that $\mathcal{F}(\Delta)$ has sliding depth in the polynomial ring $R = k[x_1, \ldots, x_n]$. We argue by induction on q.

The case $q = 1$ is the case where $\mathcal{F}(\Delta) = (M_1)$, and the Koszul complex looks like

$$0 \longrightarrow R \xrightarrow{.M_1} R \longrightarrow 0 \ .$$

which gives $\mathrm{H}_0(M_1) = R/(M_1)$, and depth $R/(M_1) = n - 1$, and so $\mathcal{F}(\Delta)$ has sliding depth.

Suppose we know that the theorem holds for any tree with up to $q - 1$ facets, $q \geq 2$. We want to show this for a simplicial complex with q facets.

Suppose without loss of generality that F_q is a leaf and

$$M_q = x_1 \ldots x_r.$$

By removing F_q from Δ (see Definition 11) we obtain a tree

$$\Delta' = \langle F_1, \ldots, F_{q-1} \rangle.$$

We let

$$Z' \subseteq \{x_1, \ldots, x_n\}$$

be the vertex set for Δ', and

$$Z'' = \{x_1, \ldots, x_r\} \setminus Z'.$$

It follows that

$$Z' \cap Z'' = \emptyset \quad \text{and} \quad Z' \cup Z'' = \{x_1, \ldots, x_n\}.$$

By the induction hypothesis, $\mathcal{F}(\Delta')$ has sliding depth in $R' = k[Z']$.

Let y' and y'' be the product of the elements of $Z' \cap \{x_1, \ldots, x_r\}$ and Z'', respectively, so that

$$M_q = x_1 \ldots x_r = y'y''.$$

Let

$$L' = \mathcal{F}(\Delta') + (y').$$

Since y' is the monomial describing the intersection of the leaf F_q with Δ', there is an M_j in the generating set of $\mathcal{F}(\Delta')$ such that $y'|M_j$, and so adding y' to $\mathcal{F}(\Delta')$ does not increase the number of generators of $\mathcal{F}(\Delta')$.

Notice that $\delta_{\mathcal{F}}(L')$ is a forest with at most $q-1$ facets. To show that it is a forest, we must show that every connected subcomplex of $\delta_{\mathcal{F}}(L')$ has a leaf. Let F' denote the facet in $\delta_{\mathcal{F}}(L')$ corresponding to the monomial y'. The only subcomplexes that we need to worry about are those that contain F', since any other one will be a connected subcomplex of Δ, and so will have a leaf by definition.

Suppose that

$$\Delta'' = \langle F', F_{i_1}, \ldots, F_{i_s} \rangle$$

is a connected subcomplex of $\delta_{\mathcal{F}}(L')$. Consider the connected subcomplex of Δ defined as

$$\overline{\Delta} = \langle F_q, F_{i_1}, \ldots, F_{i_s} \rangle.$$

Since Δ is a tree, $\overline{\Delta}$ has a leaf. There are two possibilities:
Case 1. For some j, F_{i_j} is a leaf of $\overline{\Delta}$, and $G \in \mathcal{U}_{\overline{\Delta}}(F_{i_j})$. Then

$$F_q \cap F_{i_j} \subseteq G \cap F_{i_j}.$$

But $F_q \cap F_{i_j} = F' \cap F_{i_j}$ (from the construction above), and so F_{i_j} is a leaf for Δ''.
Case 2. F_q is a leaf of $\overline{\Delta}$, and $G \in \mathcal{U}_{\overline{\Delta}}(F_q)$. Then for all j,

$$F_{i_j} \cap F_q \subseteq G \cap F_q$$

which, as above, translates into

$$F_{i_j} \cap F' \subseteq G \cap F'$$

which implies that F' is a leaf for Δ''.

Therefore $\delta_{\mathcal{F}}(L')$ is a forest. By the induction hypothesis, the facet ideal of every connected component of $\delta_{\mathcal{F}}(L')$ satisfies sliding depth, and so L' will satisfy sliding depth (this is because the Koszul homology of L' can be written as a direct sum of the tensor products, over k, of the Koszul homologies of the facet ideal of each connected component of $\delta_{\mathcal{F}}(L')$; see the discussion on page 397 of [SVV]).

If $R' = k[Z']$ and $R'' = k[Z'']$, then

$$R = R' \otimes_k R''$$

and

$$\mathcal{F}(\Delta) = \mathcal{F}(\Delta') + (y)$$

where

$$y = M_q = y'y''.$$

If $\mathbb{K}'$ denotes the Koszul complex of $\mathcal{F}(\Delta')$ over R', $|Z'| = n'$ and $|Z''| = n''$, the sliding depth condition on $\mathcal{F}(\Delta')$ implies that for all i:

$$\operatorname{depth} \mathrm{H}_i(\mathbb{K}') \geq n' - (q-1) + i = n' - q + i + 1.$$

If $\mathbb{K}$ is the Koszul complex of $\mathcal{F}(\Delta')$ in $R = R'[Z'']$, since R is a faithfully flat extension of R' by n'' variables, it follows from [BH] Proposition 1.6.7 and the inequality above that for every u,

$$\begin{aligned}\text{depth } \mathrm{H}_i(\mathbb{K}) &= \text{depth } \mathrm{H}_i(\mathbb{K}') + \text{depth } k[Z''] \\ &\geq n' - q + (i+1) + n'' \\ &= n - q + (i+1).\end{aligned}$$

Suppose that $\mathbb{L}$ is the Koszul complex of $\mathcal{F}(\Delta) = \mathcal{F}(\Delta') + (y)$. As in Proposition 3.7 of [SVV], the basic relationship between the homologies of $\mathbb{L}$ and $\mathbb{K}$ is:

$$0 \longrightarrow \overline{\mathrm{H}_i(\mathbb{K})} \longrightarrow \mathrm{H}_i(\mathbb{L}) \longrightarrow {}_y\mathrm{H}_{i-1}(\mathbb{K}) \longrightarrow 0, \tag{1}$$

where for any module E, $\overline{E} = E/Ey$ and ${}_yE = \{e \in E \mid y.e = 0\}$.

We will need the following lemma:

Lemma 2 (Depth Lemma). *If $0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$ is a short exact sequence of modules over a local ring R, then*

(a) If depth $M <$ depth L, *then* depth $N =$ depth M.

(b) If depth $M =$ depth L, *then* depth $N \geq$ depth M.

(c) If depth $M >$ depth L, *then* depth $N =$ depth $L + 1$.

Proof. This is Lemma 1.3.9 in [Vi2]. See Corollary 18.6 in [E] for a proof. □

Lemma 3. *With $\mathbb{K}'$ defined (as above) as the Koszul complex of $\mathcal{F}(\Delta')$ over the polynomial ring $R' = k[Z']$, we have:*

$$\text{depth } \overline{\mathrm{H}_i(\mathbb{K}')} \geq n' - q + i + 1$$

$$\text{depth } {}_{y'}\mathrm{H}_i(\mathbb{K}') \geq n' - q + i + 1$$

where $\overline{\mathrm{H}_i(\mathbb{K}')} = \mathrm{H}_i(\mathbb{K}')/y'\mathrm{H}_i(\mathbb{K}')$, *and* ${}_{y'}\mathrm{H}_i(\mathbb{K}') = \{x \in \mathrm{H}_i(\mathbb{K}') \mid y'.x = 0\}$.

Proof. Suppose that

$$L' = \mathcal{F}(\Delta') + (y').$$

Notice that L' is the facet ideal of a simplicial complex with n' vertices and q' facets, where $q' \leq q - 1$, since y' is part of at least one facet of Δ'. By induction L' has sliding depth, so if $\mathbb{A}'$ is the Koszul complex of L',

$$\text{depth } \mathrm{H}_i(\mathbb{A}') \geq n' - q' + i.$$

We also have an exact sequence:

$$0 \longrightarrow {}_{y'}\mathrm{H}_i(\mathbb{K}') \longrightarrow \mathrm{H}_i(\mathbb{K}') \xrightarrow{y'} \mathrm{H}_i(\mathbb{K}') \longrightarrow \overline{\mathrm{H}_i(\mathbb{K}')} \longrightarrow 0$$

which breaks into two exact sequences:

$$0 \longrightarrow y'\mathrm{H}_i(\mathbb{K}') \longrightarrow \mathrm{H}_i(\mathbb{K}') \longrightarrow \overline{\mathrm{H}_i(\mathbb{K}')} \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow {}_{y'}\mathrm{H}_i(\mathbb{K}') \longrightarrow \mathrm{H}_i(\mathbb{K}') \longrightarrow y'\mathrm{H}_i(\mathbb{K}') \longrightarrow 0\ . \tag{3}$$

Similar to Sequence (1) above, we also have the following short exact sequence:

$$0 \longrightarrow \overline{\mathrm{H}_i(\mathbb{K}')} \longrightarrow \mathrm{H}_i(\mathbb{A}') \longrightarrow {}_{y'}\mathrm{H}_{i-1}(\mathbb{K}') \longrightarrow 0\ . \tag{4}$$

We know that $\mathbb{K}'$ and $\mathbb{A}'$ have sliding depth. We prove the lemma by induction on i.

For $i = 0$, $\mathrm{H}_0(\mathbb{K}') = R'/\mathcal{F}(\Delta')$ has depth $\geq n' - q + 1$. Since the facet corresponding to the monomial y' is contained in a facet of Δ',

$$\overline{\mathrm{H}_0(\mathbb{K}')} = R'/(\mathcal{F}(\Delta'), y')$$

is the zeroth homology of a forest with at most $q-1$ facets, and so in R' one has

$$\operatorname{depth} \overline{\mathrm{H}_0(\mathbb{K}')} \geq n' - (q-1).$$

Plugging this into Sequence (2), along with Lemma 2 implies that

$$\operatorname{depth} y'\mathrm{H}_0(\mathbb{K}') \geq n' - (q-1)$$

and then Sequence (3) and Lemma 2 imply that:

$$\operatorname{depth} {}_{y'}\mathrm{H}_0(\mathbb{K}') \geq n' - (q-1).$$

Now suppose that the statement holds for all $j \leq i-1$. We verify it for $j = i$.

Since

$$\operatorname{depth} {}_{y'}\mathrm{H}_{i-1}(\mathbb{K}') \geq n' - (q-1) + i - 1$$

and

$$\operatorname{depth} \mathrm{H}_i(\mathbb{A}') \geq n' - (q-1) + i,$$

Sequence (4) and Lemma 2 imply that

$$\operatorname{depth} \overline{\mathrm{H}_i(\mathbb{K}')} \geq n' - (q-1) + i.$$

We plug this into Sequence (2) and consider the fact that depth $\mathrm{H}_i(\mathbb{K}') \geq n' - (q-1) + i$ along with Lemma 2 and conclude that

$$\operatorname{depth} y'\mathrm{H}_i(\mathbb{K}') \geq n' - (q-1) + i$$

which when plugged in Sequence (3) implies that

$$\operatorname{depth} {}_{y'}\mathrm{H}_i(\mathbb{K}') \geq n' - (q-1) + i.$$

□

Proposition 2. *If $\mathbb{K}$ is the Koszul complex of $\mathcal{F}(\Delta')$ in R and $y = M_q = y'y''$, as above, then*

$$\operatorname{depth} \overline{\mathrm{H}_i(\mathbb{K})} \geq n - q + i$$

$$\operatorname{depth} {}_{y}\mathrm{H}_i(\mathbb{K}) \geq n - q + i + 1.$$

Proof. Notice that

$$\mathrm{H}_i(\mathbb{K}) = \mathrm{H}_i(\mathbb{K}') \otimes_k k[Z'']$$

where $\mathbb{K}'$ and Z'' are defined above, and multiplication by $y = y'y''$ in $\mathrm{H}_i(\mathbb{K})$ can be factored as

$$\mathrm{H}_i(\mathbb{K}') \otimes_k k[Z''] \xrightarrow{y' \otimes_k 1} \mathrm{H}_i(\mathbb{K}') \otimes_k k[Z''] \xrightarrow{1 \otimes_k y''} \mathrm{H}_i(\mathbb{K}') \otimes_k k[Z''].$$

From here, setting $E = \mathrm{H}_i(\mathbb{K})$, $E' = \mathrm{H}_i(\mathbb{K}')$, $E'' = k[Z'']$ and $y = y'y''$ we get an exact sequence:

$$0 \to {}_{y'}E' \otimes_k E'' \to {}_yE \to E' \otimes_k {}_{y''}E'' \to E'/y'E' \otimes_k E'' \xrightarrow{f}$$

$$E/yE \to E' \otimes_k E''/y''E'' \to 0 \tag{5}$$

The map f is multiplication by $1 \otimes_k y''$ and so the image of f is isomorphic to

$$E'/y'E' \otimes_k y''E''$$

which by Lemma 3 has depth at least $n - q + (i+1)$, and

$$\text{depth } E' \otimes_k E''/y''E'' \geq n' - (q-1) + i + n'' - 1 = n - q + i,$$

and therefore by Depth Lemma:

$$\text{depth } E/yE \geq n - q + i.$$

Also, since E'' is a polynomial ring, ${}_{y''}E'' = 0$ and so

$$\text{depth } {}_yE = \text{depth } {}_{y'}E' \otimes_k E'' = n - q + (i+1)$$

as desired. □

Conclusion of Proof of Main Theorem. We now apply the result of Proposition 2 and Depth Lemma to the short exact sequence (1) from above:

$$0 \longrightarrow \overline{\mathrm{H}_i(\mathbb{K})} \longrightarrow \mathrm{H}_i(\mathbb{L}) \longrightarrow {}_y\mathrm{H}_{i-1}(\mathbb{K}) \longrightarrow 0$$

and conclude that for all i,

$$\text{depth } \mathrm{H}_i(\mathbb{L}) \geq n - q + i.$$

□

Proposition 3. *Suppose that Δ is a Cohen-Macaulay tree and $I = \mathcal{F}(\Delta) \subseteq k[x_1, \ldots, x_n]$. Let p be a prime ideal of R. Then, if $\mu(J)$ denotes the minimal number of generators of the ideal J,*

$$\mu(I_p) \leq \max \{\text{height } I, \text{height } p - 1\}.$$

Proof. If p is a minimal prime of I, then by Proposition 1 p is generated by a minimal vertex cover of Δ, say

$$p = (x_{i_1}, \dots, x_{i_r}).$$

If $I = (M_1, \dots, M_q)$, then since $\{x_{i_1}, \dots, x_{i_r}\}$ is a minimal vertex cover, for each $x_{i_e} \in \{x_{i_1}, \dots, x_{i_r}\}$ there is at least one $M_j \in \{M_1, \dots, M_q\}$ such that $x_{i_e} | M_j$ and $x_f \not| M_j$ for $x_f \in \{x_{i_1}, \dots, x_{i_r}\} \setminus \{x_{i_e}\}$. This implies that $p_p \subseteq I_p$ and since $I_p \subseteq p_p$, it follows that $I_p = p_p$.

So in particular,

$$\mu(I_p) = \mu(p) = \text{height } I$$

and therefore the theorem holds.

Now we prove the general version of the theorem by induction on the number of vertices of Δ. The case $n = 1$ follows from the argument above. Suppose that the inequality holds for any simplicial complex with less than n vertices.

Let Δ be a simplicial complex with vertices $x_1, \dots, x_n$, and let p be a prime ideal of $k[x_1, \dots, x_n]$. We know by Lemma 1 that $\delta_{\mathcal{F}}(I_p)$ is a forest. By the first paragraph of the proof of the same lemma, we can assume that p is generated by a subset of $\{x_1, \dots, x_n\}$.

Now suppose that p is not a minimal vertex cover. So, with $s > 0$, we have

$$I_p = I_1 + \dots + I_s + (x_{j_1}, \dots, x_{j_t})$$

where $\delta_{\mathcal{F}}(I_i)$ for $i = 1, \dots, s$ and $\delta_{\mathcal{F}}(x_{j_e})$ for $e = 1, \dots, t$, are the connected components of the forest $\delta_{\mathcal{F}}(I_p)$.

Notice that each $\delta_{\mathcal{F}}(I_i)$ is also a Cohen-Macaulay tree, in the sense that if R_i denotes the polynomial ring over k generated by the variables appearing in I_i, then R_i/I_i is a Cohen-Macaulay ring. This follows, for example, from [BH] Theorem 2.1.7, since R_p/I_p, which is Cohen-Macaulay, is isomorphic to the tensor product, over k, of all the R_i/I_i.

Let

$$p = p_1 + \dots + p_s + (x_{j_1}, \dots, x_{j_t})$$

where each p_i is the ideal of all vertices of $\delta_{\mathcal{F}}(I_i)$ above.

Since p_i is the ideal generated by all vertices of $\delta_{\mathcal{F}}(I_i)$, it *properly* contains the ideal generated by a minimal vertex cover of $\delta_{\mathcal{F}}(I_i)$. Therefore, by the first paragraph of this proof

$$\text{height } I_i < \text{height } p_i$$

which implies that

$$\text{height } I_i \leq \text{height } p_i - 1.$$

By the induction hypothesis for each i, since $I_i = (I_i)_{p_i}$,

$$\mu(I_i) \leq \max\{\text{height } I_i, \text{height } p_i - 1\} \leq \text{height } p_i - 1.$$

We now have (recall that $s > 0$)

$$\begin{aligned}\mu(I_p) &= \mu(I_1) + \cdots + \mu(I_s) + t \\ &\leq \text{height } p_1 - 1 + \cdots + \text{height } p_s - 1 + t \\ &= \text{height } p - s \\ &\leq \text{height } p - 1\end{aligned}$$

The assertion then follows. □

Definition 14 (Strongly Cohen-Macaulay). *An ideal I of a ring R is* strongly Cohen-Macaulay *if all Koszul homology modules of I are Cohen-Macaulay.*

Corollary 3 (The facet ideal of a C-M tree is strongly C-M). *Suppose that Δ is a Cohen-Macaulay tree. Then $\mathcal{F}(\Delta)$ is a strongly Cohen-Macaulay ideal.*

Proof. This follows from Proposition 3 above, the fact that $\mathcal{F}(\Delta)$ has sliding depth and Theorem 1.4 of [HVV], or Theorem 3.3.17 of [V]. □

Definition 15 ([V]). *An ideal I of R satisfies condition $\mathcal{F}_1$ if $\mu(I) \leq$ dim R_p for all prime ideals of p of R such that $I \subset p$.*

Proposition 4. *Suppose that Δ is a tree and $I = \mathcal{F}(\Delta) \subseteq k[x_1, \ldots, x_n]$. Then I satisfies $\mathcal{F}_1$.*

Proof. As in Proposition 3, if p is a minimal prime of I we have

$$\mu(I_p) = \mu(p) = ht(p) = \dim R_p.$$

If I represents a tree with q facets, then we proceed by induction on q. The case $q = 1$ results in $\mu(I_p) = 1 \leq \dim R_p$ for all primes p containing I. Suppose that the proposition holds for all trees that have less than q facets, and let I be the facet ideal of a tree with q facets, and p be a prime ideal of R that contains I. If p is not minimal over R, similar to the proof of Proposition 3, we have that

$$I_p = I_1 + \cdots + I_s + (x_{j_1}, \ldots, x_{j_t})$$

where $\delta_{\mathcal{F}}(I_i)$ for $i = 1, \ldots, s$ and $\delta_{\mathcal{F}}(x_{j_e})$ for $e = 1, \ldots, t$, are the connected components of the forest $\delta_{\mathcal{F}}(I_p)$.

Let

$$p = p_1 + \cdots + p_s + (x_{j_1}, \ldots, x_{j_t})$$

where each p_i is the ideal of all vertices of $\delta_{\mathcal{F}}(I_i)$ above, so for $i = 1, \ldots, s$, $I_i = (I_i)_{p_i}$. By induction hypothesis, $\mu(I_i) \leq \dim R_{p_i}$ for all i, and so

$$\mu(I_p) = \mu(I_1) + \cdots + \mu(I_s) + t \leq \dim R_{p_1} + \cdots + \dim R_{p_s} + t = \dim R_p.$$

□

Corollary 4 (The Rees ring of a tree is normal and C-M). *Suppose that Δ is a tree over vertices $x_1, \dots, x_n$ with facet ideal $I = \mathcal{F}(\Delta)$ in the polynomial ring $k[x_1, \dots, x_n]$ where k is a field. Then the Rees ring of I is normal and Cohen-Macaulay.*

Proof. From Theorem 1 and Proposition 4 it follows that I satisfies sliding depth and the condition $\mathcal{F}_1$, and so $R[It]$ and $\mathrm{gr}_I(R)$ are Cohen-Macaulay by Theorem 4.3.8 and Corollary 3.3.21 of [V].

Proposition 4 and Theorem 1 also imply that for all t, the modules I^t/I^{t+1} are torsion-free over R/I (using approximation complexes), and since I is generically complete intersection and R/I is reduced, it follows from Corollary 5.3 of [SVV] that $R[It]$ is normal. □

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References

[BH] Bruns, W. and Herzog, J. *Cohen-Macaulay Rings*, vol. 39, Cambridge studies in advanced mathematics, revised edition, 1998.

[E] Eisenbud, D. *Commutative Algebra with a View Towards Algebraic Geometry*, Graduate Texts in Mathematics, No. 150, Springer Verlag, New York 1995.

[HVV] Herzog J., Vasconcelos W., Villarreal R., *Ideals with sliding depth*, Nagoya Math. J. 99 (1985), 159-172.

[SVV] Simis A., Vasconcelos W., Villarreal R., *On the ideal theory of graphs*, J. Algebra 167 (1994), no. 2, 389–416.

[V] Vasconcelos W., *Arithmetic of blowup algebras*, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1994.

[Vi1] Villarreal R., *Cohen-Macaulay graphs*, Manuscripta Math. 66 (1990), no. 3, 277–293.

[Vi2] Villarreal R., *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics, 238. Marcel Dekker, Inc., New York, 2001.