Monomial ideals via square-free monomial ideals

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1 Introduction

In this paper we study monomial ideals using the operation "polarization" to first turn them into square-free monomial ideals. Various forms of polarization appear throughout the literature and have been used for different purposes in algebra and algebraic combinatorics (for example, Weyman [**17**], Fröberg [**8**], Schwartau [**13**], or Rota and Stein [**11**]). One of the most useful features of polarization is that the chain of substitutions that turn a given monomial ideal into a square-free one can be described in terms of a regular sequence (Fröberg [**8**]). This fact allows many properties of a monomial ideal to transfer to its polarization. Conversely, to study a given monomial ideal, one could examine its polarization. The advantage of this latter approach is that there are many combinatorial tools dealing with square-free monomial ideals. One of these tools is Stanley-Reisner theory: Schwartau's thesis [**13**] and the book by Stückrad and Vogel [**15**] discuss how the Stanley-Reisner theory of square-free monomial ideals produces results about general monomial

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ideals using polarization. Another tool for studying square-free monomial ideals, which will be our focus here, is facet ideal theory, developed by the author in [**5**], [**6**] and [**7**].

The paper is organized as follows. In Section 2 we define polarization and introduce some of its basic properties. In Section 3 we introduce facet ideals and its features that are relevant to this paper. In particular, we introduce simplicial trees, which correspond to square-free monomial ideals with exceptionally strong algebraic properties. Section 4 extends the results of facet ideal theory to general monomial ideals. Here we study a monomial ideal I whose polarization is a tree, and show that many of the properties of simplicial trees hold for such ideals. This includes Cohen-Macaulayness of the Rees ring of I (Corollary 4.8), I being sequentially Cohen-Macaulay (Corollary 4.12), and several inductive tools for studying such ideals, such as localization (see Section 4.1).

Appendix A is an independent study of primary decomposition in a sequentially Cohen-Macaulay module. We demonstrate how in a sequentially Cohen-Macaulay module M, every submodule appearing in the filtration of M can be described in terms of the primary decomposition of the 0-submodule of M. This is used to prove Proposition 4.11.

2 Polarization

Definition 2.1 Let $R = k[x_1,\dots,x_n]$ be a polynomial ring over a field k. Suppose $M = {x_1}^{a_1}\dots{x_n}^{a_n}$ is a monomial in R. Then we define the *polarization* of M to be the square-free monomial

$$\mathcal{P}(M) = x_{1,1}x_{1,2}\dots x_{1,a_1}x_{2,1}\dots x_{2,a_2}\dots x_{n,1}\dots x_{n,a_n}$$

in the polynomial ring $S = k[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq a_i]$.

If I is an ideal of R generated by monomials $M_1,\dots,M_q$, then the *polarization* of I is defined as:

$$\mathcal{P}(I) = \big(\mathcal{P}(M_1),\dots,\mathcal{P}(M_q)\big)$$

which is a square-free monomial ideal in a polynomial ring S.

Here is an example of how polarization works.

Example 2.2 Let $J = ({x_1}^2, x_1x_2, {x_2}^3) \subseteq R = k[x_1,x_2]$. Then

$$\mathcal{P}(J) = (x_{1,1}x_{1,2}, x_{1,1}x_{2,1}, x_{2,1}x_{2,2}x_{2,3})$$

is the polarization of J in the polynomial ring

$$S = k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{2,3}]$$

Note that by identifying each x_i with $x_{i,1}$, one can consider S as a polynomial extension of R. Exactly how many variables S has will always depend on what we polarize. Therefore, as long as we are interested in the polarizations of finitely many monomials and ideals, S remains a finitely generated algebra.

Below we describe some basic properties of polarization, some of which appear (without proof) in [**15**]. Here we record the proofs where appropriate.

Proposition 2.3 (basic properties of polarization) *Suppose that $R = k[x_1, \dots, x_n]$ is a polynomial ring over a field k, and I and J are two monomial ideals of R.*

1. $\mathcal{P}(I+J) = \mathcal{P}(I) + \mathcal{P}(J)$*;*
2. *For two monomials M and N in R, $M \mid N$ if and only if $\mathcal{P}(M) \mid \mathcal{P}(N)$;*
3. $\mathcal{P}(I \cap J) = \mathcal{P}(I) \cap \mathcal{P}(J)$*;*
4. *If $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ is a (minimal) prime containing I, then $\mathcal{P}(\mathfrak{p})$ is a (minimal) prime containing $\mathcal{P}(I)$;*
5. *If $\mathfrak{p}' = (x_{i_1,e_1}, \dots, x_{i_r,e_r})$ is a prime over $\mathcal{P}(I)$, then $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ is a prime over I. Moreover, if $\mathfrak{p}'$ has minimal height (among all primes containing $\mathcal{P}(I)$), then $\mathfrak{p}$ must have minimal height as well (among all primes containing I);*
6. height $I =$ height $\mathcal{P}(I)$*;*

Proof:

1. Follows directly from Definition 2.1.
2. Suppose that $M = {x_1}^{b_1} \dots {x_n}^{b_n}$ and $N = {x_1}^{c_1} \dots {x_n}^{c_n}$, and suppose that

$$\mathcal{P}(M) = x_{1,1} \dots x_{1,b_1} \dots x_{n,1} \dots x_{n,b_n}$$

   and

$$\mathcal{P}(N) = x_{1,1} \dots x_{1,c_1} \dots x_{n,1} \dots x_{n,c_n}.$$

   If $M \mid N$, then $b_i \leq c_i$ for all i, which implies that $\mathcal{P}(M) \mid \mathcal{P}(N)$. The converse is also clear using the same argument.

3. Suppose that $I = (M_1, \dots, M_q)$ and $J = (N_1, \dots, N_s)$ where the generators are all monomials. If $U = {x_1}^{b_1} \dots {x_n}^{b_n}$ is a monomial in $I \cap J$, then for some generator M_i of I and N_j and J, we have $M_i \mid U$ and $N_j \mid U$, hence by part 2, $\mathcal{P}(M_i) \mid \mathcal{P}(U)$ and $\mathcal{P}(N_j) \mid \mathcal{P}(U)$, which implies that $\mathcal{P}(U) \in \mathcal{P}(I) \cap \mathcal{P}(J)$.

   Conversely, if U' is a monomial in $\mathcal{P}(I) \cap \mathcal{P}(J)$, then for some generator $M_i = {x_1}^{b_1} \dots {x_n}^{b_n}$ of I and $N_j = {x_1}^{c_1} \dots {x_n}^{c_n}$ and J we have $\mathcal{P}(M_i) \mid U'$ and

$\mathcal{P}(N_j) \mid U'$. This means that $\mathrm{lcm}(\mathcal{P}(M_i),\mathcal{P}(N_j)) \mid U'$. It is easy to see (by an argument similar to the one in part 2) that $\mathrm{lcm}(\mathcal{P}(M_i),\mathcal{P}(N_j)) = \mathcal{P}(\mathrm{lcm}(M_i,N_j))$. Since $\mathrm{lcm}(M_i,N_j)$ is one of the generators of $I \cap J$, it follows that $\mathcal{P}(\mathrm{lcm}(M_i,N_j))$ is a generator of $\mathcal{P}(I \cap J)$ and hence $U' \in \mathcal{P}(I \cap J)$.

4. If $\mathfrak{p} = (x_{i_1},\ldots,x_{i_r})$ is a minimal prime over $I = (M_1,\ldots,M_q)$, then for each of the x_{i_j} there is a M_t such that $x_{i_j} \mid M_t$, and no other generator of $\mathfrak{p}$ divides M_t. The same holds for the polarization of the two ideals: $\mathcal{P}(\mathfrak{p}) = (x_{i_1,1},\ldots,x_{i_r,1})$ and $\mathcal{P}(I) = (\mathcal{P}(M_1),\ldots,\mathcal{P}(M_t))$, and so $\mathcal{P}(\mathfrak{p})$ is minimal over $\mathcal{P}(I)$.

5. Suppose that $\mathfrak{p}' = (x_{i_1,e_1},\ldots,x_{i_r,e_r})$ is a prime lying over $\mathcal{P}(I)$. Then for every generator M_t of I, there is a x_{i_j,e_j} in $\mathfrak{p}'$ such that $x_{i_j,e_j} \mid \mathcal{P}(M_t)$. But this implies that $x_{i_j} \mid M_t$, and therefore $I \subseteq \mathfrak{p} = (x_{i_1},\ldots,x_{i_r})$.

   Now suppose that $\mathfrak{p}'$ has minimal height r over $\mathcal{P}(I)$, and there is a prime ideal $\mathfrak{q}$ over I with height $\mathfrak{q} < r$. This implies (from part 4) that $\mathcal{P}(\mathfrak{q})$, which is a prime of height less than r, contains $\mathcal{P}(I)$, which is a contradiction.

6. This follows from parts 4 and 5.

□

Example 2.4 It is not true that every minimal prime of $\mathcal{P}(I)$ comes from a minimal prime of I. For example, let $I = (x_1^2, x_1x_2^2)$. Then

$$\mathcal{P}(I) = (x_{1,1}x_{1,2}, x_{1,1}x_{2,1}x_{2,2}).$$

The ideal $(x_{1,2},x_{2,1})$ is a minimal prime over $\mathcal{P}(I)$, but the corresponding prime (x_1,x_2) is not a minimal prime of I (however, if we had taken any minimal prime of minimal height of $\mathcal{P}(I)$, e.g. $(x_{1,1})$, then the corresponding prime over I would have been minimal; this is part 5 above).

For a monomial ideal I in a polynomial ring $R = k[x_1,\ldots,x_n]$ as above, there is a unique irredundant irreducible decomposition of the form

$$I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_m$$

where each $\mathfrak{q}_i$ is a primary ideal generated by powers of the variables $x_1,\ldots,x_n$ (see [**16**, Theorem 5.1.17]).

Proposition 2.5 (polarization and primary decomposition) *Let I be a monomial ideal in a polynomial ring $R = k[x_1,\ldots,x_n]$, and let $\mathcal{P}(I)$ be the polarization of I in $S = k[x_{i,j}]$ as described in* Definition 2.1.

1. *If $I = (x_{i_1}{}^{a_1}, \ldots, x_{i_r}{}^{a_r})$ where the a_j are positive integers, then*

$$\mathcal{P}(I) = \bigcap_{\substack{1 \le c_j \le a_j \\ 1 \le j \le r}} (x_{i_1,c_1}, \ldots, x_{i_r,c_r})$$

2. *If $I = (x_{i_1}, \ldots, x_{i_r})^m$, where $1 \le i_1, \ldots, i_r \le n$ and m is a positive integer, then $\mathcal{P}(I)$ has the following irredundant irreducible primary decomposition:*

$$\mathcal{P}(I) = \bigcap_{\substack{1 \le c_j \le m \\ \Sigma c_j \le m+r-1}} (x_{i_1,c_1}, \ldots, x_{i_r,c_r})$$

3. *Suppose that $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_m$ is the unique irredundant irreducible primary decomposition of I, such that for each $i = 1, \ldots, m$,*

$$\mathfrak{q}_i = (x_1{}^{a^i_1}, \ldots, x_n{}^{a^i_n}),$$

where the a^i_j are nonnegative integers, and if $a^i_j = 0$ we assume that $x_j{}^{a^i_j} = 0$. Then $\mathcal{P}(I)$ has the following irreducible primary decomposition (some primes might be repeated).

$$\mathcal{P}(I) = \bigcap_{1 \le i \le m} \bigcap_{\substack{1 \le c_j \le a^i_j \\ 1 \le j \le n}} (x_{1,c_1}, \ldots, x_{n,c_n})$$

where when $a^i_j = 0$, we assume that $c_j = x_{j,0} = 0$.

Proof:

1. We know that

$$\mathcal{P}(I) = (x_{i_1,1} \ldots x_{i_1,a_1}, \ldots, x_{i_1,1} \ldots x_{i_1,a_r}).$$

Clearly the minimal primes of $\mathcal{P}(I)$ are $(x_{i_1,c_1}, \ldots, x_{i_r,c_r})$ for all $c_j \le a_j$. This settles the claim.

2. Assume, without loss of generality, that $I = (x_1, \ldots, x_r)^m$. So we can write

$$I = \left(x_1{}^{b_1} \ldots x_r{}^{b_r} \mid 0 \le b_i \le m,\ b_1 + \cdots + b_r = m\right)$$

so that

$$\mathcal{P}(I) = \left(x_{1,1} \ldots x_{1,b_1} \ldots x_{r,1} \ldots x_{r,b_r} \mid 0 \le b_i \le m,\ b_1 + \cdots + b_r = m\right).$$

We first show that $\mathcal{P}(I)$ is contained in the intersection of the ideals of the form $(x_{1,c_1},\dots,x_{r,c_r})$ described above. It is enough to show this for each generator of $\mathcal{P}(I)$. So we show that

$$\mathcal{U}=x_{1,1}\dots x_{1,b_1}\dots x_{r,1}\dots x_{r,b_r}\in I=(x_{1,c_1},\dots,x_{r,c_r})$$

where $0\le b_i\le m$, $b_1+\cdots+b_r=m$, $1\le c_j\le m$ and $c_1+\cdots+c_r\le m+r-1$.

If for any i, $b_i\ge c_i$, then it would be clear that $\mathcal{U}\in I$.

Assume $b_i\le c_i-1$ for $i=1,\dots,r-1$. It follows that

$$\begin{aligned} m-b_r &= b_1+\cdots+b_{r-1}\\ &\le c_1+\cdots+c_{r-1}-(r-1)\\ &\le m+r-1-c_r-(r-1)\\ &= m-c_r \end{aligned}$$

which implies that $b_r\ge c_r$, hence $\mathcal{U}\in I$.

So far we have shown one direction of the inclusion.

To show the opposite direction, take any monomial

$$\mathcal{U}\in\bigcap(x_{1,c_1},\dots,x_{r,c_r})$$

where $1\le c_j\le m$ and $c_1+\cdots+c_r\le m+r-1$.

Notice that for some $i\le r$, $x_{i,1}\mid\mathcal{U}$; this is because $\mathcal{U}\in(x_{1,1},\dots,x_{r,1})$.

We write $\mathcal{U}$ as

$$\mathcal{U}=x_{1,1}\dots x_{1,b_1}\dots x_{r,1}\dots x_{r,b_r}\mathcal{U}'$$

where $\mathcal{U}'$ is a monomial, and the b_i are nonnegative integers such that for each $j<b_i$, $x_{i,j}\mid\mathcal{U}$ (if $x_{i,1}\nmid\mathcal{U}$ then set $b_i=0$). We need to show that it is possible to find such b_i so that $b_1+\cdots+b_r=m$.

Suppose $b_1+\cdots+b_r\le m-1$, and $x_{i,b_i+1}\nmid\mathcal{U}$ for $1\le i\le r$. Then

$$b_1+\cdots+b_r+r\le m+r-1,$$

hence

$$\mathcal{U}\in(x_{1,b_1+1},\dots,x_{r,b_r+1})$$

implying that $x_{i,b_i+1}\mid\mathcal{U}$ for some i, which is a contradiction.

Therefore $b_1,\dots,b_r$ can be picked so that they add up to m, and hence $\mathcal{U}\in\mathcal{P}(I)$; this settles the opposite inclusion.

3. This follows from part 1 and Proposition 2.3 part 3.

$\square$

Corollary 2.6 (polarization and associated primes) *Let I be a monomial ideal in a polynomial ring $R = k[x_1, \ldots, x_n]$, and let $\mathcal{P}(I)$ be its polarization in $S = k[x_{i,j}]$ as described in* Definition 2.1. *Then $(x_{i_1}, \ldots, x_{i_r}) \in \mathrm{Ass}_R(R/I)$ if and only if $(x_{i_1,c_1}, \ldots, x_{i_r,c_r}) \in \mathrm{Ass}_S(S/\mathcal{P}(I))$ for some positive integers $c_1, \ldots, c_r$. Moreover, if $(x_{i_1,c_1}, \ldots, x_{i_r,c_r}) \in \mathrm{Ass}_S(S/\mathcal{P}(I))$, then $(x_{i_1,b_1}, \ldots, x_{i_r,b_r}) \in \mathrm{Ass}_S(S/\mathcal{P}(I))$ for all b_j such that $1 \le b_j \le c_j$.*

Example 2.7 Consider the primary decomposition of $J = ({x_1}^2, {x_2}^3, x_1x_2)$:

$$J = (x_1, {x_2}^3) \cap ({x_1}^2, x_2).$$

By Proposition 2.5, $\mathcal{P}(J) = (x_{1,1}x_{1,2}, x_{2,1}x_{2,2}x_{2,3}, x_{1,1}x_{2,1})$ will have primary decomposition

$$\mathcal{P}(J) = (x_{1,1}, x_{2,1}) \cap (x_{1,1}, x_{2,2}) \cap (x_{1,1}, x_{2,3}) \cap (x_{1,2}, x_{2,1}).$$

A very useful property of polarization is that the final polarized ideal is related to the original ideal via a regular sequence. The proposition below, which looks slightly different here than the original statement in [**8**], states this fact.

Proposition 2.8 (Fröberg [8]) *Let k be a field and*

$$R = k[x_1, \ldots, x_n]/(M_1, \ldots, M_q),$$

where $M_1, \ldots, M_q$ are monomials in the variables $x_1, \ldots, x_n$, and let

$$N_1 = \mathcal{P}(M_1), \ldots, N_q = \mathcal{P}(M_q)$$

be a set of square-free monomials in the polynomial ring

$$S = k[x_{i,j} \mid 1 \le i \le n, 1 \le j \le a_i]$$

such that for each i, the variable x_{i,a_i} appears in at least one of the monomials $N_1, \ldots, N_q$. Then the sequence of elements

$$x_{i,1} - x_{i,j} \text{ where } 1 \le i \le n \text{ and } 1 < j \le a_i \tag{1}$$

forms a regular sequence in the quotient ring

$$R' = S/(N_1, \ldots, N_q)$$

and if J is the ideal of R' generated by the elements in (1)*, then*

$$R = R'/J.$$

Moreover, R is Cohen-Macaulay (Gorenstein) if and only if R' is.

Example 2.9 Let J and R be as in Example 2.2. According to Proposition 2.8, the sequence

$$x_{1,1}-x_{1,2},\ x_{2,1}-x_{2,2},\ x_{2,1}-x_{2,3}$$

is a regular sequence in $S/\mathcal{P}(J)$, and

$$R/J = S/\big(\mathcal{P}(J)+(x_{1,1}-x_{1,2},x_{2,1}-x_{2,2},x_{2,1}-x_{2,3})\big).$$

3 Square-free monomial ideals as facet ideals

Now that we have introduced polarization as a method of transforming a monomial ideal into a square-free one, we can focus on square-free monomial ideals. In particular, here we are interested in properties of square-free monomial ideals that come as a result of them being considered as facet ideals of simplicial complexes. Below we review the basic definitions and notations in facet ideal theory, as well as some of the basic concepts of Stanley-Reisner theory. We refer the reader to [**2**], [**5**], [**6**], [**7**], and [**14**] for more details and proofs in each of these topics.

Definition 3.1 (simplicial complex, facet, subcollection and more) A *simplicial complex* Δ over a set of vertices $V=\{v_1,\dots,v_n\}$ is a collection of subsets of V, with the property that $\{v_i\}\in\Delta$ for all i, and if $F\in\Delta$ then all subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ, and the *dimension* of a face F of Δ is defined as $|F|-1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively, and $\dim\emptyset=-1$. The maximal faces of Δ under inclusion are called *facets* of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets.

We denote the simplicial complex Δ with facets $F_1,\dots,F_q$ by

$$\Delta=\langle F_1,\dots,F_q\rangle$$

and we call $\{F_1,\dots,F_q\}$ the *facet set* of Δ. A simplicial complex with only one facet is called a *simplex*. By a *subcollection* of Δ we mean a simplicial complex whose facet set is a subset of the facet set of Δ.

Definition 3.2 (connected simplicial complex) A simplicial complex $\Delta=\langle F_1,\dots,F_q\rangle$ is *connected* if for every pair i,j, $1\le i<j\le q$, there exists a sequence of facets $F_{t_1},\dots,F_{t_r}$ of Δ such that $F_{t_1}=F_i$, $F_{t_r}=F_j$ and $F_{t_s}\cap F_{t_{s+1}}\neq\emptyset$ for $s=1,\dots,r-1$.

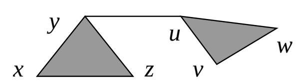
Definition 3.3 (facet/non-face ideals and complexes) Consider a polynomial ring $R=k[x_1,\dots,x_n]$ over a field k and a set of indeterminates $x_1,\dots,x_n$. Let $I=(M_1,\dots,M_q)$ be an ideal in R, where $M_1,\dots,M_q$ are square-free monomials that form a minimal set of generators for I.

- The *facet complex* of I, denoted by $\delta_{\mathcal{F}}(I)$, is the simplicial complex over a set of vertices $v_1, \ldots, v_n$ with facets $F_1, \ldots, F_q$, where for each i, $F_i = \{v_j \mid x_j | M_i, 1 \leq j \leq n\}$. The *non-face complex* or the *Stanley-Reisner complex* of I, denoted by $\delta_{\mathcal{N}}(I)$ is be the simplicial complex over a set of vertices $v_1, \ldots, v_n$, where $\{v_{i_1}, \ldots, v_{i_s}\}$ is a face of $\delta_{\mathcal{N}}(I)$ if and only if $x_{i_1} \ldots x_{i_s} \notin I$.

- Conversely, if Δ is a simplicial complex over n vertices labeled $v_1, \ldots, v_n$, we define the *facet ideal* of Δ, denoted by $\mathcal{F}(\Delta)$, to be the ideal of R generated by square-free monomials $x_{i_1} \ldots x_{i_s}$, where $\{v_{i_1}, \ldots, v_{i_s}\}$ is a facet of Δ. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by $\mathcal{N}(\Delta)$, is the ideal of R generated by square-free monomials $x_{i_1} \ldots x_{i_s}$, where $\{v_{i_1}, \ldots, v_{i_s}\}$ is not a face of Δ.

Throughout this paper we often use a letter x to denote both a vertex of Δ and the corresponding variable appearing in $\mathcal{F}(\Delta)$, and $x_{i_1} \ldots x_{i_r}$ to denote a facet of Δ as well as a monomial generator of $\mathcal{F}(\Delta)$.

Example 3.4 If Δ is the simplicial complex $\langle xyz, yu, uvw \rangle$ drawn below,



then $\mathcal{F}(\Delta) = (xyz, yu, uvw)$ and $\mathcal{N}(\Delta) = (xu, xv, xw, yv, yw, zu, zv, zw)$ are its facet ideal and nonface (Stanley-Reisner) ideal, respectively.

Facet ideals give a one-to-one correspondence between simplicial complexes and square-free monomial ideals.

Next we define the notion of a vertex cover. The combinatorial idea here comes from graph theory. In algebra, it corresponds to prime ideals lying over the facet ideal of a given simplicial complex.

Definition 3.5 (vertex covering, independence, unmixed) Let Δ be a simplicial complex with vertex set V. A *vertex cover* for Δ is a subset A of V that intersects every facet of Δ. If A is a minimal element (under inclusion) of the set of vertex covers of Δ, it is called a *minimal vertex cover*. The smallest of the cardinalities of the vertex covers of Δ is called the *vertex covering number* of Δ and is denoted by $\alpha(\Delta)$. A simplicial complex Δ is *unmixed* if all of its minimal vertex covers have the same cardinality.

A set $\{F_1, \ldots, F_u\}$ of facets of Δ is called an *independent set* if $F_i \cap F_j = \emptyset$ whenever $i \neq j$. The maximum possible cardinality of an independent set of facets in Δ, denoted by $\beta(\Delta)$, is called the *independence number* of Δ. An independent set of facets which is not a proper subset of any other independent set is called a *maximal independent set* of facets.

Example 3.6 If Δ is the simplicial complex in Example 3.4, then the vertex covers of Δ are:

$$\{\mathbf{x},\mathbf{u}\},\{\mathbf{y},\mathbf{u}\},\{\mathbf{y},\mathbf{v}\},\{\mathbf{y},\mathbf{w}\},\{\mathbf{z},\mathbf{u}\},\{x,y,u\},\{x,z,u\},\{x,y,v\},\ldots.$$

The first five vertex covers above (highlighted in bold), are the minimal vertex covers of Δ. It follows that $\alpha(\Delta) = 2$, and Δ is unmixed. On the other hand, $\{xyz, uvw\}$ is the largest maximal independent set of facets that Δ contains, and so $\beta(\Delta) = 2$.

Definition 3.7 (Alexander dual) Let I be a square-free monomial ideal in the polynomial ring $k[V]$ with $V = \{x_1,\ldots,x_n\}$, and let Δ_N be the non-face complex of I (i.e. $\Delta_N = \delta_{\mathcal{N}}(I)$). Then the *Alexander dual* of Δ_N is the simplicial complex

$$\Delta_N{}^{\vee} = \{F \subset V \mid F^c \notin \Delta_N\}$$

where F^c is the complement of the face F in V.

We call the nonface ideal of $\Delta_N{}^{\vee}$ the *Alexander dual* of I and denote it by $I^{\vee}$.

3.1 Simplicial Trees

Considering simplicial complexes as higher dimensional graphs, one can define the notion of a *tree* by extending the same concept from graph theory. Before we define a tree, we determine what "removing a facet" from a simplicial complex means. We define this idea so that it corresponds to dropping a generator from the facet ideal of the complex.

Definition 3.8 (facet removal) Suppose Δ is a simplicial complex with facets F_1, $\ldots,F_q$ and $\mathcal{F}(\Delta) = (M_1,\ldots,M_q)$ its facet ideal in $R = k[x_1,\ldots,x_n]$. The simplicial complex obtained by *removing the facet* F_i from Δ is the simplicial complex

$$\Delta \setminus \langle F_i \rangle = \langle F_1,\ldots,\hat{F}_i,\ldots,F_q\rangle$$

and $\mathcal{F}(\Delta \setminus \langle F_i \rangle) = (M_1,\ldots,\hat{M}_i,\ldots,M_q)$.

The definition that we give below for a simplicial tree is one generalized from graph theory. See [**5**] and [**6**] for more on this concept.

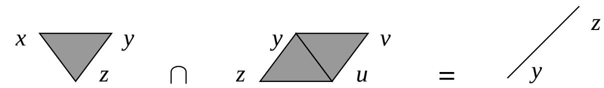
Definition 3.9 (leaf, joint) A facet F of a simplicial complex is called a *leaf* if either F is the only facet of Δ, or for some facet $G \in \Delta \setminus \langle F \rangle$ we have

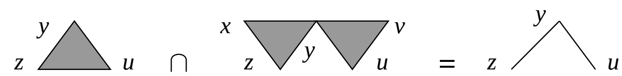
$$F \cap (\Delta \setminus \langle F \rangle) \subseteq G.$$

If $F \cap G \neq \emptyset$, the facet G above is called a *joint* of the leaf F.

Equivalently, a facet F is a leaf of Δ if $F \cap (\Delta \setminus \langle F \rangle)$ is a face of $\Delta \setminus \langle F \rangle$.

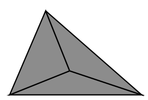
Example 3.10 Let $I = (xyz, yzu, zuv)$. Then $F = xyz$ is a leaf, but $H = yzu$ is not, as one can see in the picture below.

$F \cap (\Delta \setminus \langle F \rangle) =$ 

$H \cap (\Delta \setminus \langle H \rangle) =$ 

Definition 3.11 (tree, forest) A connected simplicial complex Δ is a *tree* if every nonempty subcollection of Δ has a leaf. If Δ is not necessarily connected, but every subcollection has a leaf, then Δ is called a forest.

Example 3.12 The simplicial complexes in examples 3.4 and 3.10 are both trees, but the one below is not because it has no leaves. It is an easy exercise to see that a leaf must contain a free vertex, where a vertex is *free* if it belongs to only one facet.



One of the most powerful properties of simplicial trees from the point of view of algebra is that they behave well under localization. This property makes it easy to use induction on the number of vertices of a tree for proving its various properties.

Lemma 3.13 (Localization of a tree is a forest) *Let $I \subseteq k[x_1, \ldots, x_n]$ be the facet ideal of a simplicial tree, where k is a field. Then for any prime ideal $\mathfrak{p}$ of $k[x_1, \ldots, x_n]$, $\delta_{\mathcal{F}}(I_{\mathfrak{p}})$ is a forest.*

Proof: See [**6**, Lemma 4.5]. $\square$

4 Properties of monomial ideals via polarization

For the purpose of all discussions in this section, unless otherwise stated, let I be a monomial ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k, whose polarization is the square-free monomial ideal $\mathcal{P}(I)$ in the polynomial ring

$$S = k[x_{i,j} \mid 1 \le i \le n, 1 \le j \le a_i].$$

We assume that the polarizing sequence (as described in (1) in Proposition 2.8) is

$$\nu = \nu_1, \dots, \nu_v$$

which is a regular sequence in $S/\mathcal{P}(I)$ and

$$R/I = S/\big(\mathcal{P}(I) + (\nu)\big).$$

4.1 Monomial ideals whose polarization is a simplicial tree

A natural question, and one that this paper is mainly concerned with, is what properties of facet ideals of simplicial trees can be extended to general (non-square-free) monomial ideals using polarization? In other words, if for a monomial ideal I in a polynomial ring $\mathcal{P}(I)$ is the facet ideal of a tree (Definition 3.11), then what properties of $\mathcal{P}(I)$ are inherited by I?

The strongest tool when dealing with square-free monomial ideals is induction– either on the number of generators, or the number of variables in the ambient polynomial ring. This is particularly the case when the facet complex of the ideal is a tree, or in some cases when it just has a leaf. In this section we show that via polarization, one can extend these tools to monomial ideals in general. For a given monomial ideal I, we show that if $\mathcal{P}(I)$ is the facet ideal of a tree, and $\mathfrak{p}$ is a prime ideal containing I, then $\mathcal{P}(I_\mathfrak{p})$ is the facet ideal of a forest (Theorem 4.1); this allows induction on number of variables. Similarly, Theorem 4.3 provides us with a way to use induction on number of generators of I.

Theorem 4.1 (localization and polarization) *If $\mathcal{P}(I)$ is the facet ideal of a tree, and $\mathfrak{p}$ is a prime ideal of R containing I, then $\mathcal{P}(I_\mathfrak{p})$ is the facet ideal of a forest.*

Proof: The first step is to show that it is enough to prove this for prime ideals of R generated by a subset of $\{x_1, \dots, x_n\}$. To see this, assume that $\mathfrak{p}$ is a prime ideal of R containing I, and that $\mathfrak{p}'$ is another prime of R generated by all $x_i \in \{x_1, \dots, x_n\}$ such that $x_i \in \mathfrak{p}$ (recall that the minimal primes of I are generated by subsets of $\{x_1, \dots, x_n\}$; see [**16**, Corollary 5.1.5]). So $\mathfrak{p}' \subseteq \mathfrak{p}$. If $I = (M_1, \dots, M_q)$, then

$$I_{\mathfrak{p}'} = ({M_1}', \dots, {M_q}')$$

where for each i, ${M_i}'$ is the image of M_i in $I_{\mathfrak{p}'}$. In other words, ${M_i}'$ is obtained by dividing $M_i = {x_1}^{a_1} \dots {x_n}^{a_n}$ by the product of all the ${x_j}^{a_j}$ such that $x_j \notin \mathfrak{p}'$. But $x_j \notin \mathfrak{p}'$ implies that $x_j \notin \mathfrak{p}$, and so it follows that ${M_i}' \in I_\mathfrak{p}$. Therefore $I_{\mathfrak{p}'} \subseteq I_\mathfrak{p}$. On the other hand since $\mathfrak{p}' \subseteq \mathfrak{p}$, $I_\mathfrak{p} \subseteq I_{\mathfrak{p}'}$, which implies that $I_{\mathfrak{p}'} = I_\mathfrak{p}$ (the equality and inclusions of the ideals here mean equality and inclusion of their generating sets).

Now suppose $I = (M_1, \dots, M_q)$, and $\mathfrak{p} = (x_1, \dots, x_r)$ is a prime containing I. Suppose that for each i, we write $M_i = M_i'.M_i''$, where

$$M_i' \in k[x_1, \dots, x_r] \text{ and } M_i'' \in k[x_{r+1}, \dots, x_n]$$

so that

$$I_{\mathfrak{p}} = (M'_1, \dots, M'_t),$$

where without loss of generality $M'_1, \dots, M'_t$ is a minimal generating set for $I_{\mathfrak{p}}$.

We would like to show that the facet complex Δ of $\mathcal{P}(I_{\mathfrak{p}})$ is a forest. Suppose that, again without loss of generality,

$$I' = (\mathcal{P}(M'_1), \dots, \mathcal{P}(M'_s))$$

is the facet ideal of a subcollection Δ' of Δ. We need to show that Δ' has a leaf.

If $s = 1$, then there is nothing to prove. Otherwise, suppose that $\mathcal{P}(M_1)$ represents a leaf of the tree $\delta_{\mathcal{F}}(\mathcal{P}(I))$, and $\mathcal{P}(M_2)$ is a joint of $\mathcal{P}(M_1)$. Then we have

$$\mathcal{P}(M_1) \cap \mathcal{P}(M_i) \subseteq \mathcal{P}(M_2) \ \textit{for all} \ i \in \{2, \dots, s\}.$$

Now let $x_{e,f}$ be in $\mathcal{P}(M'_1) \cap \mathcal{P}(M'_i)$ for some $i \in \{2, \dots, s\}$. This implies that

(*i*) $x_{e,f} \in \mathcal{P}(M_1) \cap \mathcal{P}(M_i) \subseteq \mathcal{P}(M_2)$, and

(*ii*) $e \in \{1 \dots, r\}$

From (*i*) and (*ii*) we can conclude that $x_{e,f} \in \mathcal{P}(M'_2)$, which proves that $\mathcal{P}(M'_1)$ is a leaf for Δ'. $\square$

Remark 4.2 It is not true in general that if $\mathfrak{p}$ is a (minimal) prime of I, then $\mathcal{P}(I_{\mathfrak{p}}) = \mathcal{P}(I)_{\mathcal{P}(\mathfrak{p})}$. For example, if $I = ({x_1}^3, {x_1}^2 x_2)$ and $\mathfrak{p} = (x_1)$, then $I_{\mathfrak{p}} = ({x_1}^2)$ so $\mathcal{P}(I_{\mathfrak{p}}) = (x_{1,1}x_{1,2})$, but $\mathcal{P}(I)_{\mathcal{P}(\mathfrak{p})} = (x_{1,1})$.

Another feature of simplicial trees is that they satisfy a generalization of König's theorem ([**6**, Theorem 5.3]). Below we explain how this property, and another property of trees that is very useful for induction, behave under polarization.

Recall that for a simplicial complex Δ, $\alpha(\Delta)$ and $\beta(\Delta)$ are the vertex covering number and the independence number of Δ, respectively (Definition 3.5). For simplicity of notation, if $I = (M_1, \dots, M_q)$ is a monomial ideal, we let $\beta(I)$ denote the maximum cardinality of a subset of $\{M_1, \dots, M_q\}$ consisting of pairwise coprime elements (so $\beta(\Delta) = \beta(\mathcal{F}(\Delta))$ for any simplicial complex Δ).

Theorem 4.3 (joint removal and polarization) *Suppose $M_1, \dots, M_q$ are monomials that form a minimal generating set for I, and $\mathcal{P}(I)$ is the facet ideal of a simplicial complex Δ. Assume that Δ has a leaf, whose joint corresponds to $\mathcal{P}(M_1)$. Then, if we let $I' = (M_2, \dots, M_q)$, we have*

$$\text{height}\, I = \text{height}\, I'.$$

Proof: If G is the joint of Δ corresponding to $\mathcal{P}(M_1)$, then $\mathcal{P}(I') = \mathcal{F}(\Delta \setminus \langle G \rangle)$. From [**6**, Lemma 5.1] it follows that $\alpha(\Delta) = \alpha(\Delta \setminus \langle G \rangle)$, so that height $\mathcal{P}(I) =$ height $\mathcal{P}(I')$, and therefore height $I =$ height I'. $\square$

Theorem 4.4 *Suppose $M_1, \dots, M_q$ are monomials that form a minimal generating set for I, and $\mathcal{P}(I)$ is the facet ideal of a simplicial tree Δ. Then* height $I = \beta(I)$.

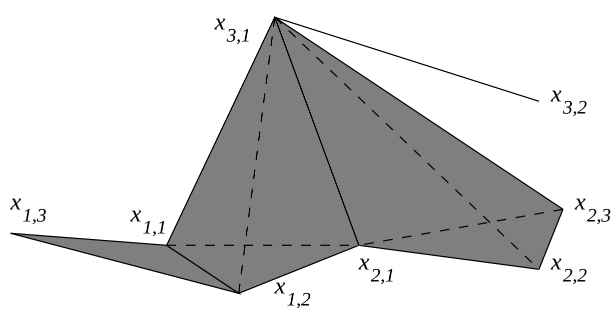
Proof: We already know that height $I =$ height $\mathcal{P}(I) = \alpha(\Delta)$. It is also clear that $\beta(I) = \beta(\mathcal{P}(I))$, since the monomials in a subset $\{M_{i_1}, \dots, M_{i_r}\}$ of the generating set of I are pairwise coprime if and only if the monomials in $\{\mathcal{P}(M_{i_1}), \dots, \mathcal{P}(M_{i_r})\}$ are pairwise coprime. On the other hand, from [**6**, Theorem 5.3] we know that $\alpha(\Delta) = \beta(\Delta)$. Our claim follows immediately. $\square$

We demonstrate how to apply these theorems via an example.

Example 4.5 Suppose $I = ({x_1}^3, {x_1}^2 x_2 x_3, {x_3}^2, {x_2}^3 x_3)$. Then

$$\mathcal{P}(I) = (x_{1,1}x_{1,2}x_{1,3}, x_{1,1}x_{1,2}x_{2,1}x_{3,1}, x_{3,1}x_{3,2}, x_{2,1}x_{2,2}x_{2,3}x_{3,1})$$

is the facet ideal of the following simplicial complex (tree) Δ.



Now $\alpha(\Delta) =$ height $I = 2$ because the prime of minimal height over I is (x_1, x_3). From Theorem 4.4 it follows that $\beta(I) = 2$. This means that you can find a set of two monomials in the generating set of I that have no common variables: for example $\{{x_1}^3, {x_3}^2\}$ is such a set.

Since the monomials ${x_1}^2 x_2 x_3$ and ${x_2}^3 x_3$ polarize into joints of Δ, by Theorem 4.3 the ideals

$$I,\ ({x_1}^3, {x_3}^2, {x_2}^3 x_3),\ ({x_1}^3, {x_1}^2 x_2 x_3, {x_3}^2),\ \text{and } ({x_1}^3, {x_3}^2)$$

all have the same height.

We now focus on the Cohen-Macaulay property. In [**6**] we showed that for a simplicial tree Δ, $\mathcal{F}(\Delta)$ is a Cohen-Macaulay ideal if and only if Δ is an unmixed simplicial complex. The condition unmixed for Δ is equivalent to all minimal primes of the ideal $\mathcal{F}(\Delta)$ (which in this case are all the associated primes of $\mathcal{F}(\Delta)$) having the same height. In general, an ideal all whose associated primes have the same height (equal to the height of the ideal) is called an *unmixed ideal*.

It now follows that

Theorem 4.6 (Cohen-Macaulay criterion for trees) *Let $\mathcal{P}(I)$ be the facet ideal of a simplicial tree Δ. Then R/I is Cohen-Macaulay if and only if I is unmixed.*

Proof: From Proposition 2.8, R/I is Cohen-Macaulay if and only if $S/\mathcal{P}(I)$ is Cohen-Macaulay. By [**6**, Corollary 8.3], this is happens if and only if $\mathcal{P}(I)$ is unmixed. Corollary 2.6 now proves the claim. $\square$

If $\mathcal{R}$ is a ring and J is an ideal of R, then the Rees ring of $\mathcal{R}$ along J is defined as

$$\mathcal{R}[Jt] = \oplus_{n \in \mathbb{N}} J^n t^n.$$

Rees rings come up in the algebraic process of "blowing up" ideals. One reason that trees were defined as they are, is that their facet ideals produce normal and Cohen-Macaulay Rees rings ([**5**]).

Proposition 4.7 *If $S[\mathcal{P}(I)t]$ is Cohen-Macaulay, then so is $R[It]$. Conversely, if we assume that R and S are localized at their irrelevant maximal ideals, then $R[It]$ being Cohen-Macaulay implies that $S[\mathcal{P}(I)t]$ is Cohen-Macaulay.*

Proof: Suppose that $\nu_1, \dots, \nu_v$ is the polarizing sequence as described before. For $i = 1, \dots, v-1$ let

$$R_i = S/(\nu_1, \dots, \nu_i),\ I_i = \mathcal{P}(I)/(\nu_1, \dots, \nu_i),\ R_v = R \text{ and } I_v = I.$$

Notice that $S[\mathcal{P}(I)t]$ and $R[It]$ are both domains. Also note that for each i,

$$S[\mathcal{P}(I)t]/(\nu_1, \dots, \nu_i) = R_i[I_i t]$$

is the Rees ring of the monomial ideal I_i in the polynomial ring R_i, and is therefore also a domain. Therefore ν_{i+1} is a regular element in the ring $S[\mathcal{P}(I)t]/(\nu_1, \dots, \nu_i)$, which means that $\nu_1, \dots, \nu_v$ is a regular sequence in $S[\mathcal{P}(I)t]$.

Similarly, we see that

$$R[It] = S[\mathcal{P}(I)t]/(\nu_1, \dots, \nu_v).$$

[**2**, Theorem 2.1.3] now implies that if $S[\mathcal{P}(I)t]$ is Cohen-Macaulay, then so is $R[It]$. The converse follows from [**2**, Exercise 2.1.28]. $\square$

Corollary 4.8 (Rees ring of a tree is Cohen-Macaulay) *Suppose that $\mathcal{P}(I)$ is the facet ideal of a simplicial tree. Then the Rees ring $R[It]$ of I is Cohen-Macaulay.*

Proof: This follows from the Proposition 4.7, and from [**5**, Corollary 4], which states that the Rees ring of the facet ideal of a simplicial tree is Cohen-Macaulay. $\square$

4.2 Polarization of sequentially Cohen-Macaulay ideals

The main result of this section is that if the polarization of a monomial ideal I is the facet ideal of a tree, then I is a sequentially Cohen-Macaulay ideal. The theorem that implies this fact (Proposition 4.11) is interesting in its own right. For a square-free monomial ideal J, Eagon and Reiner [**4**] proved that J is Cohen-Macaulay if and only if its Alexander dual $J^\vee$ has a linear resolution. Herzog and Hibi [**9**] then defined componentwise linear ideals and generalized their result, so that a square-free monomial ideal J is sequentially Cohen-Macaulay if and only if $J^\vee$ is componentwise linear (see [**9**] or [**7**]). But even though Alexander duality has been generalized to all monomial ideals from square-free ones, the criterion for sequential Cohen-Macaulayness does not generalize: it is not true that if I is any monomial ideal, then I is sequentially Cohen-Macaulay if and only of $I^\vee$ is a componentwise linear ideal; see Miller [**10**]. We show that the statement is true if $I^\vee$ is replaced by $\mathcal{P}(I)^\vee$.

Definition 4.9 ([14, Chapter III, Definition 2.9]) Let M be a finitely generated $\mathbb{Z}$-graded module over a finitely generated $\mathbb{N}$-graded k-algebra, with $R_0 = k$. We say that M is *sequentially Cohen-Macaulay* if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

of M by graded submodules M_i satisfying the following two conditions.

(a) Each quotient M_i/M_{i-1} is Cohen-Macaulay.

(b) $\dim\,(M_1/M_0) < \dim\,(M_2/M_1) < \ldots < \dim\,(M_r/M_{r-1})$, where dim denotes Krull dimension.

We define a componentwise linear ideal in the square-free case using [**9**, Proposition 1.5].

Definition 4.10 (componentwise linear) Let I be a square-free monomial ideal in a polynomial ring R. For a positive integer k, the k-th *square-free homogeneous component* of I, denoted by $I_{[k]}$ is the ideal generated by all square-free monomials in I of degree k. The ideal I above is said to be *componentwise linear* if for all k, the square-free homogeneous component $I_{[k]}$ has a linear resolution.

For our monomial ideal I, let $Q(I)$ denote the set of primary ideals appearing in a reduced primary decomposition of I. Suppose that $h = \text{height } I$ and $s = \max\{\text{height } \mathfrak{q} \mid \mathfrak{q} \in Q(I)\}$, and set

$$I_i = \bigcap_{\substack{\mathfrak{q}\in Q(I)\\ \text{height } \mathfrak{q}\leq s-i}} \mathfrak{q}.$$

So we have the following filtration for R/I (we assume that all inclusions in the filtration are proper; if there is an equality anywhere, we just drop all but one of the equal ideals).

$$0 = I = I_0 \subset I_1 \subset \ldots \subset I_{s-h} \subset R/I. \tag{2}$$

If R/I is sequentially Cohen-Macaulay, then by Theorem A.4, (2) is the appropriate filtration that satisfies Conditions (a) and (b) in Definition 4.9.

For the square-free monomial ideal $J = \mathcal{P}(I)$, we similarly define

$$Q(J) = \{\text{minimal primes over } J\} \text{ and } J_i = \bigcap_{\substack{\mathfrak{p}\in Q(J)\\ \text{height } \mathfrak{p}\leq s-i}} \mathfrak{p}$$

where the numbers h and s are the same as for I because of Proposition 2.5. It follows from Proposition 2.5 and Corollary 2.6 that for each i, $\mathcal{P}(I_i) = J_i$ and the polarization sequence that transforms I_i into J_i is a subsequence of $\nu = \nu_1, \ldots, \nu_v$.

What we have done so far is to translate, via polarization, the filtration (2) of the quotient ring R/I into one of S/J:

$$0 = J = J_0 \subset J_1 \subset \ldots \subset J_{s-h} \subset S/J. \tag{3}$$

Now note that for a given i, the sequence ν is a J_{i+1}/J_i-regular sequence in S, as ν is a regular sequence in S/J_i, which contains J_{i+1}/J_i. Also note that

$$R/I_i \simeq S/(J_i + \nu).$$

It follows that J_{i+1}/J_i is Cohen-Macaulay if and only if I_{i+1}/I_i is (see [**2**, Exercise 2.1.27(c), Exercise 2.1.28, and Theorem 2.1.3]).

Proposition 4.11 *The monomial ideal I is sequentially Cohen-Macaulay if and only if $\mathcal{P}(I)$ is sequentially Cohen-Macaulay, or equivalently, $\mathcal{P}(I)^\vee$ is a componentwise linear ideal.*

Proof: By [**7**, Proposition 4.5], $\mathcal{P}(I)^\vee$ is componentwise linear if and only if $\mathcal{P}(I)$ is sequentially Cohen-Macaulay, which by the discussion above is equivalent to I being sequentially Cohen-Macaulay. $\square$

Corollary 4.12 (Trees are sequentially Cohen-Macaulay) *Let $\mathcal{P}(I)$ be the facet ideal of a simplicial tree. Then I is sequentially Cohen-Macaulay.*

Proof: This follows from Proposition 4.11 and by [**7**, Corollary 5.5], which states that $\mathcal{P}(I)^{\vee}$ is a componentwise linear ideal. $\square$

5 Further examples and remarks

To use the main results of this paper for computations on a given monomial ideal, there are two steps. One is to compute the polarization of the ideal, which as can be seen from the definition, is a quick and simple procedure. This has already been implemented in Macaulay2. The second step is to determine whether the polarization is the facet ideal of a tree, or has a leaf. Algorithms that serve this purpose are under construction [**3**].

Remark 5.1 Let $I = (M_1, \dots, M_q)$ be a monomial ideal in a polynomial ring R. If $\mathcal{P}(I)$ is the facet ideal of a tree, then by Corollary 4.8, $R[It]$ is Cohen-Macaulay. But more is true: if you drop any generator of I, for example if you consider $I' = (M_1, \dots, \hat{M}_i, \dots, M_q)$, then $R[I't]$ is still Cohen-Macaulay. This is because $\mathcal{P}(I')$ corresponds to the facet ideal of a forest, so one can apply the same result.

A natural question is whether one can say the same with the property "Cohen-Macaulay" replaced by "normal". If I is square-free, this is indeed the case. But in general, polarization does not preserve normality of ideals.

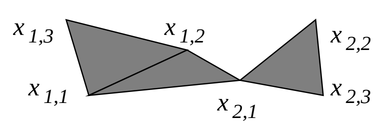
Example 5.2 (normality and polarization) A valid question is whether Proposition 4.7 holds if the word "Cohen-Macaulay" is replaced with "normal", given that simplicial trees have normal facet ideals ([**5**])?

The answer is negative. Here is an example.

Let $I = ({x_1}^3, {x_1}^2x_2, {x_2}^3)$ be an ideal of $k[x_1, x_2]$. Then I is not normal; this is because I is not even integrally closed: $x_1{x_2}^2 \in \overline{I}$ as $(x_1{x_2}^2)^3 - {x_1}^3{x_2}^6 = 0$, but $x_1{x_2}^2 \notin I$. Now

$$\mathcal{P}(I) = (x_{1,1}x_{1,2}x_{1,3}, x_{1,1}x_{1,2}x_{2,1}, x_{2,1}x_{2,2}x_{2,3})$$

is the facet ideal of the tree



which is normal by [**5**].

The reason that normality (or integral closure in general) does not pass through polarization is much more basic: polarization does not respect multiplication of ideals, or monomials. Take, for example, two monomials M and N and two monomial ideals I and J, such that $MN \in IJ$. It is not necessarily true that $\mathcal{P}(M)\mathcal{P}(N) \in \mathcal{P}(I)\mathcal{P}(J)$.

Indeed, let $I = J = (x_1x_2)$ and $M = {x_1}^2$ and $N = {x_2}^2$. Then $MN = {x_1}^2{x_2}^2 \in IJ$. But

$$\mathcal{P}(M) = x_{1,1}x_{1,2},\ \mathcal{P}(N) = x_{2,1}x_{2,2},\ \text{and } \mathcal{P}(I) = \mathcal{P}(J) = (x_{1,1}x_{2,1})$$

and clearly $\mathcal{P}(M)\mathcal{P}(N) \notin \mathcal{P}(I)\mathcal{P}(J) = ({x_{1,1}}^2{x_{2,1}}^2)$.

Remark 5.3 It is useful to think of polarization as a chain of substitutions. This way, as a monomial ideal I gets polarized, the ambient ring extends one variable at a time. All the in-between ideals before we hit the final square-free ideal $\mathcal{P}(I)$ have the same polarization.

For example let $J = (x^2, xy, y^3) \subseteq k[x,y]$. We use a diagram to demonstrate the process described in the previous paragraph. Each linear form $a - b$ stands for "replacing the variable b with a", or vice versa, depending on which direction we are going.

$$J = (x^2, xy, y^3) \xrightarrow{u-x} J_1 = (xu, uy, y^3) \xrightarrow{v-y} J_2 = (xu, uv, y^2v)$$
$$\xrightarrow{w-y} J_3 = (xu, uv, yvw)\ .$$

This final square-free monomial ideal J_3 is the polarization of J, and is isomorphic to $\mathcal{P}(J)$ as we defined it in Definition 2.1. Note that all ideals J, J_1, J_2 and J_3 have the same (isomorphic) polarization. We can classify monomial ideals according to their polarizations. An interesting question is to see what properties do ideals in the same polarization class have. A more difficult question is how far can one "depolarize" a square-free monomial ideal I, where by depolarizing I we mean finding monomial ideals whose polarization is equal to I, or equivalently, traveling the opposite direction on the above diagram.

A Appendix: Primary decomposition in a sequentially Cohen-Macaulay module

The purpose of this appendix is to study, using basic facts about primary decomposition of modules, the structure of the submodules appearing in the (unique) filtration of a sequentially Cohen-Macaulay module M. The main result (Theorem A.4) states that each submodule appearing in the filtration of M is the intersection of all primary submodules whose associated primes have a certain height and appear in an irredundant primary decomposition of the 0-submodule of M. Similar results, stated in a different language, appear in [**12**]; the author thanks Jürgen Herzog for pointing this out.

We first record two basic lemmas that we shall use later (the second one is an exercise in Bourbaki [**1**]). Throughout the discussions below, we assume that R is a finitely generated algebra over a field, and M is a finite module over R.

Lemma A.1 *Let $Q_1, \dots, Q_t, \mathcal{P}$ all be primary submodules of an R-module M, such that $\operatorname{Ass}(M/Q_i) = \{\mathfrak{q}_i\}$ and $\operatorname{Ass}(M/\mathcal{P}) = \{\mathfrak{p}\}$. If $Q_1 \cap \dots \cap Q_t \subseteq \mathcal{P}$ and $Q_i \nsubseteq \mathcal{P}$ for some i, then there is a $j \neq i$ such that $\mathfrak{q}_j \subseteq \mathfrak{p}$.*

Proof: Let $x \in Q_i \setminus \mathcal{P}$. For each j not equal to i, pick the positive integer m_j such that $\mathfrak{q}_j^{m_j} x \subseteq Q_j$. So we have that

$$\mathfrak{q}_1^{m_1} \dots \mathfrak{q}_{i-1}^{m_{i-1}} \mathfrak{q}_{i+1}^{m_{i+1}} \dots \mathfrak{q}_t^{m_t} x \subseteq Q_1 \cap \dots \cap Q_t \subseteq \mathcal{P} \implies \mathfrak{q}_1^{m_1} \dots \mathfrak{q}_{i-1}^{m_{i-1}} \mathfrak{q}_{i+1}^{m_{i+1}} \dots \mathfrak{q}_t^{m_t} \subseteq \mathfrak{p}$$

where the second inclusion is because $x \notin \mathcal{P}$. Hence for some $j \neq i$, $\mathfrak{q}_j \subseteq \mathfrak{p}$. □

Lemma A.2 *Let M be an R-module and N be a submodule of M. Then for every $\mathfrak{p} \in \operatorname{Ass}(M/N)$, if $\mathfrak{p} \nsupseteq \operatorname{Ann}(N)$, then $\mathfrak{p} \in \operatorname{Ass}(M)$.*

Proof: Since $\mathfrak{p} \in \operatorname{Ass}(M/N)$, there exists $x \in M \setminus N$ such that $\mathfrak{p} = \operatorname{Ann}(x)$; in other words $\mathfrak{p}x \subseteq N$. Suppose $\operatorname{Ann}(N) \nsubseteq \mathfrak{p}$, and let $y \in \operatorname{Ann}(N) \setminus \mathfrak{p}$. Now $y\mathfrak{p}x = 0$, and so $\mathfrak{p} \subseteq \operatorname{Ann}(yx)$ in M. On the other hand, if $z \in \operatorname{Ann}(yx)$, then $zyx = 0 \subseteq N$ and so $zy \in \mathfrak{p}$. But $y \notin \mathfrak{p}$, so $z \in \mathfrak{p}$. Therefore $\mathfrak{p} \in \operatorname{Ass}(M)$.

□

Suppose M is a sequentially Cohen-Macaulay module with filtration as in Definition 4.9. We adopt the following notation. For a given integer j, we let

$$\operatorname{Ass}(M)_j = \{\mathfrak{p} \in \operatorname{Ass}(M) \mid \text{height } \mathfrak{p} = j\}.$$

Suppose that all the j where $\operatorname{Ass}(M)_j \neq \emptyset$ form the sequence of integers

$$0 \leq h_1 < \dots < h_c \leq \dim R$$

so that $\operatorname{Ass}(M) = \bigcup_{1 \leq j \leq c} \operatorname{Ass}(M)_{h_j}$.

Proposition A.3 *For all $i = 0, \dots, r-1$, we have*

1. $\operatorname{Ass}(M_{i+1}/M_i) \cap \operatorname{Ass}(M) \neq \emptyset$;
2. $\operatorname{Ass}(M)_{h_{r-i}} \subseteq \operatorname{Ass}(M_{i+1}/M_i)$ *and* $c = r$;
3. *If* $\mathfrak{p} \in \operatorname{Ass}(M_{i+1})$, *then* $\text{height } \mathfrak{p} \geq h_{r-i}$;

4. *If* $\mathfrak{p} \in \operatorname{Ass}(M_{i+1}/M_i)$*, then* $\operatorname{Ann}(M_i) \not\subseteq \mathfrak{p}$*;*

5. $\operatorname{Ass}(M_{i+1}/M_i) \subseteq \operatorname{Ass}(M)$*;*

6. $\operatorname{Ass}(M_{i+1}/M_i) = \operatorname{Ass}(M)_{h_{r-i}}$*;*

7. $\operatorname{Ass}(M/M_i) = \operatorname{Ass}(M)_{\leq h_{r-i}}$*;*

8. $\operatorname{Ass}(M_{i+1}) = \operatorname{Ass}(M)_{\geq h_{r-i}}$.

Proof:

1. We use induction on the length r of the filtration of M. The case $r = 1$ is clear, as we have a filtration $0 \subset M$, and the assertion follows. Now suppose the statement holds for sequentially Cohen-Macaulay modules with filtrations of length less than r. Notice that M_{r-1} that appears in the filtration of M in Definition 4.9 is also sequentially Cohen-Macaulay, and so by the induction hypothesis, we have

$$\operatorname{Ass}(M_{i+1}/M_i) \cap \operatorname{Ass}(M_{r-1}) \neq \emptyset \text{ for } i = 0, \ldots, r-2$$

and since $\operatorname{Ass}(M_{r-1}) \subseteq \operatorname{Ass}(M)$ it follows that

$$\operatorname{Ass}(M_{i+1}/M_i) \cap \operatorname{Ass}(M) \neq \emptyset \text{ for } i = 0, \ldots, r-2.$$

It remains to show that $\operatorname{Ass}(M/M_{r-1}) \cap \operatorname{Ass}(M) \neq \emptyset$.

For each i, $M_{i-1} \subset M_i$, so we have ([**1**] Chapter IV)

$$\operatorname{Ass}(M_1) \subseteq \operatorname{Ass}(M_2) \subseteq \operatorname{Ass}(M_1) \cup \operatorname{Ass}(M_2/M_1) \tag{4}$$

The inclusion $M_2 \subseteq M_3$ along with the inclusions in (4) imply that

$$\begin{aligned}\operatorname{Ass}(M_2) \subseteq \operatorname{Ass}(M_3) &\subseteq \operatorname{Ass}(M_2) \cup \operatorname{Ass}(M_3/M_2)\\ &\subseteq \operatorname{Ass}(M_1) \cup \operatorname{Ass}(M_2/M_1) \cup \operatorname{Ass}(M_3/M_2).\end{aligned}$$

If we continue this process inductively, at the i-th stage we have

$$\begin{aligned}\operatorname{Ass}(M_i) &\subseteq \operatorname{Ass}(M_{i-1}) \cup \operatorname{Ass}(M_i/M_{i-1})\\ &\subseteq \operatorname{Ass}(M_1) \cup \operatorname{Ass}(M_2/M_1) \cup \operatorname{Ass}(M_3/M_2) \cup \ldots \cup \operatorname{Ass}(M_i/M_{i-1})\end{aligned}$$

and finally, when $i = r$ it gives

$$\operatorname{Ass}(M) \subseteq \operatorname{Ass}(M_1) \cup \operatorname{Ass}(M_2/M_1) \cup \ldots \cup \operatorname{Ass}(M/M_{r-1}). \tag{5}$$

Because of Condition (b) in Definition 4.9, and the fact that each M_{i+1}/M_i is Cohen-Macaulay (and hence all its associated primes have the same height; see [**2**] Chapter 2), if for every i we pick $\mathfrak{p}_i \in \mathrm{Ass}(M_{i+1}/M_i)$, then

$$h_c \geq \text{height } \mathfrak{p}_0 > \text{height } \mathfrak{p}_1 > \ldots > \text{height } \mathfrak{p}_{r-1}.$$

where the left-hand-side inequality comes from the fact that $\mathrm{Ass}(M_1) \subseteq \mathrm{Ass}(M)$. By our induction hypothesis, $\mathrm{Ass}(M)$ intersects $\mathrm{Ass}(M_{i+1}/M_i)$ for all $i \leq r-2$, and so because of (5) we conclude that

$$\text{height } \mathfrak{p}_i = h_{c-i}, \text{ and } \mathrm{Ass}(M)_{h_{c-i}} \subseteq \mathrm{Ass}(M_{i+1}/M_i) \text{ for } 0 \leq i \leq r-2.$$

And now $\mathrm{Ass}(M)_{h_0}$ has no choice but to be included in $\mathrm{Ass}(M/M_{r-1})$, which settles our claim. It also follows that $c = r$.

2. See the proof for part 1.

3. We use induction. The case $i = 0$ is clear, since for every $\mathfrak{p} \in \mathrm{Ass}(M_1) = \mathrm{Ass}(M_1/M_0)$ we know from part 2 that height $\mathfrak{p} = h_r$. Suppose the statement holds for all indices up to $i-1$. Consider the inclusion

$$\mathrm{Ass}(M_i) \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M_i) \cup \mathrm{Ass}(M_{i+1}/M_i).$$

From part 2 and the induction hypothesis it follows that if $\mathfrak{p} \in \mathrm{Ass}(M_{i+1})$ then height $\mathfrak{p} \geq h_{r-i}$.

4. Suppose $\mathrm{Ann}(M_i) \subseteq \mathfrak{p}$. Since $\sqrt{\mathrm{Ann}(M_i)} = \bigcap_{\mathfrak{p}' \in \mathrm{Ass}(M_i)} \mathfrak{p}'$, it follows that $\bigcap_{\mathfrak{p}' \in \mathrm{Ass}(M_i)} \mathfrak{p}' \subseteq \mathfrak{p}$, so there is a $\mathfrak{p}' \in \mathrm{Ass}(M_i)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$. But from parts 2 and 3 above it follows that height $\mathfrak{p}' \geq h_{r-i+1}$ and height $\mathfrak{p} = h_{r-i}$, which is a contradiction.

5. From part 4 and Lemma A.2, it follows that $\mathrm{Ass}(M_{i+1}/M_i) \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M)$.

6. This follows from parts 2 and 5, and the fact that M_{i+1}/M_i is Cohen-Macaulay, and hence all associated primes have the same height.

7. We show this by induction on $e = r - i$. The case $e = 1$ (or $i = r-1$) is clear, because by part 6 we have $\mathrm{Ass}(M/M_{r-1}) = \mathrm{Ass}(M)_{h_1} = \mathrm{Ass}(M)_{\leq h_1}$.

   Now suppose the equation holds for all integers up to $e-1$ (namely $i = r-e+1$), and we would like to prove the statement for e (or $i = r-e$). Since $M_{i+1}/M_i \subseteq M/M_i$, we have

$$\mathrm{Ass}(M_{i+1}/M_i) \subseteq \mathrm{Ass}(M/M_i) \subseteq \mathrm{Ass}(M_{i+1}/M_i) \cup \mathrm{Ass}(M/M_{i+1}) \qquad (6)$$

By the induction hypothesis and part 6 we know that

$$\mathrm{Ass}(M/M_{i+1}) = \mathrm{Ass}(M)_{\leq h_{r-i-1}} \text{ and } \mathrm{Ass}(M_{i+1}/M_i) = \mathrm{Ass}(M)_{h_{r-i}},$$

which put together with (6) implies that

$$\mathrm{Ass}(M)_{h_{r-i}} \subseteq \mathrm{Ass}(M/M_i) \subseteq \mathrm{Ass}(M)_{\leq h_{r-i}}$$

We still have to show that $\mathrm{Ass}(M/M_i) \supseteq \mathrm{Ass}(M)_{\leq h_{r-i-1}}$.
Let

$$\mathfrak{p} \in \mathrm{Ass}(M)_{\leq h_{r-i-1}} = \mathrm{Ass}(M/M_{i+1}) = \mathrm{Ass}((M/M_i)/(M_{i+1}/M_i)).$$

If $\mathfrak{p} \supseteq \mathrm{Ann}(M_{i+1}/M_i)$, then (by part 6)

$$\mathfrak{p} \supseteq \bigcap_{\mathfrak{q} \in \mathrm{Ass}(M)_{h_{r-i}}} \mathfrak{q} \implies \mathfrak{p} \supseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in \mathrm{Ass}(M)_{h_{r-i}}$$

which is a contradiction, as height $\mathfrak{p} \leq h_{r-i-1} <$ height $\mathfrak{q}$. It follows from Lemma A.2 that $\mathfrak{p} \in \mathrm{Ass}(M/M_i)$.

8. The argument is based on induction, and exactly the same as the one in part 4, using more information; from the inclusions

$$\mathrm{Ass}(M_i) \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M_i) \cup \mathrm{Ass}(M_{i+1}/M_i),$$

the induction hypothesis, and part 6 we deduce that

$$\mathrm{Ass}(M)_{\geq h_{r-i+1}} \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M)_{\geq h_{r-i+1}} \cup \mathrm{Ass}(M)_{h_{r-i}},$$

which put together with part 4, along with Lemma A.2 produces the equality.

$\square$

Now suppose that as a submodule of M, $M_0 = 0$ has an irredundant primary decomposition of the form:

$$M_0 = 0 = \bigcap_{1 \leq j \leq r} Q_1^{h_j} \cap \ldots \cap Q_{s_j}^{h_j} \tag{7}$$

where for a fixed $j \leq r$ and $e \leq s_j$, $Q_e^{h_j}$ is a primary submodule of M with $\mathrm{Ass}(M/Q_e^{h_j}) = \{\mathfrak{p}_e^{h_j}\}$ and $\mathrm{Ass}(M)_{h_j} = \{\mathfrak{p}_1^{h_j}, \ldots, \mathfrak{p}_{s_j}^{h_j}\}$.

Theorem A.4 *Let M be a sequentially Cohen-Macaulay module with filtration as in Definition 4.9, and suppose that $M_0 = 0$ has a primary decomposition as in (7). Then for each $i = 0, \ldots, r-1$, M_i has the following primary decomposition*

$$M_i = \bigcap_{1 \leq j \leq r-i} Q_1^{h_j} \cap \ldots \cap Q_{s_j}^{h_j}. \tag{8}$$

Proof: We prove this by induction on r (length of the filtration). The case $r = 1$ is clear, as the filtration is of the form $0 = M_0 \subset M$. Now consider M with filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M.$$

Since M_{r-1} is a sequentially Cohen-Macaulay module of length $r-1$, it satisfies the statement of the theorem. We first show that M_{r-1} has a primary decomposition as described in (8). From part 7 of Proposition A.3 it follows that

$$\operatorname{Ass}(M/M_{r-1}) = \operatorname{Ass}(M)_{h_1}$$

and so for some $\mathfrak{p}_e^{h_1}$-primary submodules $\mathcal{P}_e^{h_1}$ of M ($1 \leq e \leq s_j$), we have

$$M_{r-1} = \mathcal{P}_1^{h_1} \cap \ldots \cap \mathcal{P}_{s_1}^{h_1}. \tag{9}$$

We would like to show that $Q_e^{h_1} = \mathcal{P}_e^{h_1}$ for $e = 1, \ldots, s_1$.

Fix $e = 1$ and assume $Q_1^{h_1} \not\subset \mathcal{P}_1^{h_1}$. From the inclusion $M_0 \subset \mathcal{P}_1^{h_1}$ and Lemma A.1 it follows that for some e and j (with $e \neq 1$ if $j = 1$), we have $\mathfrak{p}_e^{h_j} \subseteq \mathfrak{p}_1^{h_1}$. Because of the difference in heights of these ideals the only conclusion is $\mathfrak{p}_e^{h_j} = \mathfrak{p}_1^{h_1}$, which is not possible. With a similar argument we deduce that $Q_e^{h_1} \subset \mathcal{P}_e^{h_1}$, for $e = 1, \ldots, s_1$.

Now fix $j \in \{1, \ldots, r\}$ and $e \in \{1, \ldots, s_j\}$. If $M_{r-1} = Q_e^{h_j}$ we are done. Otherwise, note that for every j and $\mathfrak{p}_e^{h_j}$-primary submodule $Q_e^{h_j}$ of M,

$$Q_e^{h_j} \cap M_{r-1}$$

is a $\mathfrak{p}_e^{h_j}$-primary submodule of M_{r-1} (as $\emptyset \neq \operatorname{Ass}(M_{r-1}/(Q_e^{h_j} \cap M_{r-1})) = \operatorname{Ass}((M_{r-1} + Q_e^{h_j})/Q_e^{h_j}) \subseteq \operatorname{Ass}(M/Q_e^{h_j}) = \{\mathfrak{p}_e^{h_j}\}$). So $M_0 = 0$ as a submodule of M_{r-1} has a primary decomposition

$$M_0 \cap M_{r-1} = 0 = \bigcap_{1 \leq j \leq r} (Q_1^{h_j} \cap M_{r-1}) \cap \ldots \cap (Q_{s_j}^{h_j} \cap M_{r-1}).$$

From Proposition A.3 part 8 it follows that

$$\operatorname{Ass}(M_{r-1}) = \operatorname{Ass}(M)_{\geq h_2}$$

so the components $Q_t^{h_1} \cap M_{r-1}$ are redundant for $t = 1, \ldots, s_1$, so for each such t we have

$$\bigcap_{Q_e^{h_j} \neq Q_t^{h_1}} (Q_1^{h_j} \cap M_{r-1}) \subseteq Q_t^{h_1} \cap M_{r-1}.$$

If $Q_e^{h_j} \cap M_{r-1} \nsubseteq Q_t^{h_1} \cap M_{r-1}$ for some e and j (with $Q_e^{h_j} \neq Q_t^{h_1}$), then by Lemma A.1 for some such e and j we have $\mathfrak{p}_e^{h_j} \subseteq \mathfrak{p}_t^{h_1}$, which is a contradiction (because of the difference of heights).

Therefore, for each t ($1 \le t \le s_1$), there exists indices e and j (with $Q_e^{h_j} \neq Q_t^{h_1}$) such that

$$Q_e^{h_j} \cap M_{r-1} \subseteq Q_t^{h_1} \cap M_{r-1}.$$

It follows now, from the primary decomposition of M_{r-1} in (9) that for a fixed t

$$\mathcal{P}_1^{h_1} \cap \ldots \cap \mathcal{P}_{s_1}^{h_1} \cap Q_e^{h_j} \subseteq Q_t^{h_1}.$$

Assume $\mathcal{P}_t^{h_1} \not\subseteq Q_t^{h_1}$. Applying Lemma A.1 again, we deduce that

$$\mathfrak{p}_e^{h_j} \subseteq \mathfrak{p}_t^{h_1}, \text{ or there is } t' \neq t \text{ such that } \mathfrak{p}_{t'}^{h_1} \subseteq \mathfrak{p}_t^{h_1}.$$

Neither of these is possible, so $\mathcal{P}_t^{h_1} \subseteq Q_t^{h_1}$ for all t.

We have therefore proved that

$$M_{r-1} = Q_1^{h_1} \cap \ldots \cap Q_{s_1}^{h_1}.$$

By the induction hypothesis, for each $i \le r-2$, M_i has the following primary decomposition

$$M_i = \bigcap_{2 \le j \le r-i} (Q_1^{h_j} \cap M_{r-1}) \cap \ldots \cap (Q_{s_j}^{h_j} \cap M_{r-1}) = \bigcap_{1 \le j \le r-i} Q_1^{h_j} \cap \ldots \cap Q_{s_j}^{h_j}$$

which proves the theorem.

$\square$

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