The projective dimension of sequentially Cohen-Macaulay monomial ideals

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Since posting this paper we have found that the same result regarding projective dimension of square-free monomial ideals appears in the paper of Morey and Villarreal [MV].

In this short note we prove that the projective dimension of a sequentially Cohen-Macaulay square-free monomial ideal is equal to the maximal height of its minimal primes (also known as the big height), or equivalently, the maximal cardinality of a minimal vertex cover of its facet complex. Along the way we also give bounds for the depth and the dimension of any monomial ideal. This in particular gives a formula for the projective dimension of facet ideals of these classes of ideals, which are known to be sequentially Cohen-Macaulay: graph trees and simplicial trees and forests [F1], chordal graphs and some cycles [FV], chordal clutters and graphs [W], some path ideals [SKT] to mention a few. Since polarization preserves projective dimension, our result also gives the projective dimension of any sequentially Cohen-Macaulay monomial ideal. Our result is a precise and simple description of the projective dimension that is not dependent on the ground field.

There has been much activity surrounding combinatorial characterizations of the projective dimension, see for example [C, DHS, DS, Ku, LM]. The authors in [KhM] and [DS] in particular prove the same result for graphs that are forests and some other classes of graphs.

A **simplicial complex** Δ over a set of vertices V is a set of subsets of V with the property that if $F \in \Delta$ then all subsets of F are also in Δ. An element of Δ is called a **face**, the **dimension** of a face F is $|F| - 1$, and the dimension of Δ is the largest dimension of a face of Δ. The maximal faces of Δ under inclusion are called **facets**, and the set of facets of Δ is denoted by Facets(Δ). If Facets$(\Delta) = \{F_1, \ldots, F_q\}$ we write $\Delta = \langle F_1, \ldots, F_q \rangle$.

Let k be any field. To a square-free monomial ideal I in a polynomial ring $R = k[x_1, \ldots, x_n]$ one can associate two unique simplicial complexes $\mathcal{N}(I)$ and $\mathcal{F}(I)$ on the vertex set labeled $\{x_1, \ldots, x_n\}$. Conversely given a simplicial complex Δ with vertices labeled $x_1, \ldots, x_n$ one can associate two unique square-free monomials $\mathcal{N}(\Delta)$ and $\mathcal{F}(\Delta)$ in the polynomial ring $k[x_1, \ldots, x_n]$; these are all defined below.

Facet complex of I	$\mathcal{F}(I) = \langle \{x_{a_1}, \ldots, x_{a_m}\} \mid x_{a_1} \ldots x_{a_m}$ minimal generator of $I \rangle$
Stanley-Reisner complex of I	$\mathcal{N}(I) = \{\{x_{a_1}, \ldots, x_{a_m}\} \mid x_{a_1} \ldots x_{a_m} \notin I\}$.
Facet ideal of Δ	$\mathcal{F}(\Delta) = (x_{a_1} \ldots x_{a_m} \mid \{x_{a_1}, \ldots, x_{a_m}\} \in$ Facets$(\Delta))$
Stanley-Reisner ideal of Δ	$\mathcal{N}(\Delta) = (x_{a_1} \ldots x_{a_m} \mid \{x_{a_1}, \ldots, x_{a_m}\} \notin \Delta)$.

Sequentially Cohen-Macaulay ideals were introduced by Stanley in relation to the concept of nonpure shellability by Björner and Wachs [BW].

Definition 1 ([S] Chapter III, Definition 2.9)**.** Let M be a finitely generated $\mathbb{Z}$-graded module over a finitely generated $\mathbb{N}$-graded k-algebra, with $R_0 = k$. We say that M is **sequentially Cohen-Macaulay** if there exists a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$$

of M by graded submodules M_i satisfying the following two conditions.

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(a) Each quotient M_i/M_{i-1} is Cohen-Macaulay.

(b) $\dim(M_1/M_0) < \dim(M_2/M_1) < \ldots < \dim(M_r/M_{r-1})$, where $\dim$ denotes Krull dimension.

Our goal is to find the projective dimension of sequentially Cohen-Macaulay square-free monomial ideal. In this setting Duval gave an equivalent characterization in terms of simplicial complexes.

Given a simplicial complex Δ and an integer i let

- $\Delta^i = \{F \in \Delta \mid \dim F \leq i\}$ =i-skeleton of Δ,
- $\Delta_i = \langle F \in \Delta \mid \dim F = i\rangle$ =pure i-skeleton of Δ.

Theorem 2 ([D] Theorem 3.3)**.** *Let I be square-free monomial ideal I in a polynomial ring R over a field k, and let $\Delta = \mathcal{N}(I)$. Then R/I is sequentially Cohen-Macaulay if and only if $R/\mathcal{N}(\Delta_i)$ is Cohen-Macaulay for all $i \in \{-1, \ldots, \dim \Delta\}$.*

The main tool we will use is a result of Fröberg. For a given simplicial complex Δ, let

Proposition 3 ([Fr] Theorem 8)**.** *Let Δ be the Stanley-Reisner complex of I. Then*

$$\text{depth}\,(R/I) = \max\{i \mid \Delta^i \textit{ is Cohen-Macaulay }\} + 1.$$

Theorem 4. *Let I be a monomial ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$. Suppose that the maximal height of an associated prime of I is d. Then*

$$\text{depth}\,(R/I) \leq n - d \textit{ and } pd(R/I) \geq d.$$

In particular, if R/I is sequentially Cohen-Macaulay then

$$\text{depth}\,(R/I) = n - d \textit{ and } pd(R/I) = d.$$

Proof. If I is a monomial ideal it has the same projective dimension as its polarization, which is square-free. Moreover by Proposition 2.5 of [F2] the maximal height of an associated prime of I is the same as the maximal height of an associated prime of its polarization, so we only consider square-free monomial ideals.

Suppose I is a square-free monomial ideal in $R = k[x_1, \ldots, x_n]$ with primary decomposition written as

$$I = (p_1 \cap \cdots \cap p_{a_1}) \cap \cdots \cap (p_{a_{s-1}+1} \cap \cdots \cap p_{a_s})$$

where

- height $p_{a_{i-1}+1} = \cdots =$ height $p_{a_i} = d_i$ for $i \in \{1, \ldots, s\}$, assuming $a_0 = 0$, and
- $d_1 < d_2 < \ldots < d_s$.

We know then that $\dim(R/I) = n - d_1$. We wish to prove that if R/I is sequentially Cohen-Macaulay, then $\text{depth}\,(R/I) = n - d_s$.

Note that since I is a square-free monomial ideal, each of the p_i is generated by a subset of $\{x_1, \ldots, x_n\}$ and the generating set of each p_i corresponds uniquely to a minimal vertex cover of $\mathcal{F}(I)$; that is, a set of vertices A such that every facet of $\mathcal{F}(I)$ contains one of the elements of this set, and no proper subset of A has this property.

Suppose Δ is the Stanley-Reisner complex of I. Then by the Proposition 2.4 of [F1] the facets of are complements of the p_i and therefore Δ have dimensions

$$n - d_1 - 1 > \ldots > n - d_s - 1.$$

Now consider Δ^i. If $i > n - d_s - 1$, then Δ^i will have facets of dimension greater than $n - d_s - 1$ as well as facets of dimension $n - d_s - 1$. So the largest i where Δ^i is pure is $n - d_s - 1$. Since Δ^i being pure is a necessary condition for the Cohen-Macaulayness of $R/\mathcal{N}(\Delta^i)$, we already know that $R/\mathcal{N}(\Delta^i)$ is not Cohen-Macaulay for $i > n - d_s - 1$. It follows immediately from Proposition 3 that

$$\text{depth}\,(R/I) \leq n - d_s.$$

Now by applying the Auslander-Buchsbaum formula [AB] we have

$$\text{pd}(R/I) = n - \text{depth}\ (R/I) \geq d_s.$$

Also notice that $\Delta^{n-d_s-1} = \Delta_{n-d_s-1}$, so if I is sequentially Cohen-Macaulay by Theorem 2 we know $R/\mathcal{N}(\Delta_{n-d_s})$ is Cohen-Macaulay. It follows that $\text{depth}\ (R/I) = n - d_s$.

The Auslander-Buchsbaum formula [AB] gives

$$\text{pd}(R/I) = n - \text{depth}\ (R/I) = d_s.$$

□

Remark 5. It is natural to expect that the same statement holds for any sequentially Cohen-Macaulay module, since their primary components behave in the same way as described in the proof above; see the appendix and in particular Theorem A4 of [F2] for more details of primary decomposition of sequentially Cohen-Macaulay modules. Indeed, after showing our result to Jürgen Herzog he pointed out that this follows from his joint work in [HP].

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