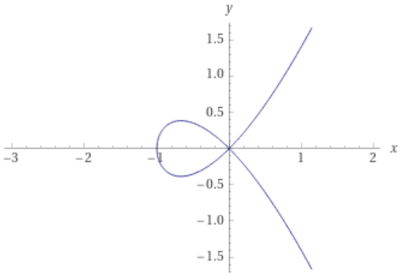
Algebraic geometry and Tangent categories

Geoff Cruttwell, Mount Allison University
(Based on joint work with J.S. Lemay)

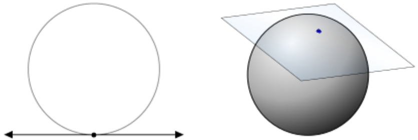
@cat seminar, March 15th, 2022



Overview: tangent categories

Tangent categories axiomatize the existence of a "tangent bundle" for each object in a category.

- The canonical example is the category of smooth manifolds.
- At a point of a smooth manifold, the **tangent space** at that point is the collection of all tangent vectors to that point:



  the **tangent bundle** is the collection of all the tangent spaces; it can itself be given the structure of a smooth manifold.

This construction is functorial; the action of this functor on smooth maps is the derivative (in each local chart).

- Moreover, there are several natural transformations and categorical limit properties related to this functor.
- These categorical properties are axiomatized into a **tangent category**.

Properties and examples

The tangent category axioms are quite powerful; over the last few years we've discovered how to define many differential geometry concepts in an arbitrary tangent category, such as

- Vector fields and vector bundles
- Differential forms and de Rham cohomology
- Connections and curvature

Many of our initial examples of tangent categories were generalizations of the category of smooth manifolds, such as

- Convenient manifolds (a type of infinite-dimensional manifold)
- C^∞-rings
- Models of synthetic differential geometry (SDG)

But there are also non-standard examples such as:

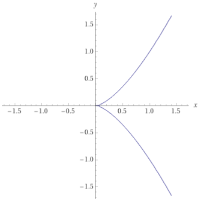
- The differential λ-calculus
- Goodwilllie functor calculus (an "∞-tangent category"[1])

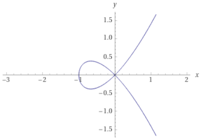
[1] "Tangent infinity-categories and Goodwillie calculus" by Bauer, Burke, and Ching

Tangent categories and Algebraic Geometry

We've also known for quite a while that there are examples of tangent categories in *algebraic* geometry...but not done anything with them (or really understood what was going on with their structure).

- But they are very interesting, because their "tangent bundles" can have singularities: places where the dimension of the tangent space changes!
- For example, the solutions of $y^2 - x^3 = 0$ and $y^2 - x^3 - x^2 = 0$





both have tangent spaces of "dimension 1" at all points except the origin, where the tangent space is of "dimension 2".

The fact that this can be seen as an example of a tangent category thus shows that tangent categories can capture both "smooth" and "non-smooth" examples of tangent functors!

Plan for the talk(s)

Goal: better understand how tangent categories relate to algebraic geometry; specifically, understand the (tangent) category structure of

$$\text{affine schemes over } R = (\text{commutative } R\text{-algebras})^{op}.$$

Plan for today:

1. Brief review of tangent categories
2. Quick entry into algebraic geometry via (commutative R-algebras)^{op}
3. The tangent structure of this category, and "tangent spaces" of some of its objects

Next time: how tangent category theory (eg., vector fields, differential bundles, connections) applied to this example relates to existing work in algebraic geometry. Still on-going, lots to do here!

Tangent category definition

A **tangent category**[2] consists of a category $\mathbb{X}$ with:

- (Tangent functor) an endofunctor $T : \mathbb{X} \to \mathbb{X}$;

[2] Rosický 1984, modified Cockett/Cruttwell 2014

Tangent category definition

A **tangent category**[2] consists of a category $\mathbb{X}$ with:

- (Tangent functor) an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- (Projection) a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- (Zero tangent vector) a natural transform $0 : 1_{\mathbb{X}} \longrightarrow T$;
- (Addition of tangent vectors) a natural transformation $+ : T_2 \longrightarrow T$, where $T_2 M$ is the pullback of p_M along itself:

$$\begin{array}{ccc} T_2 M & \longrightarrow & TM \\ \downarrow & & \downarrow {\scriptstyle p_M} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

[2] Rosický 1984, modified Cockett/Cruttwell 2014

Tangent category definition

A **tangent category**[2] consists of a category $\mathbb{X}$ with:

- (Tangent functor) an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$;
- (Projection) a natural transformation $p : T \longrightarrow 1_{\mathbb{X}}$;
- (Zero tangent vector) a natural transform $0 : 1_{\mathbb{X}} \longrightarrow T$;
- (Addition of tangent vectors) a natural transformation $+ : T_2 \longrightarrow T$, where $T_2 M$ is the pullback of p_M along itself:

$$\begin{array}{ccc} T_2 M & \longrightarrow & TM \\ \downarrow & & \downarrow{\scriptstyle p_M} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

- (Vertical lift) a natural transformation $\ell : T \longrightarrow T \circ T =: T^2$;
- (Canonical flip) a natural transformation $c : T^2 \longrightarrow T^2$;

satisfying various axioms.

---

[2] Rosický 1984, modified Cockett/Cruttwell 2014

Example: "multivariable calculus"

There is a tangent category consisting of objects open subsets of $\mathbb{R}^n$'s, and maps smooth maps between them. It has tangent functor T given by:

- for $U \subseteq \mathbb{R}^n$, $TU := U \times \mathbb{R}^n$.
- for $f : (U \subseteq \mathbb{R}^n) \to (V \subseteq \mathbb{R}^m)$, $T(f)(x, v) := J[f](x) \cdot v$ (where J is the Jacobian)

Example: "multivariable calculus"

There is a tangent category consisting of objects open subsets of $\mathbb{R}^n$'s, and maps smooth maps between them. It has tangent functor T given by:

- for $U \subseteq \mathbb{R}^n$, $TU := U \times \mathbb{R}^n$.
- for $f : (U \subseteq \mathbb{R}^n) \to (V \subseteq \mathbb{R}^m)$, $T(f)(x, v) := J[f](x) \cdot v$ (where J is the Jacobian)

And natural transformations:

- (Projection) $p(x, v) := x$
- (Zero) $0(x) := (x, 0)$
- (Addition) $+(x, v_1, v_2) := (x, v_1 + v_2)$

Example: "multivariable calculus"

There is a tangent category consisting of objects open subsets of $\mathbb{R}^n$'s, and maps smooth maps between them. It has tangent functor T given by:

- for $U \subseteq \mathbb{R}^n$, $TU := U \times \mathbb{R}^n$.
- for $f : (U \subseteq \mathbb{R}^n) \to (V \subseteq \mathbb{R}^m)$, $T(f)(x, v) := J[f](x) \cdot v$ (where J is the Jacobian)

And natural transformations:

- (Projection) $p(x, v) := x$
- (Zero) $0(x) := (x, 0)$
- (Addition) $+(x, v_1, v_2) := (x, v_1 + v_2)$
- (Lift) $\ell(x, v) := (x, 0, 0, v)$
- (Flip) $c(x, v_1, v_2, w) := (x, v_2, v_1, w)$

Many of the axioms follow from rules of calculus, eg., functoriality of T is the chain rule, and naturality of the flip is the symmetry of mixed partial derivatives: $\frac{\partial^2 f}{dxdy} = \frac{\partial^2 f}{dydx}$.

Other Examples

1. Smooth manifolds with their tangent bundle (with structure locally given as on the previous slide - but this structure need not persist globally, ie., TM is usually not just $M \times R^n$).
2. Convenient manifolds (a certain type of infinite-dimensional manifold) with their *kinematic* tangent bundle.
3. The infinitesimally linear objects in a model of synthetic differential geometry (SDG).
4. The category of C^∞-rings.
5. Abelian functor calculus gives a tangent category, and Goodwillie functor calculus gives an (infinity) tangent category.
6. The vector fields in any tangent category form a new tangent category (as do many other constructions).

Tangent spaces

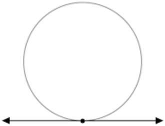
If we have a tangent category, how can we recover the individual tangent spaces of an object?

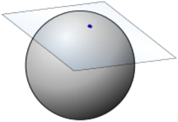
Definition

If $(\mathbb{X}, T)$ is a tangent category with a terminal object 1, and $a : 1 \longrightarrow M$ is a (categorical) point of an object M, then the **tangent space of M at a**, T_aM, is the pullback

$$\begin{array}{ccc} T_aM & \longrightarrow & TM \\ \downarrow & & \downarrow{\scriptstyle p_M} \\ 1 & \xrightarrow[a]{} & M \end{array}$$

(assuming it exists and is preserved by T).



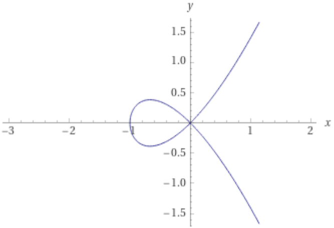


What is Algebraic Geometry?

Classically, it is about the geometric properties of the set of common zeroes of some polynomial equations in some base ring R, eg., solutions to

$$y^2 - x^3 - x^2 = 0$$

in $\mathbb{R}$ look like

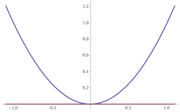


How can we represent these objects?

- Classically, these objects were literally seen as their set of solutions: this is called the **variety** associated to the polynomial equations.
- *However*, this doesn't keep track of multiplicity: eg., $x = 0$ and $x^2 = 0$ have the same set of solutions, so define the same variety.

Category of Algebraic Geometry?

But multiplicity is important! For example, Bezout's theorem, which talks about the intersection of two equations, counts the intersections with multiplicity, eg., $y = x^2$ and $y = 0$ have an intersection of multiplicity 2 ($x^2 = 0$, not $x = 0$).



- So it can be very useful to view $x = 0$ and $x^2 = 0$ as separate objects.

How to do this? The standard answer is to build the category of "affine schemes" (over a ring R)

- But this definition is fairly complicated (and unintuitive, at least to me).
- It involves prime ideals, the Zariski topology, locally ringed spaces...
- But at the end of the day, the category that is built is equivalent to (commutative R-algebras)^{op}.

Commutative R-algebras

So we'll represent these objects directly as objects in (commutative R-algebras)^{op}. Recall:

Definition

For a commutative unital ring R, a **commutative R-algebra** is a commutative unital ring A with an R-module structure which is compatible with A's ring structure.

- Equivalently, an R-algebra consists of a commutative unital ring A with a ring homomorphism $R \longrightarrow A$.
- We'll write $cAlg_R$ for the category of these objects.

Varieties as objects of $cAlg_R^{op}$

Given a set of polynomials, we view the associated variety as an object of $cAlg_R^{op}$ by sending it to its **co-ordinate ring**

$$R[x_1, x_2, \ldots x_n]/(\text{ideal generated by the polynomials})$$

For example, we represent $y^2 - x^3 - x^2 = 0$ as the algebra

$$R[x, y]/(y^2 - x^3 - x^2)$$

or $y - x^2 = 0, yz = 0$ as the algebra

$$R[x, y, z]/(y - x^2, yz)$$

But why should this be the right category to represent these objects? Two answers:

1. Points
2. Subobjects more generally.

Points of the co-ordinate ring of a variety

The (categorical) points of a co-ordinate ring are exactly the solutions of its equations!

- R is an initial object in $cAlg_R$, hence a terminal object in $cAlg_R^{op}$
- So a (categorical) point of $R[x,y]/(y^2 - x^3 - x^2)$ is a map

$$R \longrightarrow R[x,y]/(y^2 - x^3 - x^2) \text{ in } cAlg_R^{op},$$

Points of the co-ordinate ring of a variety

The (categorical) points of a co-ordinate ring are exactly the solutions of its equations!

- R is an initial object in $cAlg_R$, hence a terminal object in $cAlg_R^{op}$
- So a (categorical) point of $R[x,y]/(y^2 - x^3 - x^2)$ is a map

$$R \longrightarrow R[x,y]/(y^2 - x^3 - x^2) \text{ in } cAlg_R^{op},$$

  so a map

$$R[x,y]/(y^2 - x^3 - x^2) \longrightarrow R \text{ in } cAlg_R.$$

- But such a map is entirely determined by where it sends x and y, and it can send them to any points of R (say a and b) so along as the associated map is well-defined, ie., $b^2 - a^3 - a^2 = 0$.
- So indeed the categorical points are precisely the points of the associated variety.

I wish algebraic geometry books put this more up front! It's usually buried in an exercise somewhere...

More examples of points

Some more examples of this idea:

- Note that the categorical points of $R[x_1, x_2, \ldots x_n]$ (viewed as an object of $cAlg_R^{op}$) are the set of *all* n-tuples of elements of R.
- Hence why $R[x_1, x_2, \ldots x_n]$ is referred to as "affine n-space".
- Note that the points of

$$R[x]/(x) \text{ and } R[x]/(x^2)$$

  are the same (both just $x = 0$). However, they are different objects in this category: $R[x]/(x)$ is just R, while

$$R[x]/(x^2) = \{a + bx : a, b \in R, x^2 = 0\}.$$

Thus viewing the equation(s) as an object of $cAlg_R^{op}$ keeps the information of its associated variety, but also contains much more.

Subobjects

Another example of why $cAlg_R^{op}$ is the "right" category: monomorphisms.

- As an example, there should be a natural inclusion of $y^2 - x^3 - x^2 = 0$ into affine 2-space
- That is, there should be a natural monomorphism

$$R[x, y]/(y^2 - x^3 - x^2) \hookrightarrow R[x, y] \text{ in } cAlg_R^{op},$$

Subobjects

Another example of why $cAlg_R^{op}$ is the “right” category: monomorphisms.

- As an example, there should be a natural inclusion of $y^2 - x^3 - x^2 = 0$ into affine 2-space
- That is, there should be a natural monomorphism

$$R[x,y]/(y^2 - x^3 - x^2) \hookrightarrow R[x,y] \text{ in } cAlg_R^{op},$$

  that is, a natural epimorphism

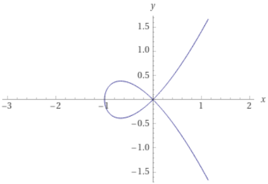
$$R[x,y] \twoheadrightarrow R[x,y]/(y^2 - x^3 - x^2) \text{ in } cAlg_R.$$

- But of course there is: the quotient map!

So quotients of algebras = inclusions of sub varieties/schemes.

Recap

So, we're thinking of varieties as objects of $cAlg_R^{op}$ ("affine schemes") via their co-ordinate ring, eg.,



that is, $y^2 - x^3 - x^2 = 0$, is represented by the object in $cAlg_{\mathbb{R}}^{op}$

$$\mathbb{R}[x,y]/(y^2 - x^3 - x^2)$$

Our main question is: **what is the tangent bundle of such an object, ie., of an affine scheme**?

Tangent bundle of an affine scheme

For an affine scheme $A \in cAlg_R^{op}$, define

$$TA := \text{Symmetric } A\text{-algebra of (Kahler differentials of } A \text{ over } R)^3$$

which is much more complicated than $TU = U \times \mathbb{R}^n$! So we'll go into it in a bit more detail:

1. I'll begin by introducing/reviewing both the Kahler differentials and the symmetric algebra, and look at some examples.
2. Then see what the tangent category structure is.
3. Then we'll look at the tangent spaces of examples.

[3] Grothendieck himself called this the "fibré tangent" (tangent bundle) of A in EGA IV (16.5.12). However, this name doesn't appear in textbooks much anymore.

Kahler differentials

Definition

If A is a commutative R-algebra, the **Kahler differentials of A (over R)**, $\Omega_R(A)$, is the free A-module generated by symbols da (for each $a \in A$), subject to the relations

- (d is R-linear): $d(0) = 0$, $d(a + b) = da + db$, $d(ra) = rd(a)$,
- (Leibniz rule(s)): $d(1) = 0$, $d(ab) = adb + bda$.

Example

If $A = R[x]$, then $\Omega_R(A)$ is just the free A-module on one generator (dx), since for example

$$d(x^2) = d(x \cdot x) = xdx + xdx = 2xdx$$

and more generally for any $p(x) \in R[x]$,

$$d[p(x)] = p'(x)dx$$

More Kahler examples

Example

Similarly, if $A = R[x_1, x_2, \ldots x_n]$, $\Omega_R(A)$ is the free A-module on the generators $dx_1, dx_2, \ldots dx_n$.

Kahlers work well with quotients of polynomial algebras: the Kahler of

$$R[x_1, x_2, \ldots x_n]/(p_1, p_2, \ldots p_k)$$

is the free A-module on the generators $dx_1, dx_2, \ldots dx_n$ subject to the *derivatives* of the p_i's.

Example

If $A = R[x, y]/(y^2 - x^3 - x^2)$, $\Omega_R(A)$ is the free A-module on dx, dy, subject to the relation $2ydy - 3x^2dx - 2xdx = 0$.

Kahlers can also be completely trivial:

Example

$\Omega_{\mathbb{R}}(\mathbb{C}) = 0$.

Symmetric algebra and tangent bundles

Definition

If M is an A-module, the **symmetric algebra** of M, $\mathbf{Sym}(M)$, is the free A-algebra generated by M.

Putting these together gives the tangent bundle:

Example

The tangent bundle of $A = R[x_1, x_2, \ldots x_n]$ is

$$TA = R[x_1, x_2, \ldots x_n, dx_1, dx_2, \ldots dx_n]$$

Example

The tangent bundle of $A = R[x, y]/(y^2 - x^3 - x^2)$ is

$$TA = R[x, y, dx, dy]/(y^2 - x^3 - x^2, 2ydy - 3x^2dx - 2xdx)$$

Example

The tangent bundle of $\mathbb{C}$ (over $\mathbb{R}$) is $T(\mathbb{C}) = \mathbb{C}$.

Tangent structure of affine schemes

What is the rest of the tangent structure?

- (Action on arrows) For $f : B \longrightarrow A$ in $cAlg_R^{op}$, $Tf : TB \longrightarrow TA$ is the algebra map $TA \longrightarrow TB$ defined by $a \mapsto f(a), da \mapsto df(a)$.

Tangent structure of affine schemes

What is the rest of the tangent structure?

- (Action on arrows) For $f : B \longrightarrow A$ in $cAlg_R^{op}$, $Tf : TB \longrightarrow TA$ is the algebra map $TA \longrightarrow TB$ defined by $a \mapsto f(a), da \mapsto df(a)$.
- (Projection) $p : TA \longrightarrow A$ is the algebra map $A \longrightarrow TA$ defined by $a \mapsto a$.
- (Zero) $0 : A \longrightarrow TA$ is the algebra map $TA \longrightarrow A$ defined by $a \mapsto a, da \mapsto 0$.
- (Addition) $+ : T_2A \longrightarrow TA$ is the algebra map $TA \longrightarrow TA \otimes_A TA$ defined by

$$a \mapsto a, da \mapsto da \otimes 1 + 1 \otimes da.$$

Tangent structure of affine schemes

What is the rest of the tangent structure?

- (Action on arrows) For $f : B \longrightarrow A$ in $cAlg_R^{op}$, $Tf : TB \longrightarrow TA$ is the algebra map $TA \longrightarrow TB$ defined by $a \mapsto f(a), da \mapsto df(a)$.
- (Projection) $p : TA \longrightarrow A$ is the algebra map $A \longrightarrow TA$ defined by $a \mapsto a$.
- (Zero) $0 : A \longrightarrow TA$ is the algebra map $TA \longrightarrow A$ defined by $a \mapsto a, da \mapsto 0$.
- (Addition) $+ : T_2A \longrightarrow TA$ is the algebra map $TA \longrightarrow TA \otimes_A TA$ defined by
$$a \mapsto a, da \mapsto da \otimes 1 + 1 \otimes da.$$
- (Lift) $\ell : TA \longrightarrow T^2A$ is the algebra map $T^2A \longrightarrow TA$ defined by
$$a \mapsto a, da \mapsto 0, d'a \mapsto 0, d'da \mapsto da$$
(T^2A is generated by symbols $a, da, d'a, d'da$)
- (Flip) $c : T^2A \longrightarrow T^2A$ is the algebra map $T^2A \longrightarrow T^2A$ defined by
$$a \mapsto a, da \mapsto d'a, d'a \mapsto da, d'da \mapsto d'da.$$

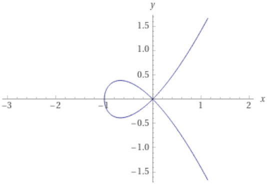
Some well-defined checking is required here, but it does work out.

Tangent spaces

But why should the tangent bundle of $A = R[x,y]/(y^2 - x^3 - x^2)$ be

$$TA = R[x,y,dx,dy]/(y^2 - x^3 - x^2, 2ydy - 3x^2dx - 2xdx)?$$

Let's look at some tangent spaces to get a better idea of what's going on. In particular, let's start with the tangent space of A at the point $(-1,0)$.



By definition, this is the pullback

$$\begin{array}{ccc} T_{(-1,0)}A & \longrightarrow & TA \\ \downarrow & & \downarrow p_A \\ R & \xrightarrow[(-1,0)]{} & A \end{array}$$

in $cAlg_R^{op}$.

Tangent spaces continued

So it is the pushout

$$\begin{array}{ccc} A & \xrightarrow{p_A} & TA \\ {\scriptstyle (-1,0)}\downarrow & & \downarrow \\ R & \longrightarrow & T_{(-1,0)}A \end{array}$$

(where recall p_A is the algebra map $a \mapsto a$) in $cAlg_R$, that is,

$$T_{(-1,0)}A = R \otimes_A TA$$

which is given by "evaluating"

$$TA = R[x, y, dx, dy]/(y^2 - x^3 - x^2, 2ydy - 3x^2dx - 2xdx)$$

at the point $(-1, 0)$, ie.,

$$T_{(-1,0)}A \cong R[dx, dy]/(-3dx + 2dx) \cong R[dx, dy]/(-dx) \cong R[dy]$$

That is, 1-dimensional; completely free in the y-direction, as the picture suggests.

Singular example

What is the tangent space of $A = R[x,y]/(y^2 - x^3 - x^2)$ at the point $(0,0)$?

- Similarly, we evaluate

$$TA = R[x,y,dx,dy]/(y^2 - x^3 - x^2, 2ydy - 3x^2dx - 2xdx)$$

  at the point $(0,0)$

- This gives

$$T_{(0,0)}A = R[dx,dy]$$

  ie., 2-dimensional, completely free in both dx and dy. The tangent space here is different than all other tangent spaces!

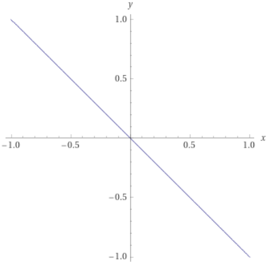
Again, note the difference with smooth manifolds, where the tangent spaces are all isomorphic.

Even more singular example

For something even more extreme, consider

$$A = R[x,y]/((x+y)^2) = R[x,y]/(x^2+2xy+y^2),$$

whose associated variety is just the line



But has tangent bundle

$$TA = R[x,y]/((x+y)^2, 2(x+y)(dx+dy))$$

whose tangent space at *any* point is 2-dimensional! Algebraic geometry books draw this kind of object as a "(infinitesimally) thickened line".

Conclusions

What does the existence of this tangent structure tell us?

- For tangent categories: they aren't just about categories of "smooth" objects; they are more flexible, and in some sense are just about "tangents", generally.
- For algebraic geometry: the idea that (Symmetric algebra of Kahler differentials) is a tangent bundle has additional theoretical support: it satisfies the same abstract properties as the tangent bundle for smooth manifolds (and generalizations of it).

And this leads to lots of further possibilities:

- Many things you can do in a tangent category (eg., define differential (vector) bundles, connections, etc.): what do they look like in this example? That's where we'll pick up next time.
- Has the potential to lead to all sorts of interesting generalizations in non-commutative algebraic geometry as well (Marcello is working on this)

References

Two basic references for tangent categories:

- (1984) Rosický, J. **Abstract tangent functors**. *Diagrammes*, 12, Exp. No. 3.
- (2014) Cockett, R. and Cruttwell, G. **Differential structure, tangent structure, and SDG**. *Applied Categorical Structures*, Vol. 22 (2), pg. 331–417.

Two algebraic geometry texts I like (in particular, they give more intuition/pictures than most others):

- (2000) Eisenbud, D. and Harris, J. *The geometry of schemes*, Springer.
- (2017) Vakil, R. *The rising sea: foundations of algebraic geometry* (draft lecture notes, available online).