Anomalies for Noninvertible Surfaces and the 2-Deligne Theorem

M. Yu

Perimeter Institute

2/14/23

Based on [2211.08436] w/ Thibault Décoppet
and WIP with Ryan Thorngren

Outline

- Motivations for the work – physical and mathematical
  - Anomalies
  - Noninvertible symmetries
  - Topological Theories
- Review properties of 2-categories
  - braided, sylleptic, symmetric
  - strongly fusion
- Main theorems about condensing in 2-categories
- Explicit examples using the theorems
- Cohomology computations for braided strongly fusion 2 categories
- 2-Deligne theorem (the symmetric case)

Motivations

- Theories of quantum gravity should have no global (categorical) symmetries.
- There should be no anomalies for the symmetries, or categorical/noninvertible symmetries.
- In the context of gravity, this mean that the topological part of the theory (described by some higher category) should be condensible to the vacuum.
- Generalize the notion of an anomaly for a symmetry, to an anomaly for a noninvertible symmetry.

Motivations cont.: Review of Anomalies

- Given a QFT, the first thing to do is identify its global symmetries.
  - Flavor
  - Spacetime
  - Duality
- Once you have a list of symmetries, you can ask "which symmetries can I *gauge*?"
- Why do you want to gauge? One of the first reasons for doing so is so that certain fields have the right number of degrees of freedom.

Motivations cont.: Review of Anomalies

- Given a QFT, the first thing to do is identify its global symmetries.
  - Flavor
  - Spacetime
  - Duality
- Once you have a list of symmetries, you can ask "which symmetries can I *gauge*?"
- Why do you want to gauge? One of the first reasons for doing so is so that certain fields have the right number of degrees of freedom.
- But not being able to gauge is OK. But the obstruction to gauging is a useful thing to remember.

Motivations cont.: Gauging in (0+1)d

- Consider QM with a state space $\mathcal{H}$ and a hamiltonian $\widehat{H}$. The system is modeled by $(\mathcal{H}, \widehat{H})$.

Motivations cont.: Gauging in (0+1)d

- Consider QM with a state space $\mathcal{H}$ and a hamiltonian $\widehat{H}$. The system is modeled by $(\mathcal{H}, \widehat{H})$.
- The Hilbert space $\mathcal{H}$ is not physical, but its projectivization is. We should work with the projective unitary group PU on this space.
- You can ask about automorphisms of $(\mathcal{H}, \widehat{H})$, which we will take to be a group G.

Motivations cont.: Gauging in (0+1)d

- Consider QM with a state space $\mathcal{H}$ and a hamiltonian $\widehat{H}$. The system is modeled by $(\mathcal{H}, \widehat{H})$.
- The Hilbert space $\mathcal{H}$ is not physical, but its projectivization is. We should work with the projective unitary group PU on this space.
- You can ask about automorphisms of $(\mathcal{H}, \widehat{H})$, which we will take to be a group G.
- If G really is represented linearly on $\mathcal{H}$, then gauging the G-symmetry would be modelled by the Hilbert space of cofixed points.

Motivations cont.: Gauging in (0+1)d

- Consider QM with a state space $\mathcal{H}$ and a hamiltonian $\widehat{H}$. The system is modeled by $(\mathcal{H}, \widehat{H})$.
- The Hilbert space $\mathcal{H}$ is not physical, but its projectivization is. We should work with the projective unitary group PU on this space.
- You can ask about automorphisms of $(\mathcal{H}, \widehat{H})$, which we will take to be a group G.
- If G really is represented linearly on $\mathcal{H}$, then gauging the G-symmetry would be modelled by the Hilbert space of cofixed points.
- But, we have $G \to \mathrm{PU}$. Then in order to define the gauged theory, you need to lift G along $\mathrm{U} \to \mathrm{PU}$.

Motivations cont.: Gauging in (0+1)d

- Consider QM with a state space $\mathcal{H}$ and a hamiltonian $\widehat{H}$. The system is modeled by $(\mathcal{H}, \widehat{H})$.
- The Hilbert space $\mathcal{H}$ is not physical, but its projectivization is. We should work with the projective unitary group PU on this space.
- You can ask about automorphisms of $(\mathcal{H}, \widehat{H})$, which we will take to be a group G.
- If G really is represented linearly on $\mathcal{H}$, then gauging the G-symmetry would be modelled by the Hilbert space of cofixed points.
- But, we have $G \to \mathrm{PU}$. Then in order to define the gauged theory, you need to lift G along $\mathrm{U} \to \mathrm{PU}$.
- The fiber of this map is a $\mathrm{U}(1)$, and so the obstruction to doing this lifting is a class $\omega \in \mathrm{H}^2(BG; \mathrm{U}(1))$.

Motivations cont.: Noninvertible symmetries

- Suppose some operators in your theory have fusion that goes $a \times b = \sum_c N_{ab}^c c$.
- The operators (defects) satisfy some crossing relations, i.e. F-symbols.
- The fusion rules are noninvertible, hence the set of defects which enact the fusion ring symmetry are noninvertible.

Motivations cont.: Noninvertible symmetries

- Suppose some operators in your theory have fusion that goes $a \times b = \sum_c N_{ab}^c c$.
- The operators (defects) satisfy some crossing relations, i.e. F-symbols.
- The fusion rules are noninvertible, hence the set of defects which enact the fusion ring symmetry are noninvertible.

Example

Consider $\mathbb{Z}/2$ Tambara Yamagami. It has a $\mathbb{Z}/2$ line η and a noninvertible topological line $\mathcal{N}$.

$$\eta \times \eta = 1, \quad \eta \times \mathcal{N} = \mathcal{N} \times \eta = \mathcal{N}, \quad \mathcal{N} \times \mathcal{N} = 1 + \eta.$$

This type of fusion category structure occurs in the Ising CFT. The line $\mathcal{N}$ implements a duality symmetry in Ising.

Motivations: Topologically Ordered Phases

Motivations: Topologically Ordered Phases

- Extended operators, with long range entanglement. The phase is locally trivial, i.e. there are no nontrivial local operators in the IR.

Motivations: Topologically Ordered Phases

- Extended operators, with long range entanglement. The phase is locally trivial, i.e. there are no nontrivial local operators in the IR.
- Extended operators are supported on a line or surface, or higher dimensional surface. Can build higher dimensional surfaces from lower dimensional ones.

Motivations: Topologically Ordered Phases

- Extended operators, with long range entanglement. The phase is locally trivial, i.e. there are no nontrivial local operators in the IR.
- Extended operators are supported on a line or surface, or higher dimensional surface. Can build higher dimensional surfaces from lower dimensional ones.
- This is the idea of a condensation, or gauging. In particular we can build a phase by starting with the vacuum and flooding with a network of interfaces.

Motivations: Topologically Ordered Phases

- Extended operators, with long range entanglement. The phase is locally trivial, i.e. there are no nontrivial local operators in the IR.
- Extended operators are supported on a line or surface, or higher dimensional surface. Can build higher dimensional surfaces from lower dimensional ones.
- This is the idea of a condensation, or gauging. In particular we can build a phase by starting with the vacuum and flooding with a network of interfaces.
- The extended operators need to be able to detect each other through a means of linking (physical requirement).

Motivations: Topologically Ordered Phases

- Extended operators, with long range entanglement. The phase is locally trivial, i.e. there are no nontrivial local operators in the IR.
- Extended operators are supported on a line or surface, or higher dimensional surface. Can build higher dimensional surfaces from lower dimensional ones.
- This is the idea of a condensation, or gauging. In particular we can build a phase by starting with the vacuum and flooding with a network of interfaces.
- The extended operators need to be able to detect each other through a means of linking (physical requirement).
- We say two phases are equivalent if we can build a gapped interface between them via a condensation (Morita equivalent).

Motivations cont.: Topological Phases in Physics

- Many physical theories, as well as supersymmetric theories, flow to topological phases.

Motivations cont.: Topological Phases in Physics

- Many physical theories, as well as supersymmetric theories, flow to topological phases.
- To determine which one, it is useful to compute anomalies and match them.
- Obtaining the actual value might involving computing something like an η-invariant, or writing down an anomaly indicator.

Motivations cont.: Topological Phases in Physics

- Many physical theories, as well as supersymmetric theories, flow to topological phases.
- To determine which one, it is useful to compute anomalies and match them.
- Obtaining the actual value might involving computing something like an η-invariant, or writing down an anomaly indicator.
- It is also important to consider the topological content of the theory, i.e. the objects and morphisms of the corresponding category.

Motivations cont.: Topological Phases in Physics

- Many physical theories, as well as supersymmetric theories, flow to topological phases.
- To determine which one, it is useful to compute anomalies and match them.
- Obtaining the actual value might involving computing something like an η-invariant, or writing down an anomaly indicator.
- It is also important to consider the topological content of the theory, i.e. the objects and morphisms of the corresponding category.

Motivations cont.

- The anomaly for noninvertible symmetries is something like the obstruction to a fibre functor

$$F : k\mathrm{Rep}(\mathcal{G}) \to k\mathrm{Vec}\,.$$

- I will focus only on 2-categories for this talk, and consider fibre functors here.

Motivations cont.

- The anomaly for noninvertible symmetries is something like the obstruction to a fibre functor

$$F : k\mathrm{Rep}(\mathcal{G}) \to k\mathrm{Vec}\,.$$

- I will focus only on 2-categories for this talk, and consider fibre functors here.
- If the category is fully symmetric and fermionic we can upgrade to

$$\mathit{Fib} : \mathfrak{C} \to 2\,\mathbf{SVec}\,.$$

Motivations cont.

- The anomaly for noninvertible symmetries is something like the obstruction to a fibre functor

$$F : k\mathrm{Rep}(\mathcal{G}) \to k\mathrm{Vec}\,.$$

- I will focus only on 2-categories for this talk, and consider fibre functors here.
- If the category is fully symmetric and fermionic we can upgrade to

$$Fib : \mathfrak{C} \to 2\,\mathbf{SVec}\,.$$

For a physical theory: every (primary) operator in the category of operators must be detectable by a topological operator. So we do not want the fully category to be $k\,\mathrm{Rep}(\mathcal{G})$.

Motivations cont.

Questions

Consider fusion 2-category $\mathfrak{C}$ with some level of monoidality:

1. What are the structures of the 2-category obtained by *condensing* a suitable algebra object?
2. In which cases can we condense to the vacuum? What are the obstructions?

Motivations cont.

Questions

Consider fusion 2-category $\mathfrak{C}$ with some level of monoidality:

1. What are the structures of the 2-category obtained by *condensing* a suitable algebra object?
2. In which cases can we condense to the vacuum? What are the obstructions?

Caveat: Certainly if the category can be condensed to the vacuum then there is no anomaly to talk about.

In the case where we cannot condense to the vacuum, we do not know how to define the anomaly.

2-Category Background

We start with some axioms for our 2-category

Definition

A fusion 2-category is a finite semisimple rigid monoidal 2-category with simple monoidal unit.

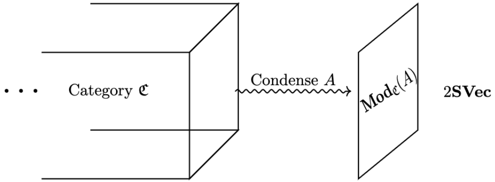
- Rigid, finite semisimple. This is a property to make it topological and useful for physics applications.

Examples

- The 2-category 2Vec of finite semisimple 1-categories
- The 2-category 2Vec[G] of finite dimensional G-graded 2-vector spaces.
- The 2-category 2Rep[G] of 2-representations of G.
- For $\mathcal{B}$ a braided fusion 1-category, Mod($\mathcal{B}$) is a 2-category of finite semisimple $\mathcal{B}$-module 1-categories.

Physical Picture

The physical picture for condensing surfaces in a 2-category involves finding some gapped boundary of the initial 2-category $\mathfrak{C}$, and then possibly triggering another condensation in order to map to $2\,\mathbf{SVec}$.



Condensation along a specific direction of spacetime builds modules which usually causes the resulting 2-category to lose a level of monoidality.

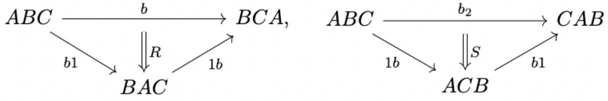
2-Category Background cont.

Now include more levels of monoidality, start with a braiding:

- A braided monoidal 2-category comes equipped with a braiding b which gives an equivalence

$$b_{A,B} : A\Box B \to B\Box A$$

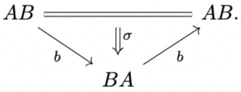
- There are two invertible modifications R and S



and they satisfy certain relations.

2-Category Background cont.

We can equip a braided monoidal 2-categories with an additional structure called a *syllepsis*.



σ is an invertible modification that satisfies certain relations.

We can define a *symmetric* 2-category as one where the syllepsis satisfies

$$\sigma_{B,A} \circ b_{A,B} = b_{A,B} \circ \sigma_{A,B} \,.$$

Statement of Theorems

Assigning extra properties to our category we can prove the following theorems:

Theorem

For $\mathfrak{B}$ a braided multifusion 2-category, and B a braided separable algebra in $\mathfrak{B}$, $\mathrm{Mod}_{\mathfrak{B}}(B)$ is a multifusion 2-category.

Statement of Theorems

Assigning extra properties to our category we can prove the following theorems:

Theorem

For $\mathfrak{B}$ a braided multifusion 2-category, and B a braided separable algebra in $\mathfrak{B}$, $\mathrm{Mod}_{\mathfrak{B}}(B)$ is a multifusion 2-category.

Example

Let $\mathcal{B}$ be a braided multifusion 1-category, that is a braided separable algebra in $2\mathbf{Vec}$. Then, $\mathrm{Mod}_{2\mathbf{Vec}}(\mathcal{B}) = \mathrm{Mod}(\mathcal{B})$ is the multifusion 2-category of finite semisimple right $\mathcal{B}$-module 1-categories.

Statement of Theorems cont.

Theorem

Let $\mathfrak{S}$ be a monoidal 2-category, and B a symmetric separable algebra in $\mathfrak{S}$. Then, $\mathrm{Mod}_{\mathfrak{S}}(B)$ is a braided monoidal 2-category.

Statement of Theorems cont.

Theorem

Let $\mathfrak{S}$ be a monoidal 2-category, and B a symmetric separable algebra in $\mathfrak{S}$. Then, $\mathrm{Mod}_{\mathfrak{S}}(B)$ is a braided monoidal 2-category.

Definition

The symmetric center of $\mathfrak{S}$, denoted by $\mathcal{Z}_{(3)}(\mathfrak{S})$ is the full sub-2-category of $\mathfrak{S}$ on those objects C such that

$$\sigma_{D,C} \circ b_{C,D} = b_{C,D} \circ \sigma_{C,D}, \quad \forall D \in \mathfrak{S}.$$

Statement of Theorems cont.

Theorem

Let $\mathfrak{S}$ be a monoidal 2-category, and B a symmetric separable algebra in $\mathfrak{S}$. Then, $\mathrm{Mod}_{\mathfrak{S}}(B)$ is a braided monoidal 2-category.

Definition

The symmetric center of $\mathfrak{S}$, denoted by $\mathcal{Z}_{(3)}(\mathfrak{S})$ is the full sub-2-category of $\mathfrak{S}$ on those objects C such that

$$\sigma_{D,C} \circ b_{C,D} = b_{C,D} \circ \sigma_{C,D}, \quad \forall D \in \mathfrak{S}.$$

Theorem

Let $\mathfrak{S}$ be a sylleptic monoidal 2-category, and B a symmetric separable algebra in $\mathcal{Z}_{(3)}(\mathfrak{S})$. Then, $\mathrm{Mod}_{\mathfrak{S}}(B)$ is a sylleptic monoidal 2-category.

Strongly Fusion 2-categories

- Suppose we condensed to a phase with no lines i.e. $\Omega\mathfrak{B} = \mathbf{SVec}$.
- It still has surface operators, which form a fusion 2-category. We call these categories *strongly (super) fusion*.

Strongly Fusion 2-categories

- Suppose we condensed to a phase with no lines i.e. $\Omega\mathfrak{B} = \mathbf{SVec}$.
- It still has surface operators, which form a fusion 2-category. We call these categories *strongly (super) fusion*.

Theorem:

Every indecomposable object of a strongly (super) fusion 2-category is invertible. The equivalence classes of indecomposable objects form a finite group.

Strongly Fusion 2-categories

- Suppose we condensed to a phase with no lines i.e. $\Omega\mathfrak{B} = \textbf{SVec}$.
- It still has surface operators, which form a fusion 2-category. We call these categories *strongly (super) fusion*.

Theorem:

Every indecomposable object of a strongly (super) fusion 2-category is invertible. The equivalence classes of indecomposable objects form a finite group.

- The surfaces have fusion rules described by a finite group E.

Strongly Fusion 2-categories

- Suppose we condensed to a phase with no lines i.e. $\Omega\mathfrak{B} = \textbf{SVec}$.
- It still has surface operators, which form a fusion 2-category. We call these categories *strongly (super) fusion*.

Theorem:

Every indecomposable object of a strongly (super) fusion 2-category is invertible. The equivalence classes of indecomposable objects form a finite group.

- The surfaces have fusion rules described by a finite group E.
- We can use strongly fusion in conjunction with other facts about the category that take into account the ambient dimensions.

Statement of Theorems cont.

- If we can make the resulting category after condensing strongly fusion, we can use cohomology to see if we can condense to $2\mathbf{SVec}$.
- The nicest case is when we find a subcategory $2\,\mathrm{Rep}(G)$ inside our original category.
- The idea is then to condense something in this subcategory and see how it affects the ambient category.

Statement of Theorems cont.

- If we can make the resulting category after condensing strongly fusion, we can use cohomology to see if we can condense to $2\mathbf{SVec}$.
- The nicest case is when we find a subcategory $2\,\mathrm{Rep}(G)$ inside our original category.
- The idea is then to condense something in this subcategory and see how it affects the ambient category.

Theorem

- Let $\mathfrak{B}$ be a braided fusion 2-category with $2\,\mathrm{Rep}(G) \simeq \mathfrak{B}^0$ as braided fusion 2-categories.
- Condensing $\varphi = \mathbf{Vec}[G]$ in $\mathfrak{B}$ yields a strongly fusion 2-category $\mathrm{Mod}_{\mathfrak{B}}(\varphi)$.

Statement of Theorems cont.

There are implication of the last theorem for sylleptic categories:

Statement of Theorems cont.

There are implication of the last theorem for sylleptic categories:

- Let $\mathfrak{S}$ be a sylleptic multifusion 2-category. Any inclusion $2\,\mathrm{Rep}(G) \subseteq \mathfrak{S}$ of sylleptic fusion 2-categories automatically includes in the symmetric center of $\mathfrak{S}$.
- Namely, $2\,\mathrm{Rep}(G)$ is necessarily contained in the component of the identity of $\mathfrak{S}$.

Corollary

Suppose that there is an inclusion of $2\,\mathrm{Rep}(G)$ in $\mathfrak{S}$, then $\mathrm{Mod}_{\mathfrak{S}}(\varphi)$ is a sylleptic strongly fusion 2-category. The canonical monoidal 2-functor $\mathfrak{S} \to \mathrm{Mod}_{\mathfrak{S}}(\varphi)$ is sylleptic.

Examples

Connected Fermionic

- Consider an ambient category $2\,\mathrm{Rep}\,(G, z)$. Take modules wrt **SVec**
- As left $2\,\mathrm{Rep}\,(G, z)$-module 2-categories, we have $2\,\mathbf{SVec} \simeq \mathrm{Mod}_{2\,\mathrm{Rep}\,(G,z)}(\mathbf{SVec})$
- **SVec** is viewed as an algebra in $2\,\mathrm{Rep}\,(G, z)$ via $\mathrm{Rep}\,(G, z) \to \mathbf{SVec}$.

Strongly fusion example

- Let $\mathfrak{B}$ be a braided fusion 2-category, and let $F : 2\,\mathrm{Rep}(G) \hookrightarrow \mathfrak{B}$ be a braided monoidal inclusion.

Examples

Connected Fermionic

- Consider an ambient category $2\,\mathrm{Rep}\,(G, z)$. Take modules wrt $\mathbf{SVec}$
- As left $2\,\mathrm{Rep}\,(G, z)$-module 2-categories, we have $2\,\mathbf{SVec} \simeq \mathrm{Mod}_{2\,\mathrm{Rep}\,(G,z)}(\mathbf{SVec})$
- $\mathbf{SVec}$ is viewed as an algebra in $2\,\mathrm{Rep}\,(G, z)$ via $\mathrm{Rep}\,(G, z) \to \mathbf{SVec}$.

Strongly fusion example

- Let $\mathfrak{B}$ be a braided fusion 2-category, and let $F : 2\,\mathrm{Rep}(G) \hookrightarrow \mathfrak{B}$ be a braided monoidal inclusion.
- Condense the algebra $\varphi = \mathbf{Vec}[G]$ in $\mathfrak{B}$.

Examples

Connected Fermionic

- Consider an ambient category $2\,\mathrm{Rep}\,(G, z)$. Take modules wrt $\mathbf{SVec}$
- As left $2\,\mathrm{Rep}\,(G, z)$-module 2-categories, we have $2\,\mathbf{SVec} \simeq \mathrm{Mod}_{2\,\mathrm{Rep}\,(G,z)}(\mathbf{SVec})$
- $\mathbf{SVec}$ is viewed as an algebra in $2\,\mathrm{Rep}\,(G, z)$ via $\mathrm{Rep}\,(G, z) \to \mathbf{SVec}$.

Strongly fusion example

- Let $\mathfrak{B}$ be a braided fusion 2-category, and let $F : 2\,\mathrm{Rep}(G) \hookrightarrow \mathfrak{B}$ be a braided monoidal inclusion.
- Condense the algebra $\varphi = \mathbf{Vec}[G]$ in $\mathfrak{B}$.
- Strongly fusion $\mathrm{Mod}_{\mathfrak{B}}(\varphi)$ equipped with a monoidal 2-functor $\mathfrak{B} \to \mathrm{Mod}_{\mathfrak{B}}(\varphi)$

Examples

Connected Fermionic

- Consider an ambient category $2\,\mathrm{Rep}\,(G, z)$. Take modules wrt $\mathbf{SVec}$
- As left $2\,\mathrm{Rep}\,(G, z)$-module 2-categories, we have $2\,\mathbf{SVec} \simeq \mathrm{Mod}_{2\,\mathrm{Rep}\,(G,z)}(\mathbf{SVec})$
- $\mathbf{SVec}$ is viewed as an algebra in $2\,\mathrm{Rep}\,(G, z)$ via $\mathrm{Rep}\,(G, z) \to \mathbf{SVec}$.

Strongly fusion example

- Let $\mathfrak{B}$ be a braided fusion 2-category, and let $F : 2\,\mathrm{Rep}(G) \hookrightarrow \mathfrak{B}$ be a braided monoidal inclusion.
- Condense the algebra $\varphi = \mathbf{Vec}[G]$ in $\mathfrak{B}$.
- Strongly fusion $\mathrm{Mod}_{\mathfrak{B}}(\varphi)$ equipped with a monoidal 2-functor $\mathfrak{B} \to \mathrm{Mod}_{\mathfrak{B}}(\varphi)$

Examples: Disconnected

- Suppose the fusion 2-category of surface operators and their interactions is given by $2\mathbf{Vec}[G]$.
- We can again consider the algebra $\mathbf{Vec}[G]$ in $2\mathbf{Vec}[G]$, the sum of the equivalence classes of simple objects.
- $\mathbf{Vec}[G]$ is the fusion 1-category of G-graded vector spaces viewed as an algebra in $2\mathbf{Vec}[G]$ with the canonical grading.

Lemma

The left $2\mathbf{Vec}[G]$-module 2-category $2\mathbf{Vec}$ is equivalent to $\mathrm{Mod}_{2\mathbf{Vec}[G]}(\mathbf{Vec}[G])$.

Examples: Disconnected cont.

Theorem

- Let $\mathfrak{B}$ be a braided fusion 2-category with $2\mathbf{Vec}[E] \subseteq \mathfrak{B}$ a braided monoidal inclusion

Examples: Disconnected cont.

Theorem

- Let $\mathfrak{B}$ be a braided fusion 2-category with $2\mathbf{Vec}[E] \subseteq \mathfrak{B}$ a braided monoidal inclusion
- The 2-category $\mathrm{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])$ obtained by condensing $\mathbf{Vec}[E]$ is a fusion 2-category

Examples: Disconnected cont.

Theorem

- Let $\mathfrak{B}$ be a braided fusion 2-category with $2\mathbf{Vec}[E] \subseteq \mathfrak{B}$ a braided monoidal inclusion
- The 2-category $\mathrm{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])$ obtained by condensing $\mathbf{Vec}[E]$ is a fusion 2-category
- $\pi_0(\mathrm{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])) \cong \pi_0(\mathfrak{B})/E$. Moreover, the canonical 2-functor $\mathfrak{B} \to \mathrm{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])$ is monoidal.

Examples: Disconnected cont.

Theorem

- Let $\mathfrak{B}$ be a braided fusion 2-category with $2\mathbf{Vec}[E] \subseteq \mathfrak{B}$ a braided monoidal inclusion
- The 2-category $\mathrm{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])$ obtained by condensing $\mathbf{Vec}[E]$ is a fusion 2-category
- $\pi_0(\mathrm{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])) \cong \pi_0(\mathfrak{B})/E$. Moreover, the canonical 2-functor $\mathfrak{B} \to \mathrm{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])$ is monoidal.

Disconnected Category

- Consider $2\mathbf{Vec}[\mathbb{Z}_4]$, with simple objects labeled by $\{\mathrm{Vec}_0, \mathrm{Vec}_1, \mathrm{Vec}_2, \mathrm{Vec}_3\}$ and fusion given by addition mod 4.
- $\mathrm{Mod}_{2\mathbf{Vec}[\mathbb{Z}_4]}(\mathbf{Vec}[\mathbb{Z}_2])$ has two connected components. On the other hand, one sees that $\pi_0(2\mathbf{Vec}[\mathbb{Z}_4])/\mathbb{Z}_2$ has the same two connected components.

Cohomology

- In general, a theory with (only) grouplike p-spacetime dimensional objects with q-ambient dimensions should be classified by degree $(p + q + 1)$ cohomology of $E[q]$.

Cohomology

- In general, a theory with (only) grouplike p-spacetime dimensional objects with q-ambient dimensions should be classified by degree $(p+q+1)$ cohomology of $E[q]$.
- Fermionic braided strongly fusion category $\mathfrak{B}$ can be classified by $\mathsf{SH}^5(E[2])$

$$\mathsf{SH}^0(\mathrm{pt}) = \mathbb{C}^\times\,, \quad \mathsf{SH}^1(\mathrm{pt}) = \mathbb{Z}_2\,, \quad \mathsf{SH}^2(\mathrm{pt}) = \mathbb{Z}_2\,.$$

Cohomology

- In general, a theory with (only) grouplike p-spacetime dimensional objects with q-ambient dimensions should be classified by degree $(p+q+1)$ cohomology of $E[q]$.
- Fermionic braided strongly fusion category $\mathfrak{B}$ can be classified by $\mathsf{SH}^5(E[2])$

$$\mathsf{SH}^0(\mathrm{pt}) = \mathbb{C}^\times\,, \quad \mathsf{SH}^1(\mathrm{pt}) = \mathbb{Z}_2\,, \quad \mathsf{SH}^2(\mathrm{pt}) = \mathbb{Z}_2\,.$$

- The (3+1)d theory associated to $\mathfrak{B}$ is condensible to the vacuum if $\mathcal{SW}^5(E[2])$ is trivial.
- $\mathcal{SW}^\bullet(\mathrm{pt})$ is a spectrum that gives the fermionic gapped theories up to morita equivalence.

$$\mathrm{H}^i(E[2]; \mathcal{SW}^j(\mathrm{pt})) \Rightarrow \mathcal{SW}^{i+j}(E[2])$$

Cohomology cont.

- In the bosonic case we compute $\mathcal{W}^5(\mathrm{pt})$ by a Galois descent procedure.
- The way to think about the bosonic case, is to treat it fermionically and equipped with an action of the categorified Galois group Gal(**SVec** /**Vec**)$= \mathbb{Z}_2^F[1]$.

Cohomology cont.

- In the bosonic case we compute $\mathcal{W}^5(\mathrm{pt})$ by a Galois descent procedure.
- The way to think about the bosonic case, is to treat it fermionically and equipped with an action of the categorified Galois group Gal(**SVec** /**Vec**)$= \mathbb{Z}_2^F[1]$.
- Galois descent says that the algebra of a bosonic higher category can be considered as the algebra of a $\mathbb{Z}_2^F[1]$-equivariant higher supercategory.
- We wish to understand the twisted $\mathcal{SW}^\bullet$-cohomology with E_2 page given by:

$$\mathrm{H}^i(B\mathbb{Z}_2^F[1];\mathcal{SW}^j(\mathrm{pt})) \Rightarrow \mathcal{SW}^{i+j}(B\mathbb{Z}_2^F[1]) = \mathcal{W}^{i+j}(\mathrm{pt}).$$

2-Deligne

Since we have been talking about braided and sylleptic F2Cs, let's discuss symmetric ones.

2-Deligne

Since we have been talking about braided and sylleptic F2Cs, let's discuss symmetric ones.

The 1-Deligne theorem:

- By working fermionically, there exists a fibre functor $F : \text{Rep}(G) \to \textbf{SVec}$.
- Implication: In a 4d/5d topological theories with line operators, we can condense them away as they are symmetric monoidal.

2-Deligne

Since we have been talking about braided and sylleptic F2Cs, let's discuss symmetric ones.

The 1-Deligne theorem:

- By working fermionically, there exists a fibre functor $F : \mathrm{Rep}(G) \to \mathbf{SVec}$.
- Implication: In a 4d/5d topological theories with line operators, we can condense them away as they are symmetric monoidal.

Theorem

1. Every symmetric fermionic fusion 2-category admits a fibre 2-functor to 2**SVec**.
2. Every symmetric fermionic *strongly fusion* 2-category admits a fibre 2-functor to 2**SVec**.

2-Deligne cont.

Strategy for part 2: Start with symmetric strongly fusion 2-category $\mathfrak{S}$, with E its abelian group of connected components.

- We begin by describing $\mathfrak{S}^\times$ the Picard sub-2-category of $\mathfrak{S}$
- We will recover $\mathfrak{S}$ through some properties of $\mathfrak{S}^\times$, which is easier to work with.

2-Deligne cont.

Strategy for part 2: Start with symmetric strongly fusion 2-category $\mathfrak{S}$, with E its abelian group of connected components.

- We begin by describing $\mathfrak{S}^\times$ the Picard sub-2-category of $\mathfrak{S}$
- We will recover $\mathfrak{S}$ through some properties of $\mathfrak{S}^\times$, which is easier to work with.
- The Picard 2-category $\mathfrak{S}^\times$ fits into the following fibre sequence of spectra

$$2\,\mathbf{SVec}^\times \to \mathfrak{S}^\times \to \mathrm{H}E \to \Sigma 2\,\mathbf{SVec}^\times,$$

- In particular, $\mathfrak{S}^\times$ is completely determined by the map of spectra $\mathrm{H}E \to \Sigma 2\,\mathbf{SVec}^\times$. Up to homotopy, such maps are classified by the group $\mathrm{SH}^7(E[4])$.

2-Deligne cont.

- Via a spectral sequence computation, we show $\mathrm{SH}^7(E[4]) = 0$.
- This implies that $\mathfrak{S}^\times \cong 2\,\mathbf{SVec}^\times \times E$ as symmetric monoidal 2-categories.
- In particular $\mathrm{B}\,\mathbf{SVec} \times E$ is a full symmetric monoidal sub-2-category of $\mathfrak{S}$. It contains an object in every connect component of $\mathfrak{S}$.

2-Deligne cont.

- Via a spectral sequence computation, we show $\mathrm{SH}^7(E[4]) = 0$.
- This implies that $\mathfrak{S}^\times \cong 2\,\mathbf{SVec}^\times \times E$ as symmetric monoidal 2-categories.
- In particular $\mathrm{B}\,\mathbf{SVec} \times E$ is a full symmetric monoidal sub-2-category of $\mathfrak{S}$. It contains an object in every connect component of $\mathfrak{S}$.
- Cauchy completing $\mathrm{B}\,\mathbf{SVec} \times E$ recovers $\mathfrak{S} = 2\,\mathbf{SVec}[E]$.
- But as we discussed before, $2\,\mathbf{SVec}[E]$ is condensable to $2\,\mathbf{SVec}$.
- In the bosonic case, such strongly fusion 2 categories are classified by $\mathrm{H}^7(E[4]; \mathbb{C}^\times)$.
- This is nonvanishing if E has 2-torsion.

More on generalizing anomalies

- There are certain categories which admit multiple fibre functors.
- Anomalies in d-dimensions should be realized as interfaces between fibre functors in $d+1$ dimensions.
- This is an analogue of the fact that anomalies for grouplike symmetries reside at the boundary of SPTs.
- It is possible to work out a noninvertible cocycle like condition for what takes the place of an anomaly.

More on generalizing anomalies

- There are certain categories which admit multiple fibre functors.
- Anomalies in d-dimensions should be realized as interfaces between fibre functors in $d+1$ dimensions.
- This is an analogue of the fact that anomalies for grouplike symmetries reside at the boundary of SPTs.
- It is possible to work out a noninvertible cocycle like condition for what takes the place of an anomaly.

Conclusion

- We wanted to simplify a 2-category as far as possible.
- The nicest case is when we find a subcategory that looks like $2\,\mathrm{Rep}(G)$ in the connected case.
- We found modules at a purely algebraic level.
- When we can make the category strongly fusion, we can compute the obstruction to condensing to the vacuum.
- This helped ups establish a generalization of the 1-Deligne theorem.

Fin.