Graphical CSS Code Transformation Using ZX Calculus

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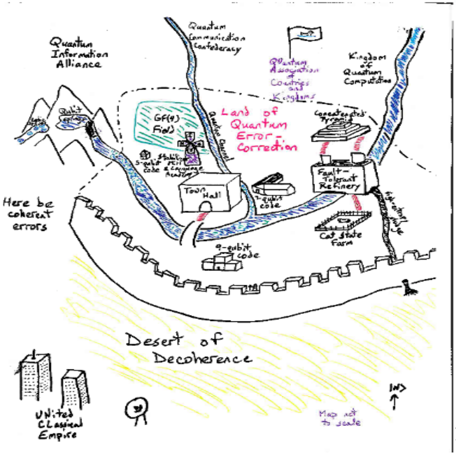
Towards a fully operational and scalable quantum computer



[1]

- Understand environmental decoherence processes and model them properly.
- Error correction to protect quantum information against decoherence.

Quantum Error Correcting Codes (QECCs)

[1] Gottesman, D. Quantum Error Correction and Fault Tolerance (Part 1) - CSSQI 2012.

Basic States & Gates

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |+\rangle = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |-\rangle = \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle = |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = iXZ, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad CX = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

What is a subspace code?

Consider a two-dimensional subspace in a large Hilbert space. The basis of this subspace is given by the encoded states $|\bar{0}\rangle$ and $|\bar{1}\rangle$. The number of encoded qubit is 1.

Example: Three-qubit code against a bit-flip error

$$|\bar{0}\rangle = |000\rangle \qquad |\bar{1}\rangle = |111\rangle$$

$$|\bar{\psi}\rangle = \alpha\,|\bar{0}\rangle + \beta\,|\bar{1}\rangle = \alpha\,|000\rangle + \beta\,|111\rangle\,, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Conside a 2^k-dimensional subspace in a larger space. The number of encoded qubits is k. The basis of this subspace is given by the encoded states

$$|\bar{0}\cdots\bar{0}\rangle\,, \quad |\bar{0}\cdots\bar{1}\rangle\,, \ldots, \quad |\bar{1}\cdots\bar{1}\rangle\,.$$

What are logical operators?

Logical operators should have the algebra of the Pauli operators on the encoded qubits (aka, logical qubits).

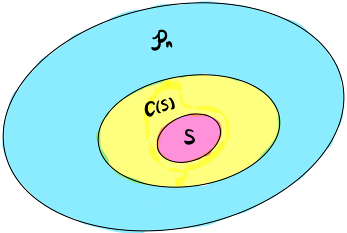
Define logical operators $\bar{X}_i, \bar{Z}_i$, $1 \leq i \leq k$.

$$
\begin{gathered}
\bar{X}_i \bar{Z}_i = -\bar{Z}_i \bar{X}_i, \quad 1 \leq i \leq k \\
\bar{X}_i \bar{Z}_j = \bar{Z}_j \bar{X}_i, \quad 1 \leq i, j \leq k, \quad i \neq j \\
\bar{Z} |\bar{0}\rangle = |\bar{0}\rangle, \quad \bar{Z} |\bar{1}\rangle = -|\bar{1}\rangle \\
\bar{X} |\bar{+}\rangle = |\bar{+}\rangle, \quad \bar{X} |\bar{-}\rangle = -|\bar{-}\rangle
\end{gathered}
$$

What is a stabilizer code? [2]

Consider three groups of Pauli operators

1. Pauli group on n qubits: $\mathcal{P}_n := \{i^c \left(\bigotimes_{i=1}^n P_i\right), P_i \in \{X, Y, Z, I\}, 0 \leq c \leq 3\}$.
2. Stabilizer group: $\mathcal{S} = \langle M_1, M_2, \ldots, M_{n-k} \rangle$, $-I \notin \mathcal{S}$. $\mathcal{S} \subset \mathcal{P}_n$. $\mathcal{S}$ Abelian.
3. Centralizer of $\mathcal{S}$: $C(\mathcal{S}) := \{U \in \mathcal{P}_n; [U, M] = 0, \forall M \in \mathcal{S}\}$.



Definition

Stabilizer codes are a class of quantum error-correcting codes used in quantum computing. Its code space $\mathcal{C}$ is the joint $+1$ eigenspace of $\mathcal{S}$.

[2] Gottesman, D. (1997). Stabilizer codes and quantum error correction. California Institute of Technology.

What is the code space in the stabilizer formalism?

$|\bar{\psi}\rangle$ is called a *codeword* in $\mathcal{C}$, where

$$\mathcal{C} := \{n\text{-qubit state } |\bar{\psi}\rangle \,;\, M|\bar{\psi}\rangle = |\bar{\psi}\rangle \,, \forall M \in \mathcal{S}\}$$

There are three important parameters for a stabilizer code: $[[n, k, d]]$.

- n is the number of physical qubits.
- k is the number of logical (or encoded) qubits.
- d is the code distance.

Example

Consider $\mathcal{S} = \langle XX, ZZ \rangle$ on two qubits. Then $\mathcal{C} = \left\{ \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \right\}$.

What are logical operators in the stabilizer code?

Consider the centralizer of $\mathcal{S}$,

$$C(\mathcal{S}) \coloneqq \{U \in \mathcal{P}_n; [U, M] = 0, \forall M \in \mathcal{S}\}.$$

- Since $\mathcal{S}$ is Abelian, $\mathcal{S} \subset C(\mathcal{S})$. They act trivially on $|\bar{\psi}\rangle$.
- $C(\mathcal{S}) \setminus \mathcal{S} = \langle \bar{X}_1, \bar{Z}_1, \ldots, \bar{X}_k, \bar{Z}_k \rangle$, up to stabilizer generators of $\mathcal{S}$.

  They act non-trivially on $|\bar{\psi}\rangle$.
- All other operators in $\mathcal{P}_n$ anti-commute with at least one element in $\mathcal{S}$ and map a codeword $|\bar{\psi}\rangle$ onto a state outside the code space $\mathcal{C}$.

Fundamental Theorem of Stabilizer Theory

Theorem

If $\mathcal{S} \subset \mathcal{P}_n$ has m generators, then $\mathcal{C}$ is a 2^k dimensional subspace of $(\mathbb{C}^2)^{\otimes n}$, $k = n - m$.

- $\mathcal{S}$ is *maximal* when $m = n$. $\mathcal{S}$ fixes a $2^0 = 1$ dimensional subspace, i.e. a quantum state, up to scalar factor.
- More generally, we think of non-maximal stabiliser groups as a description for the embedding of $k = n - m$ "logical" qubits into a space of n "physical" qubits.

Example: Four-qubit code [[4, 2, 2]]

$$\mathcal{S} = \langle XXXX, ZZZZ \rangle$$

- What is the dimension of the code space?

Code Distance

Definition

Let d be the distance of a stabilizer code $\mathcal{C}(\mathcal{S})$, $|P|$ denotes the weight of $P \in \mathcal{P}_n$, the number of physical qubits on which P acts nontrivially. Then

$$d := \min_{P \in C(\mathcal{S}) \setminus \mathcal{S}} |P|.$$

The code distance is the **minimum weight** of any logical operator.

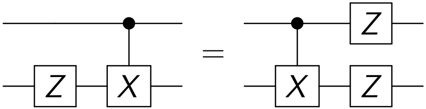
Example: Four-qubit code $[[4, 2, 2]]$

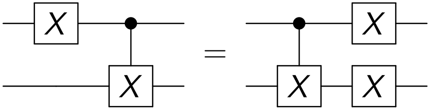
$$\mathcal{S} = \langle XXXX, ZZZZ \rangle$$

- Find pairs of mutually anti-commuting Paulis which commute with $XXXX$, $ZZZZ$.
- What is the code distance?

Fault-tolerant Technique: Transversality

- The large-scale execution of quantum algorithms requires basic logical quantum operations to be implemented fault-tolerantly.





Definition

A transversal logical operator is not implemented by any multi-qubit physical operation acting on the same code block[3].

- Transversality prevents any errors from spreading within a block, so a single physical error cannot cause a whole block of codes to go bad.

[3] Gottesman, D. (2000). Fault-tolerant quantum computation with local gates. Journal of Modern Optics, 47(2-3), 333-345.

Code Construction

Definition

Let $\mathcal{P}$ be the single-qubit Pauli group. If M is a $k \times n$ binary matrix and $T \in \mathcal{P}$, then

$$M^T := \left\{ \bigotimes_{j=1}^{n} T^{[M]_{ij}} : 1 \leq i \leq k \right\} \subset \mathcal{P}^{\otimes n}.$$

- Example: Let $T = X$, $k = 3$, $n = 7$.

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}$$

- Then $M^T = \{M_1, M_2, M_3\}$, where

$M_1 = X \otimes I \otimes X \otimes I \otimes X \otimes I \otimes X = X_1X_3X_5X_7,\ M_2 = X_2X_3X_6X_7,\ M_2 = X_4X_5X_6X_7.$

Calderbank-Shor-Steane (CSS) Codes

Definition

CSS codes are stabilizer codes whose stabilizer generators are defined by two orthogonal binary matrices G, H, $GH^T = 0$:

$$S = \langle G^X, H^Z \rangle.$$

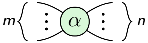
- The stabilizer generators can be divided into two types: X type and Z type.
- $GH^T = 0$ implies that each X generator overlaps with a Z generator in an even number of places.
- Example: The $[\![7,1,3]\!]$ Steane code

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}$$

$$\begin{aligned} S &= \langle M^X, M^Z \rangle \\ &= \langle M_1^X, M_2^X, M_3^X, M_1^Z, M_2^Z, M_3^Z \rangle. \end{aligned}$$

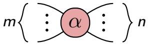
The ZX Calculus

- An intuitive graphical language for quantum computation.
- Every ZX diagram is composed of two types of generators:
  - Z spiders, which sum over the eigenbasis of the Z operator:



$$:= |0\rangle^{\otimes n}\langle 0|^{\otimes m} + e^{i\alpha}|1\rangle^{\otimes n}\langle 1|^{\otimes m},$$

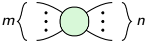
  - X spiders, which sum over the eigenbasis of the X operator:



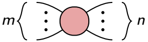
$$:= |+\rangle^{\otimes n}\langle +|^{\otimes m} + e^{i\alpha}|-\rangle^{\otimes n}\langle -|^{\otimes m}.$$

Definition

A ZX diagram is *phase-free* if its spiders have no phases.



$$:= |0\rangle^{\otimes n}\langle 0|^{\otimes m} + |1\rangle^{\otimes n}\langle 1|^{\otimes m}$$

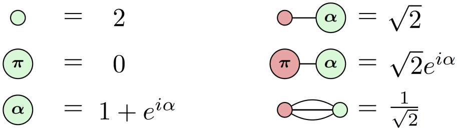


$$:= |+\rangle^{\otimes n}\langle +|^{\otimes m} + |-\rangle^{\otimes n}\langle -|^{\otimes m}$$

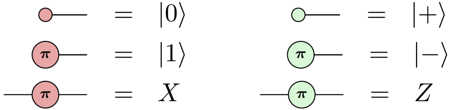
The ZX Calculus is Universal

Any linear map from *m* to *n* qubits corresponds exactly to a ZX diagram.

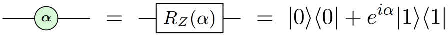
- A ZX diagram with 0 input and output represents a scalar.



- A ZX diagram with 0 input and 1 output represents a state.

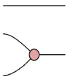


- A ZX diagram with the same number of inputs and outputs represents a unitary.

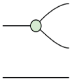


Represent the CNOT Gate in ZX

Proof. Horizontally composing the two diagrams below,

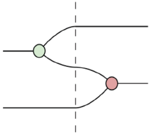


$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\end{pmatrix}$$

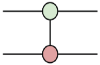


$$= \begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

we get



$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

Therefore, $CNOT = \sqrt{2}$ 

The ZX Calculus is Complete

If two ZX diagrams represent the same linear map, then there should be a sequence of rewrites that transforms one diagram into the other.

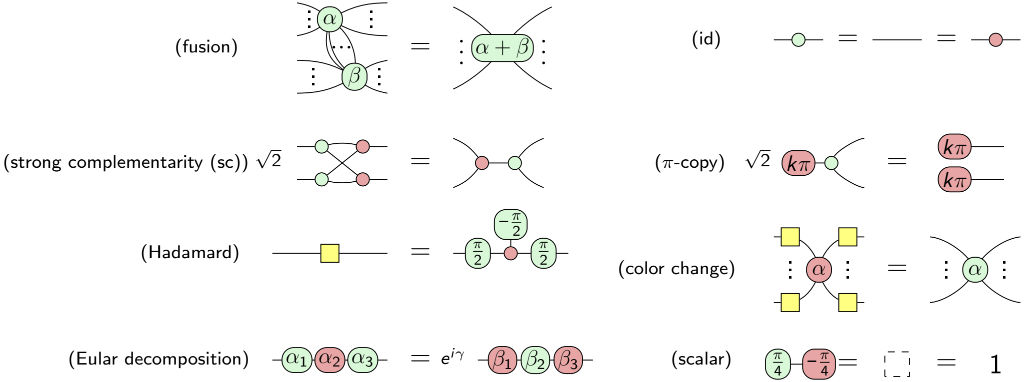


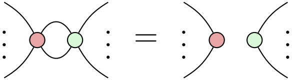
Figure: The minimal complete rule set for ZX calculus.[4]

[4] Vilmart, R. (2019, June). A near-minimal axiomatisation of zx-calculus for pure qubit quantum mechanics. In 2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS) (pp. 1-10). IEEE.

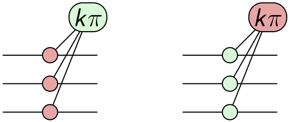
Additional ZX Rules

These rules are derivable from the minimal rule set. Used extensively in this work.

(Hopf)

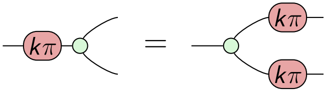


(XXX measurement)



(ZZZ measurement)

(π-copy') $\sqrt{2}$



Phase-free ZX Diagrams are CSS Codes

Consider an $[\![n,k,d]\!]$ CSS code with X-type stabilizers $\{S_1^X, S_2^X, \ldots, S_{m_1}^X\}$ and logical operators $\{\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_k\}$, it has a unique ZX normal form [5].

Example: The Steane code

- $n = 7,\ k = 1,\ d = 3.$
- 3 X-type stabilizers:

$$S_1^X = X_2X_3X_6X_7$$
$$S_2^X = X_1X_3X_5X_7$$
$$S_3^X = X_4X_5X_6X_7$$

- 1 logical X operator: $\overline{X} = X_1X_4X_5$

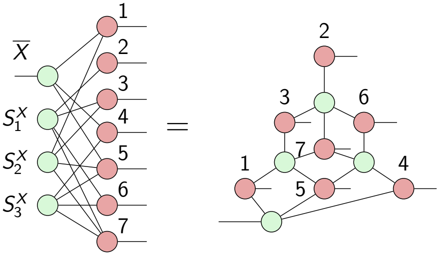


Figure: The Steane code encoder in ZX normal form.

[5] Kissinger, A. (2022). Phase-free ZX diagrams are CSS codes (... or how to graphically grok the surface code). arXiv preprint arXiv:2204.14038.

CSS Subsystem Codes[6]

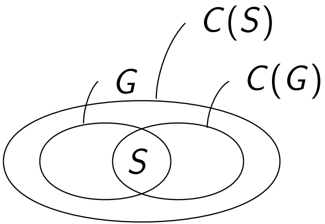
Subsystem codes are stabilizer codes where some of the logical qubits are not used for information storage and processing. These logical qubits are called *gauge qubits*.

Definition

Let G be an arbitrary subgroup of the Pauli group $\mathcal{P}$. A subsystem code defined by G has a group S of stabilizers and a set L_g of gauge operators, where

$$S = C(G) \cap G, \text{ where } C(G) = \{P \in \mathcal{P} : PM = MP, \forall M \in G\}, \qquad L_g = G \setminus S.$$

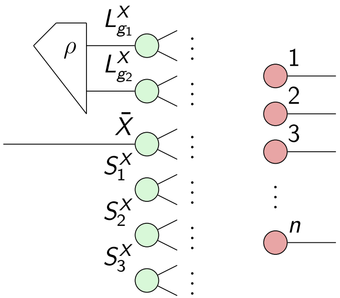
- If G is Abelian, $G = S$ and $L_g = \emptyset$.
- When $L_g \neq \emptyset$, it contains pairs of anticommuting Pauli operators.



[6] David Kribs, Raymond Laflamme & David Poulin (2005): Unified and Generalized Approach to Quantum Error Correction. Physical Review Letters, 94.

Normal Form for CSS Subsystem Codes

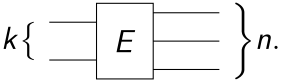
Given a stabilizer tableau of a CSS subsystem code, the corresponding ZX normal form can be constructed through the following steps:



1. For each physical qubit, introduce an X spider.
2. For each X-type stabilizer S^x_i, logical operator $\overline{X}_j$ and gauge operator $L^x_{g_t}$, introduce a Z spider and connect it to all X spiders where this operator has support.
3. Give each X spider an output wire.
4. Give each Z spider representing $\overline{X}_j$ an input wire.
5. Give all Z spiders representing $L^x_{g_t}$ a joint arbitrary input state (i.e., a density operator ρ).

Pushing through the Encoder

For any $[\![n, k, d]\!]$ CSS code, its encoder map E is of the form:

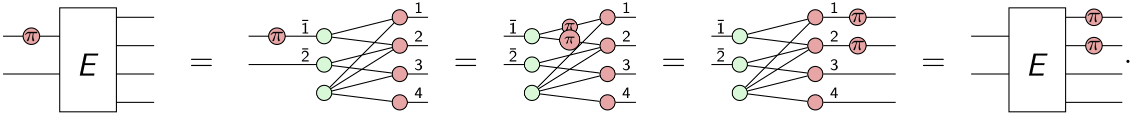


E is an isometry. $E^\dagger E = I$.

Lemma

In any CSS code, all $\overline{X_i}$ and $\overline{Z_i}$ must be multi-qubit Pauli operators.

Example: For the $[\![4, 2, 2]\!]$ code, $\overline{X_1} = X_1 X_2$:

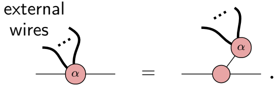


Physically Implement a Logical Operator

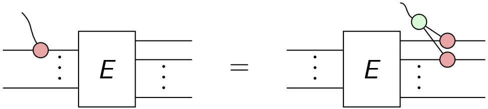
Proposition

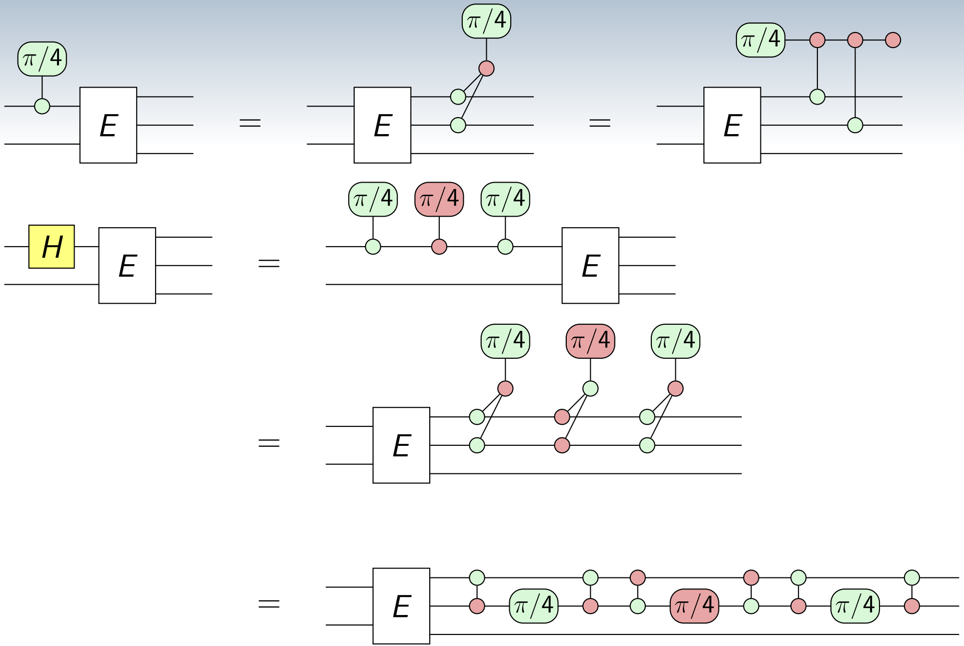
Let E be the encoder of a CSS code. For any ZX diagram L on the left-hand side of E, one can write down a corresponding ZX diagram P on the right-hand side of E, such that $EL = PE$. In other words, P is a valid physical implementation of L on that CSS code.

- Unfuse all spiders on logical qubit wires of L, whenever they are not phase-free or have more than one external wire.



- For each X spider on logical qubit wires, rewriting E to be in ZX normal form and then applying the strong complementarity (sc) rule.





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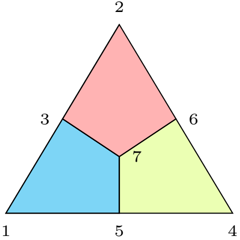
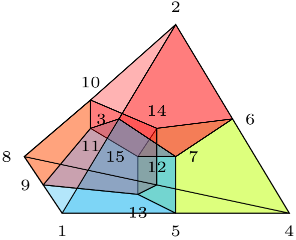
Switch between Two CSS Codes

	QRM(3)	**QRM(4)**
Code parameters	$[\![7, 1, 3]\!]$	$[\![15, 1, 3]\!]$
Logical operators	$\overline{X} = X_1X_4X_5$, $\overline{Z} = Z_1Z_4Z_5$	$\overline{X} = X_1X_4X_5$, $\overline{Z} = Z_1Z_4Z_5$
Other transversal gates	$CX, S,$ **H**	$CX, S,$ **T**
Towards universality	Need transversal logical T	Need transversal logical H
Topology	Triangle, 2D	Tetrahedron, 3D



(a) QRM(3) as a 2D color code.



(b) QRM(4) as a 3D color code.

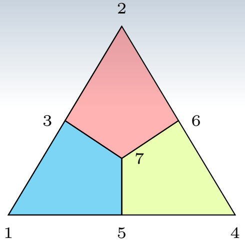
Steane Code & Quantum Reed-Muller Code

QRM(3) & QRM(4) are stabilizer codes defined by the stabilizers $\mathbb{S}_3$ & $\mathbb{S}_4$ respectively.

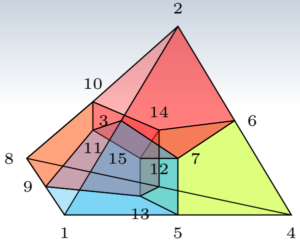
$$\mathbb{S}_3 = \langle M^X, M^Z \rangle, \qquad \mathbb{S}_4 = \langle N^X, N^Z, H^Z, T^Z \rangle, \qquad \text{where}$$

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}, \quad N = \begin{bmatrix} M & 0 & M \\ \mathbf{0} & 1 & \mathbf{1} \end{bmatrix}_{4\times 15}, \quad H = \begin{bmatrix} M & \mathbf{0} \end{bmatrix}_{3\times 15}$$

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 15}$$



(a) QRM(3) as a 2D color code.



(b) QRM(4) as a 3D color code.

- $\mathbb{S}_3 = \langle M^X, M^Z \rangle$.
- Coloured face $\mapsto X/Z$ stabilizer generator.
- QRM(3) is self-dual.
- $\overline{X} = X_1X_4X_5$, $\overline{Z} = Z_1Z_4Z_5$.
- $d = 3$.

- $\mathbb{S}_4 = \langle N^X, N^Z, H^Z, T^Z \rangle$.
- Coloured face $\mapsto Z$ stabilizer generator.
- Coloured cell $\mapsto X/Z$ stabilizer generator.
- $\overline{X} = X_1X_4X_5$, $\overline{Z} = Z_1Z_4Z_5$.
- $d = 3$.

Extended Quantum Reed-Muller Code: EQRM

EQRM is defined by the stabilizer group $\mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X \rangle$, where

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}, \qquad N = \begin{bmatrix} M & 0 & M \\ \mathbf{0} & 1 & \mathbf{1} \end{bmatrix}_{4\times 15}, \qquad H = \begin{bmatrix} M & \mathbf{0} \end{bmatrix}_{3\times 15}.$$

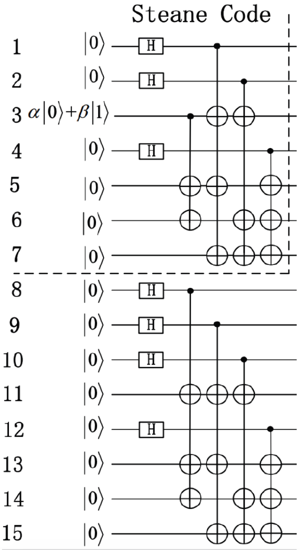
- Let $|\overline{0}\rangle$ and $|\overline{1}\rangle$ be the logical 0 and 1 encoded in QRM(3).
- Let $|\overline{\psi}\rangle = \alpha\,|\overline{0}\rangle + \beta\,|\overline{1}\rangle$ be the single-qubit logical information encoded in QRM(3).

Lemma

An EQRM codeword $|\overline{\Phi}\rangle$ can be decomposed as

$$|\overline{\Phi}\rangle = |\overline{\psi}\rangle \otimes |\phi\rangle\,, \ \textit{where}$$

$|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|\overline{0}\rangle + |1\rangle\,|\overline{1}\rangle)$.



$$|\bar{\psi}\rangle = \alpha|\bar{0}\rangle_3 + \beta|\bar{1}\rangle_3$$

$\mathbb{S}_3$ stabilizes $|\bar{\psi}\rangle$

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\bar{0}\rangle_3 + |1\rangle|\bar{1}\rangle_3)$$

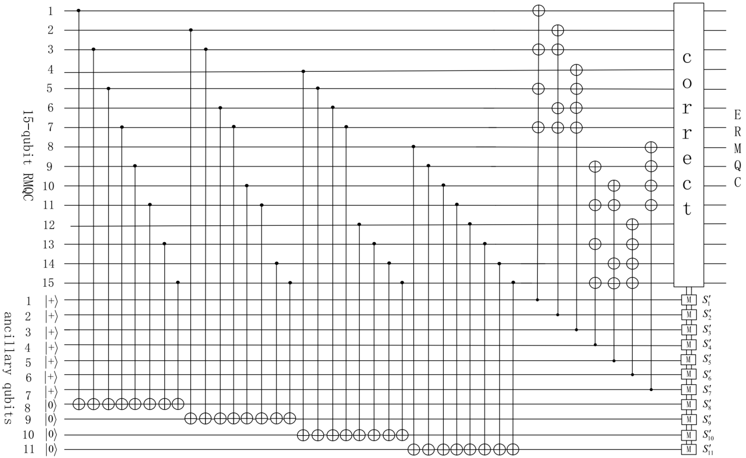
No information is stored on the last eight qubits.

$$|\bar{\Phi}\rangle = |\bar{\psi}\rangle|\phi\rangle$$

$\mathbb{S}_E$ stabilizes $|\bar{\Phi}\rangle$

Code Switching[7,8]

- Codes with complementary fault-tolerant gate sets are switched between each other to realize a universal set of logical operations.
- Fault-tolerantly switch between QRM(3) and QRM(4)



[7] Anderson, J. T., Duclos-Cianci, G., & Poulin, D. (2014). Fault-tolerant conversion between the steane and reed-muller quantum codes. Physical review letters, 113(8), 080501.

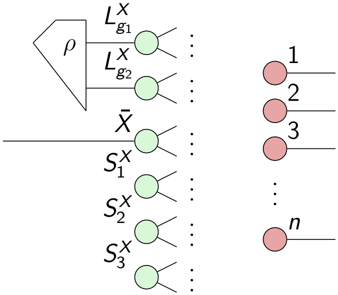
[8] Quan, D. X., Zhu, L. L., Pei, C. X., & Sanders, B. C. (2018). Fault-tolerant conversion between adjacent Reed–Muller quantum codes based on gauge fixing. Journal of Physics A: Mathematical and Theoretical, 51(11), 115305.

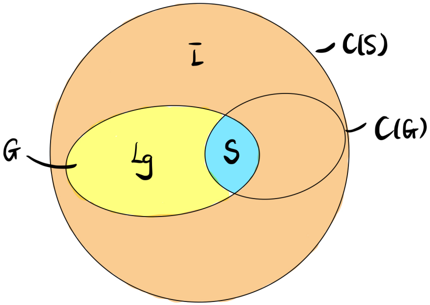
Recap: Normal Form for CSS Subsystem Codes

Definition

Let G be an arbitrary subgroup of the Pauli group $\mathcal{P}$. A subsystem code defined by G has a group S of stabilizers and a set L_g of gauge operators, where

$$S = C(G) \cap G, \text{ where } C(G) = \{P \in \mathcal{P} : PM = MP, \forall M \in G\}, \qquad L_g = G \setminus S.$$





Subsystem Quantum Reed-Muller Code: SQRM

Definition

SQRM is defined by the gauge group:

$$G = \langle N^X, N^Z, H^Z, H^X, T^Z \rangle.$$

- The associated stabilizer group, gauge operators and logical operators are:

$$\mathbb{S}_S = \langle N^X, N^Z, H^Z \rangle_{11}, \qquad L_g = \langle H^X, T^Z \rangle_6, \qquad \overline{L} = \langle \overline{X}, \overline{Z} \rangle_2.$$

- For brevity, we will use $L_g^X = H^X$ and $L_g^Z = T^Z$.

$\Rightarrow$ SQRM has 1 logical qubit and 3 gauge qubits.

$\Rightarrow$ Alternatively, $\mathbb{S}_S$ stabilizes the $[\![15, 4, 3]\!]$ CSS code, with logical operators $\{L_g, \overline{L}\}$.

Construct SQRM from QRM(4) & EQRM

- QRM(4): $\mathbb{S}_4 = \langle N^X, N^Z, H^Z, T^Z \rangle$
- EQRM: $\mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X \rangle$
- SQRM: $(G,\ \mathbb{S}_S,\ L_g,\ \overline{L})$, where

$$
\begin{aligned}
G &= \mathbb{S}_4 \cup \mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X, T^Z \rangle \\
\mathbb{S}_S &= \mathbb{S}_4 \cap \mathbb{S}_E = \langle N^X, N^Z, H^Z \rangle \\
L_g &= G \setminus \mathbb{S}_S = \langle H^X, T^Z \rangle \\
\overline{L} &= \langle \overline{X}, \overline{Z} \rangle
\end{aligned}
$$

Construct SQRM from QRM(4) & EQRM

- QRM(4): $\mathbb{S}_4 = \langle N^X, N^Z, H^Z, T^Z \rangle$
- EQRM: $\mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X \rangle$
- SQRM: $(G,\ \mathbb{S}_S,\ L_g,\ \overline{L})$, where

$$
\begin{aligned}
G &= \mathbb{S}_4 \cup \mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X, T^Z \rangle \\
\mathbb{S}_S &= \mathbb{S}_4 \cap \mathbb{S}_E = \langle N^X, N^Z, H^Z \rangle \\
L_g &= G \setminus \mathbb{S}_S = \langle H^X, T^Z \rangle \\
\overline{L} &= \langle \overline{X}, \overline{Z} \rangle
\end{aligned}
$$

Construct SQRM from QRM(4) & EQRM

- QRM(4): $\mathbb{S}_4 = \langle N^X, N^Z, H^Z, T^Z \rangle$
- EQRM: $\mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X \rangle$
- SQRM: $(G,\ \mathbb{S}_S,\ L_g,\ \overline{L})$, where

$$
\begin{aligned}
G &= \mathbb{S}_4 \cup \mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X, T^Z \rangle \\
\mathbb{S}_S &= \mathbb{S}_4 \cap \mathbb{S}_E = \langle N^X, N^Z, H^Z \rangle \\
L_g &= G \setminus \mathbb{S}_S = \langle T^Z, H^X \rangle \\
\overline{L} &= \langle \overline{X}, \overline{Z} \rangle
\end{aligned}
$$

Construct SQRM from QRM(4) & EQRM

- QRM(4): $\mathbb{S}_4 = \langle N^X, N^Z, H^Z, T^Z \rangle$
- EQRM: $\mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X \rangle$
- SQRM: $(G,\ \mathbb{S}_S,\ L_g,\ \overline{L})$, where

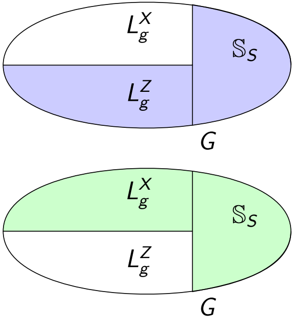
$$
\begin{aligned}
G &= \mathbb{S}_4 \cup \mathbb{S}_E = \langle N^X, N^Z, H^Z, H^X, T^Z \rangle \\
\mathbb{S}_S &= \mathbb{S}_4 \cap \mathbb{S}_E = \langle N^X, N^Z, H^Z \rangle \\
L_g &= G \setminus \mathbb{S}_S = \langle T^Z, H^X \rangle \\
\overline{L} &= \langle \overline{\mathbf{X}}, \overline{\mathbf{Z}} \rangle
\end{aligned}
$$

Forward & Backward Switching [9]

- Forward conversion: EQRM $\Rightarrow$ QRM(4) $\mathbb{S}_4$
- Backward conversion: QRM(4) $\Rightarrow$ EQRM $\mathbb{S}_E$

$$\mathbb{S}_S \xrightarrow{\mathbb{M}\left(L_g^Z\right)} \mathbb{S}_4 = \mathbb{S}_S \cup L_g^Z$$

$$\mathbb{S}_S \xrightarrow{\mathbb{M}\left(L_g^X\right)} \mathbb{S}_E = \mathbb{S}_S \cup L_g^X$$



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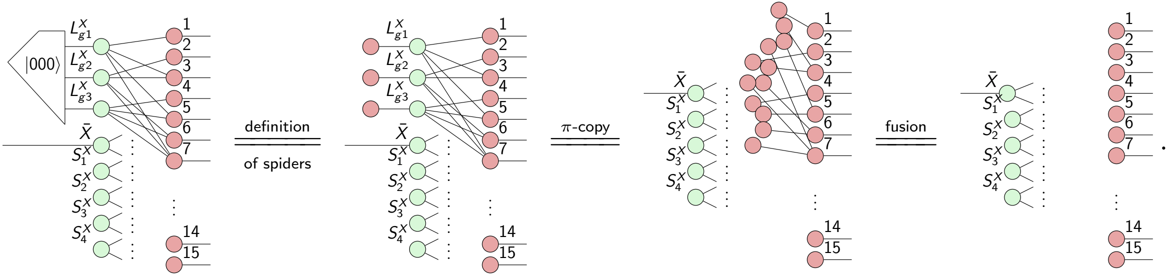
Gauge Fixing of the SQRM Code

Lemma

When the three gauge qubits are in the $|\overline{000}\rangle$ state, SQRM is fixed to QRM(4).



- When the three gauge qubits are in the $|\overline{+++}\rangle$ state, SQRM is fixed to EQRM.

$\Rightarrow$ Start with the XZ normal form of the SQRM encoder.

Gauge Fixing SQRM Switches between QRM(4) and EQRM

Step 1: Measure a commuting subset of gauge operators.
E.g., measure three X-type gauge operators L_g^X and obtain the corresponding outcomes $k_1, k_2, k_3 \in \{0, 1\}$.

Remark: $0 \mapsto +1$ and $1 \mapsto -1$.

Step 2: When $k_i = 1$, apply some correction operator $L_{g_i}^Z$ as the gauge qubit i has collapsed to the wrong state $|-\rangle$.

$\Rightarrow$ All three gauge qubits are set to the desired state $|+\rangle$.

`Example 1:` Measure an X-type stabilizer

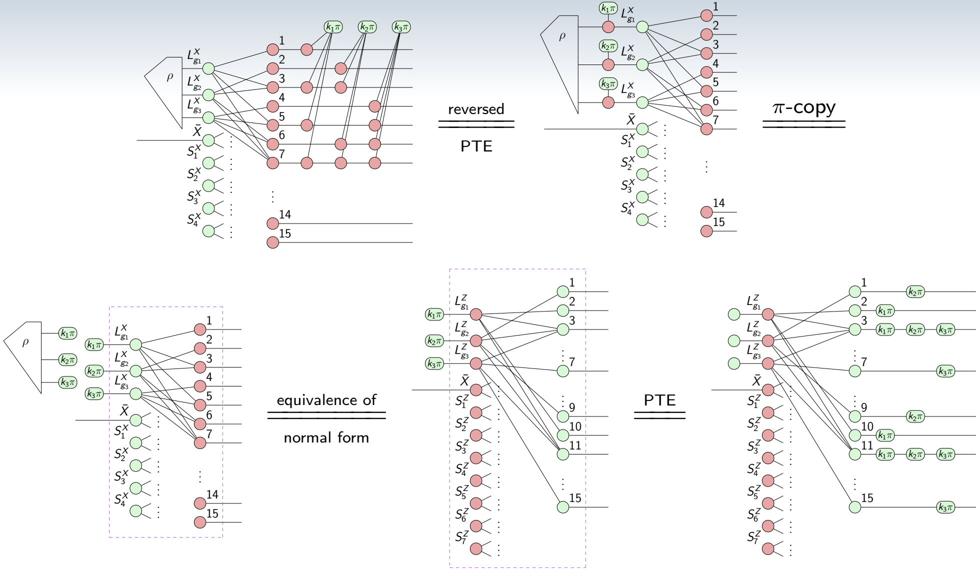
$$X|+\rangle = |+\rangle, \; X|-\rangle = -|-\rangle$$

$$|+\rangle \overset{Z}{\longleftrightarrow} |-\rangle$$

`Example 2:` Measure a Z-type stabilizer

$$Z|0\rangle = |0\rangle, \; Z|1\rangle = -|1\rangle$$

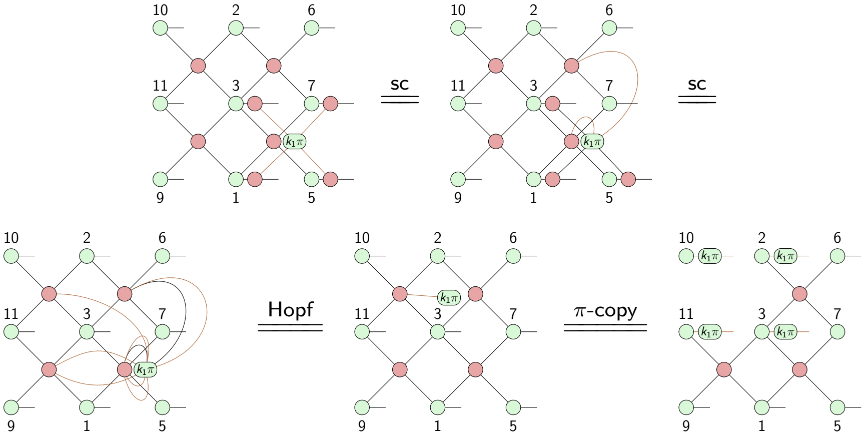
$$|0\rangle \overset{X}{\longleftrightarrow} |1\rangle$$



ρ
$L^X_{g_1}$
$L^X_{g_2}$
$L^X_{g_3}$
$\bar{X}$
S^X_1
S^X_2
S^X_3
S^X_4
$k_1\pi$
$k_2\pi$
$k_3\pi$
reversed
PTE
π-copy
equivalence of
normal form
PTE
$L^Z_{g_1}$
$L^Z_{g_2}$
$L^Z_{g_3}$
S^Z_1
S^Z_2
S^Z_3
S^Z_4
S^Z_5
S^Z_6
S^Z_7

Syndrome-determined Fixing Operation

- Measuring L_g^X adds these operators into the stabilizer group and removes stabilizers L_g^Z. Moreover, the fixing operations be readily read-off from the graphical derivation.



Open Problems

Through the lens of ZX calculus, we will

- **Present** CSS code deformations [10].
- **Understand** code concatenation [11].
- **Derive** new good QECCs from the existing QECCs[12].
- **Unify** fault-tolerant protocols for stabilizer codes [13],[14].

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Thank you!

Example: $\overline{X} = X_4X_5X_6$