Enriched structure-semantics adjunctions and monad-theory equivalences for a subcategory of arities

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ATCAT Seminar
March 8, 2022[1]

Natural Sciences and Engineering Research Council of Canada
Conseil de recherches en sciences naturelles et en génie du Canada
Canada

[1] We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC). Nous remercions le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG) de son soutien.

Motivation

- Several structure-semantics adjunctions and monad-theory equivalences have been established in category theory.
- In [7], Lawvere established a structure-semantics adjunction between Lawvere theories and *tractable* **Set**-valued functors, which was later generalized by Linton [8]. For a complete and well-powered closed category $\mathscr{V}$, Dubuc [4] proved a structure-semantics adjunction between *$\mathscr{V}$-theories* and *tractable* $\mathscr{V}$-valued $\mathscr{V}$-functors.

Motivation

- Linton [8] also showed that there is an equivalence between Lawvere theories and finitary monads on **Set**. Lucyshyn-Wright [10] generalized this by showing that if $\mathscr{J} \hookrightarrow \mathscr{V}$ is any *eleutheric system* of arities in a closed category $\mathscr{V}$, then there is an equivalence between $\mathscr{J}$-theories and $\mathscr{J}$*-ary* $\mathscr{V}$-monads on $\mathscr{V}$.
- Building on earlier work of Power and Nishizawa [14, 13], Bourke and Garner [2] recently showed that if $\mathscr{J} \hookrightarrow \mathscr{C}$ is any small subcategory of arities in a locally presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally presentable closed category $\mathscr{V}$, then there is an equivalence between $\mathscr{J}$*-theories* and $\mathscr{J}$*-nervous* $\mathscr{V}$-monads on $\mathscr{C}$.
- Neither equivalence subsumes the other; can both equivalences, along with the aforementioned structure-semantics adjunctions, be captured by a common framework that also yields new examples?

Objectives

- That is the subject of this talk: we have developed a general framework for studying enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities, which specializes to recover the aforementioned results and also yields new examples.

- More specifically, given a subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ over a closed category $\mathscr{V}$, we will identify hypotheses on these data that entail a structure-semantics adjunction, a monad-theory equivalence, a rich theory of *presentations* for monads and theories, and more.

Basic definitions

- We fix a **subcategory of arities** $j : \mathscr{J} \hookrightarrow \mathscr{C}$, i.e. a full and dense sub-$\mathscr{V}$-category, in a $\mathscr{V}$-category $\mathscr{C}$ over a symmetric monoidal closed category $\mathscr{V}$. Since we do not assume that $\mathscr{J}$ is small or that $\mathscr{V}$ is (co)complete, we also fix a suitable universe extension $\mathscr{V} \hookrightarrow \mathscr{V}'$.

- We have a fully faithful $\mathscr{V}'$-functor

$$N_j : \mathscr{C} \to [\mathscr{J}^{\mathbf{op}}, \mathscr{V}]$$

$$N_j C = \mathscr{C}(j-, C)$$

  that we call the j-**nerve** $\mathscr{V}'$-functor. The presheaves in its essential image are called j-**nerves**.

Pretheories and their algebras

- (Linton [8], Diers [3], Bourke-Garner [2]) A $\mathscr{J}$-**pretheory** is just an identity-on-objects $\mathscr{V}$-functor $\tau : \mathscr{J}^{\mathbf{op}} \to \mathscr{T}$, while a $\mathscr{J}$-**theory** is a $\mathscr{J}$-pretheory $\mathscr{T}$ such that each $\mathscr{T}(J, \tau-) : \mathscr{J}^{\mathbf{op}} \to \mathscr{V}$ ($J \in \mathbf{ob}\,\mathscr{J}$) is a j-nerve. We have the category $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-pretheories and its full subcategory $\mathbf{Th}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-theories.

- Let $\mathscr{T}$ be a $\mathscr{J}$-pretheory. The $\mathscr{V}'$-category $\mathscr{T}$-Alg of **(concrete)** $\mathscr{T}$-**algebras** is defined by the following pullback in $\mathscr{V}'$-CAT:

$$\begin{array}{ccc} \mathscr{T}\text{-Alg} & \longrightarrow & [\mathscr{T}, \mathscr{V}] \\ {\scriptstyle U^{\mathscr{T}}}\downarrow & & \downarrow{\scriptstyle [\tau,1]} \\ \mathscr{C} & \xrightarrow[N_j]{} & [\mathscr{J}^{\mathbf{op}}, \mathscr{V}]. \end{array}$$

Amenable subcategories of arities

- A $\mathscr{J}$-pretheory $\mathscr{T}$ is **admissible** if the $\mathscr{V}'$-category $\mathscr{T}$-Alg is actually a $\mathscr{V}$-category, and $U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}$ has a left adjoint.
- The subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **amenable** if every $\mathscr{J}$-theory is admissible, and is **strongly amenable** if every $\mathscr{J}$-pretheory $\mathscr{T}$ is admissible.

$\mathscr{J}$-tractable $\mathscr{V}$-categories

- A $\mathscr{J}$**-tractable** $\mathscr{V}$**-category over** $\mathscr{C}$ is a $\mathscr{V}$-category $G : \mathscr{A} \to \mathscr{C}$ over $\mathscr{C}$ such that $\mathscr{C}$ admits the weighted limit $\{\mathscr{C}(J, G-), G\}$ for each $J \in \mathbf{ob}\, \mathscr{J}$. Then $\mathscr{J}$-**Tract**$(\mathscr{C})$ is the full subcategory of $\mathscr{V}$-CAT$/\mathscr{C}$ consisting of the $\mathscr{J}$-tractable $\mathscr{V}$-categories over $\mathscr{C}$.

- Let $\mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})$ be the full subcategory of $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ consisting of the *admissible* $\mathscr{J}$-pretheories. We define a **semantics** functor

$$\mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$$

by

$$\mathbf{Sem}\,\mathscr{T} = \left(U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}\right)$$

for each admissible $\mathscr{J}$-pretheory $\mathscr{T}$.

$\mathscr{J}$-structure

- Let $G : \mathscr{A} \to \mathscr{C}$ be a $\mathscr{J}$-tractable $\mathscr{V}$-category over $\mathscr{C}$. We define a $\mathscr{J}$-theory $\tau_G : \mathscr{J}^{\mathbf{op}} \to \mathbf{Str}G$, the $\mathscr{J}$-**structure of** G, by taking the (bijective-on-objects, fully faithful) factorization of the composite $\mathscr{V}'$-functor

$$\begin{array}{ccccc} \mathscr{J}^{\mathbf{op}} & \xrightarrow{j^{\mathbf{op}}} & \mathscr{C}^{\mathbf{op}} & \xrightarrow{N_{G^{\mathbf{op}}}} & [\mathscr{A}, \mathscr{V}]. \\ & {}_{\tau_G}\searrow & & \nearrow & \\ & & \mathbf{Str}G & & \end{array}$$

(Since G is $\mathscr{J}$-tractable, $\mathbf{Str}G$ is indeed a $\mathscr{V}$-category and moreover a $\mathscr{J}$-theory).

The structure-semantics adjunction

A $\mathscr{J}$**-algebraic** $\mathscr{V}$**-category over** $\mathscr{C}$ is a $\mathscr{V}$-category over $\mathscr{C}$ in the essential image of **Sem**; we let $\mathscr{J}$-$\mathbf{Alg}(\mathscr{C})$ be the full subcategory of $\mathscr{J}$-$\mathbf{Tract}(\mathscr{C})$ consisting of these objects.

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be an amenable subcategory of arities. The semantics functor $\mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$ *has a left adjoint* **Str** *that sends each $\mathscr{J}$-tractable $\mathscr{V}$-category over $\mathscr{C}$ to its $\mathscr{J}$-structure. This adjunction is idempotent, and restricts to an adjoint equivalence*

$$\mathbf{Th}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \underset{\mathbf{Str}}{\overset{\mathbf{Sem}}{\rightleftarrows}} \mathscr{J}\text{-}\mathbf{Alg}(\mathscr{C}).$$

The monad-pretheory adjunction

- Given an admissible $\mathscr{J}$-pretheory $\mathscr{T}$, the $\mathscr{V}$-functor $U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}$ is strictly monadic, and hence **Sem** corestricts to the full subcategory $\mathbf{Monadic}^!(\mathscr{C}) \hookrightarrow \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$ of strictly monadic $\mathscr{V}$-categories over $\mathscr{C}$.

- Let $\mathscr{J}$ be amenable. Because $\mathbf{Monadic}^!(\mathscr{C}) \simeq \mathbf{Mnd}(\mathscr{C})^{\mathbf{op}}$, the structure-semantics adjunction yields an idempotent adjunction

$$\mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Mnd}(\mathscr{C}), \quad \Phi \dashv \Psi$$

where Φ sends a $\mathscr{V}$-monad $\mathbb{T}$ to its **Kleisli** $\mathscr{J}$**-theory**, while Ψ sends an admissible $\mathscr{J}$-pretheory $\mathscr{T}$ to the free $\mathscr{T}$-algebra $\mathscr{V}$-monad on $\mathscr{C}$.

The monad-theory equivalence

A $\mathscr{V}$-monad $\mathbb{T}$ on $\mathscr{C}$ is $\mathscr{J}$**-nervous** if $\mathbb{T} \cong \Psi\mathscr{T}$ for some admissible $\mathscr{J}$-pretheory $\mathscr{T}$ (there is also a more technical definition that does not involve pretheories).

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be an amenable subcategory of arities. The idempotent monad-pretheory adjunction $\Psi \dashv \Phi$ restricts to an adjoint equivalence

$$\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$$

between $\mathscr{J}$-theories and $\mathscr{J}$-nervous $\mathscr{V}$-monads, which commutes with semantics in an appropriate sense. Also $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})$ is reflective, while $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Mnd}(\mathscr{C})$ is coreflective.

Additional consequences of strong amenability

We now suppose that $\mathscr{V}$ is complete and cocomplete, that $\mathscr{C}$ is cocomplete and cotensored, and that $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is small and strongly amenable.

Proposition

$\mathbf{Preth}_{\mathscr{J}}(\mathscr{C}), \mathbf{Th}_{\mathscr{J}}(\mathscr{C})$, *and* $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ *are all cocomplete, and small colimits therein are sent to limits in* $\mathscr{V}$-CAT$/\mathscr{C}$ *by the respective semantics functors.*

Monadicity over signatures

A $\mathscr{J}$-**signature** is a $\mathscr{V}$-functor $\Sigma : \mathbf{ob}\,\mathscr{J} \to \mathscr{C}$, i.e. an $\mathbf{ob}\,\mathscr{J}$-indexed family of objects of $\mathscr{C}$. We have a category $\mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-signatures, and a forgetful functor $\mathcal{U} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ defined by

$$\mathcal{U}\mathbb{T} = (TJ)_{J \in \mathscr{J}}\,.$$

Theorem

The functor $\mathcal{U} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ is monadic, and hence every $\mathscr{J}$-nervous $\mathscr{V}$-monad has a $\mathscr{J}$-presentation. Moreover, every $\mathscr{J}$-presentation P presents a $\mathscr{J}$-nervous $\mathscr{V}$-monad $\mathbb{T}_P$ with $\mathbb{T}_P$-Alg $\cong P$-Alg in $\mathscr{V}$-CAT$/\mathscr{C}$.

Some other nice consequences

We now also suppose that $\mathscr{T}$-Alg has conical coequalizers of reflexive pairs for each $\mathscr{J}$-pretheory $\mathscr{T}$.

Theorem

Let $H : \mathscr{T} \to \mathscr{U}$ be a morphism of $\mathscr{J}$-pretheories. Then the algebraic $\mathscr{V}$-functor $H^* = \mathbf{Sem}\ H : \mathscr{U}\text{-Alg} \to \mathscr{T}\text{-Alg}$ is strictly monadic.

Theorem

Let $\mathscr{T}$ be a $\mathscr{J}$-pretheory. Then the full sub-$\mathscr{V}$-category $\mathscr{T}\text{-Alg} \hookrightarrow [\mathscr{T}, \mathscr{V}]$ is reflective.

Summary so far...

- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is amenable, then we have a structure-semantics adjunction $\mathbf{Str} \dashv \mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$; a monad-theory equivalence $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \simeq \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$; and the reflectivity of $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})$ and coreflectivity of $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Mnd}(\mathscr{C})$.
- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is small and strongly amenable and $\mathscr{C}, \mathscr{V}$ are sufficiently (co)complete, then we also have a monad-pretheory adjunction $\Psi \dashv \Phi : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$; the (algebraic) cocompleteness of $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C}), \mathbf{Th}_{\mathscr{J}}(\mathscr{C}), \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$; a rich theory of presentations for $\mathscr{J}$-nervous $\mathscr{V}$-monads (and hence $\mathscr{J}$-theories); the strict monadicity of algebraic $\mathscr{V}$-functors; and the reflectivity of $\mathscr{T}$-algebras in presheaves (and more).

First example: eleutheric subcategories of arities

- A subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **eleutheric** [10, 12] if every $\mathscr{V}$-functor $H : \mathscr{J} \to \mathscr{C}$ has a left Kan extension along j that is preserved by each $\mathscr{C}(J, -) : \mathscr{C} \to \mathscr{V}$ ($J \in \mathbf{ob}\,\mathscr{J}$). For example:
  - The full sub-$\mathscr{V}$-category of enriched α-presentable objects in a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally α-presentable $\mathscr{V}$.
  - The "strongly finitary" subcategory of arities $j : \mathbf{SF}(\mathscr{V}) \hookrightarrow \mathscr{V}$ consisting of the finite copowers of the terminal object in a complete and cocomplete cartesian closed $\mathscr{V}$.
  - Just the unit object $\{I\} \hookrightarrow \mathscr{V}$ in any closed category $\mathscr{V}$.
  - The "unrestricted" subcategory of arities $1_{\mathscr{C}} : \mathscr{C} \hookrightarrow \mathscr{C}$ in any $\mathscr{V}$-category $\mathscr{C}$.
  - The Yoneda embedding $\mathbf{y} : \mathscr{A}^{\mathbf{op}} \hookrightarrow [\mathscr{A}, \mathscr{V}]$ for any small $\mathscr{V}$-category $\mathscr{A}$.
  - Any free Ψ-cocompletion $j : \mathscr{J} \hookrightarrow \mathscr{C}$ of a small $\mathscr{V}$-category $\mathscr{J}$ under a class of small weights Ψ.

First example: eleutheric subcategories of arities

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be an eleutheric subcategory of arities. Then $\mathscr{J}$ is amenable.

- We will observe below that most of the above examples satisfy an additional *boundedness* property that also makes them **strongly** amenable.
- If $j = 1_{\mathscr{V}} : \mathscr{V} \to \mathscr{V}$, then we recover Dubuc's structure-semantics adjunction [4] between $\mathscr{V}$-theories and tractable $\mathscr{V}$-valued $\mathscr{V}$-functors, and his equivalence between $\mathscr{V}$-theories and arbitrary $\mathscr{V}$-monads on $\mathscr{V}$.
- If $\mathscr{C} = \mathscr{V}$ and $j : \mathscr{J} \hookrightarrow \mathscr{V}$ is an eleutheric **system** of arities (i.e. contains I and is closed under $\otimes$), then we recover Lucyshyn-Wright's equivalence [10] between $\mathscr{J}$-theories and $\mathscr{J}$*-ary* $\mathscr{V}$-monads on $\mathscr{V}$.

Second example: bounded subcategories of arities

For this example, we make the following background assumptions:

- $\mathscr{V}$ is complete and cocomplete and has an enriched factorization system $(\mathscr{E}, \mathscr{M})$ [9].
- $\mathscr{C}$ is cocomplete and cotensored and has a *compatible* enriched factorization system $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ [12], and $\mathscr{C}$ has arbitrary $\mathscr{E}_{\mathscr{C}}$-cointersections; moreover, $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ is proper or $\mathscr{C}$ is $\mathscr{E}_{\mathscr{C}}$-cowellpowered.

A small subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **bounded** if each $J \in \mathbf{ob}\,\mathscr{J}$ is bounded (in the sense of [12]). If $\mathscr{C}$ is a locally bounded $\mathscr{V}$-category [11] over a locally bounded closed category $\mathscr{V}$, then any small $\mathscr{J} \hookrightarrow \mathscr{C}$ is automatically bounded.

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be a (small) subcategory of arities that is contained in some bounded and eleutheric subcategory of arities. Then $\mathscr{J}$ is strongly amenable, and $\mathscr{T}$-Alg has small conical colimits for every $\mathscr{J}$-pretheory $\mathscr{T}$.

Second example: bounded subcategories of arities

- For example: most of the above examples of eleutheric subcategories of arities are also bounded, and hence strongly amenable. Also, any small subcategory of arities in a locally presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally presentable $\mathscr{V}$ is contained in a bounded and eleutheric subcategory of arities, from which we recover the monad-pretheory adjunction and monad-theory equivalence of Bourke and Garner [2].

- By dropping the requirement of eleuthericity and strengthening the notion of boundedness in certain ways, we can also obtain further examples of strongly amenable subcategories of arities.

Locally bounded examples

A $\mathscr{V}$-category $\mathscr{C}$ is $\mathscr{V}$-**sketchable** if $\mathscr{C}$ is equivalent to the $\mathscr{V}$-category Φ-$\mathbf{Cts}(\mathcal{T}, \mathscr{V})$ of models of a small Φ-theory $\mathcal{T}$ for a class of small weights Φ.

Theorem

*Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be any small subcategory of arities in a $\mathscr{V}$-sketchable $\mathscr{V}$-category $\mathscr{C}$ over a locally bounded closed category $\mathscr{V}$. Then $\mathscr{J}$ is strongly amenable. If $\mathscr{V}$ is $\mathscr{E}$-cowellpowered, then $\mathscr{T}$-*Alg *is locally bounded (and hence cocomplete) for any $\mathscr{J}$-pretheory $\mathscr{T}$, and* $\mathbb{T}$-Alg *is locally bounded for any $\mathscr{J}$-nervous $\mathscr{V}$-monad $\mathbb{T}$.*

This provides a second method for recovering the main results of Bourke-Garner [2], because every locally presentable $\mathscr{V}$ is locally bounded and every locally presentable $\mathscr{V}$-category $\mathscr{C}$ is $\mathscr{V}$-sketchable [6].

Locally bounded examples

Since $\mathscr{V}$ itself is $\mathscr{V}$-sketchable, we may take $\mathscr{C} = \mathscr{V}$ and obtain the following:

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{V}$ be any small subcategory of arities in a locally bounded closed category $\mathscr{V}$. Then $\mathscr{J}$ is strongly amenable, and $\mathscr{T}$-Alg is cocomplete for each $\mathscr{J}$-pretheory $\mathscr{T}$.

As shown in [11], we have the following examples of locally bounded closed categories: any locally presentable closed category; any cocomplete locally cartesian closed category with a small generator (e.g. Dubuc's concrete quasitoposes [5] and the convenient categories of smooth spaces of [1]); any topological category over **Set** with its canonical ("separate continuity") symmetric monoidal closed structure (e.g. **Top** and **Meas**); and many convenient (cartesian closed) categories of topological spaces.

In summary...

- We have developed a general framework for enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities. If $\mathscr{J}$ is amenable (every $\mathscr{J}$-theory has free algebras), then we have a structure-semantics adjunction

$$\mathbf{Str} \dashv \mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$$

  and a monad-theory equivalence $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \simeq \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$.

- If $\mathscr{C}, \mathscr{V}$ are sufficiently (co)complete and $\mathscr{J}$ is small and strongly amenable (every $\mathscr{J}$-pretheory has free algebras), then we also have a monad-pretheory adjunction $\Psi \dashv \Phi : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ and a rich theory of presentations and algebraic colimits for $\mathscr{J}$-theories and $\mathscr{J}$-nervous $\mathscr{V}$-monads.

In summary...

- Many previously studied subcategories of arities are (strongly) amenable, from which we obtain many of the enriched structure-semantics adjunctions and monad-theory equivalences already established in the literature.

- Every small subcategory of arities in a $\mathscr{V}$-sketchable $\mathscr{V}$-category $\mathscr{C}$ over a locally bounded closed category $\mathscr{V}$ is strongly amenable; in particular, we may take $\mathscr{C} = \mathscr{V}$ itself. Examples of such $\mathscr{V}$ include many convenient categories of spaces.

Thank you!

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