Enriched structure-semantics adjunctions and monad-theory equivalences for a subcategory of arities

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J. Parker (joint with R. Lucyshyn-Wright) Enriched structure-semantics adjunctions and monad-theory...

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Motivation

- Several structure-semantics adjunctions and monad-theory equivalences have been established in category theory.
- In [7], Lawvere established a structure-semantics adjunction between Lawvere theories and *tractable* Set-valued functors, which was later generalized by Linton [8]. For a complete and well-powered closed category *V*, Dubuc [4] proved a structure-semantics adjunction between *V*-theories and tractable *V*-valued *V*-functors.

Motivation

- Linton [8] also showed that there is an equivalence between Lawvere theories and finitary monads on Set. Lucyshyn-Wright [10] generalized this by showing that if *J* → *V* is any *eleutheric system* of arities in a closed category *V*, then there is an equivalence between *J*-theories and *J*-ary *V*-monads on *V*.
- Building on earlier work of Power and Nishizawa [14, 13], Bourke and Garner [2] recently showed that if *J* → *C* is any small subcategory of arities in a locally presentable *V*-category *C* over a locally presentable closed category *V*, then there is an equivalence between *J*-theories and *J*-nervous *V*-monads on *C*.
- Neither equivalence subsumes the other; can both equivalences, along with the aforementioned structure-semantics adjunctions, be captured by a common framework that also yields new examples?

Objectives

- That is the subject of this talk: we have developed a general framework for studying enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities, which specializes to recover the aforementioned results and also yields new examples.
- More specifically, given a subcategory of arities 𝓕 → 𝔅 in a 𝒱-category 𝔅 over a closed category 𝒱, we will identify hypotheses on these data that entail a structure-semantics adjunction, a monad-theory equivalence, a rich theory of *presentations* for monads and theories, and more.

Basic definitions

- We fix a subcategory of arities j: 𝒴 → 𝔅, i.e. a full and dense sub-𝒴-category, in a 𝒴-category 𝔅 over a symmetric monoidal closed category 𝒴. Since we do not assume that 𝒴 is small or that 𝒴 is (co)complete, we also fix a suitable universe extension 𝒴 → 𝒴'.
- We have a fully faithful \mathscr{V}' -functor

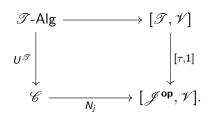
$$N_j: \mathscr{C} \to [\mathscr{J}^{\mathrm{op}}, \mathscr{V}]$$

 $N_j C = \mathscr{C}(j-, C)$

that we call the *j*-nerve \mathscr{V}' -functor. The presheaves in its essential image are called *j*-nerves.

Pretheories and their algebras

- (Linton [8], Diers [3], Bourke-Garner [2]) A *J*-pretheory is just an identity-on-objects *V*-functor *τ* : *J*^{op} → *T*, while a *J*-theory is a *J*-pretheory *T* such that each *T*(*J*,*τ*-) : *J*^{op} → *V*(*J*∈ ob *J*) is a *j*-nerve. We have the category **Preth**_{*J*}(*C*) of *J*-pretheories and its full subcategory **Th**_{*J*}(*C*) of *J*-theories.
- Let *T* be a *J*-pretheory. The *V*'-category *T*-Alg of (concrete)
 T-algebras is defined by the following pullback in *V*'-CAT:



Amenable subcategories of arities

- A \mathscr{J} -pretheory \mathscr{T} is **admissible** if the \mathscr{V}' -category \mathscr{T} -Alg is actually a \mathscr{V} -category, and $U^{\mathscr{T}}: \mathscr{T}$ -Alg $\rightarrow \mathscr{C}$ has a left adjoint.
- The subcategory of arities j : J → C is amenable if every J-theory is admissible, and is strongly amenable if every J-pretheory T is admissible.

\mathscr{J} -tractable \mathscr{V} -categories

- A *J*-tractable *V*-category over *C* is a *V*-category *G* : *A* → *C* over *C* such that *C* admits the weighted limit {*C*(*J*, *G*-), *G*} for each *J* ∈ ob *J*. Then *J*-Tract(*C*) is the full subcategory of *V*-CAT/*C* consisting of the *J*-tractable *V*-categories over *C*.
- Let Preth^a_{\$\mathcal{I}\$}(\$\mathcal{C}\$) be the full subcategory of Preth_{\$\mathcal{I}\$}(\$\mathcal{C}\$) consisting of the admissible \$\mathcal{J}\$-pretheories. We define a semantics functor

$$\mathsf{Sem}:\mathsf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathsf{op}} o\mathscr{J}\operatorname{-}\mathsf{Tract}(\mathscr{C})$$

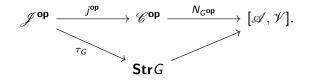
by

$$\mathsf{Sem}\mathscr{T} = \left(U^{\mathscr{T}}:\mathscr{T} ext{-}\mathsf{Alg} o \mathscr{C}
ight)$$

for each admissible $\mathscr{J}\operatorname{-pretheory}\, \mathscr{T}.$

J − structure

• Let $G : \mathscr{A} \to \mathscr{C}$ be a \mathscr{J} -tractable \mathscr{V} -category over \mathscr{C} . We define a \mathscr{J} -theory $\tau_G : \mathscr{J}^{\operatorname{op}} \to \operatorname{Str} G$, the \mathscr{J} -structure of G, by taking the (bijective-on-objects, fully faithful) factorization of the composite \mathscr{V}' -functor



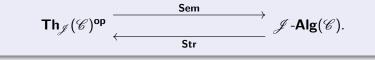
(Since G is \mathcal{J} -tractable, **Str**G is indeed a \mathcal{V} -category and moreover a \mathcal{J} -theory).

The structure-semantics adjunction

A \mathscr{J} -algebraic \mathscr{V} -category over \mathscr{C} is a \mathscr{V} -category over \mathscr{C} in the essential image of **Sem**; we let \mathscr{J} -**Alg**(\mathscr{C}) be the full subcategory of \mathscr{J} -**Tract**(\mathscr{C}) consisting of these objects.

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be an amenable subcategory of arities. The semantics functor **Sem** : **Preth**^a_{\mathscr{J}}(\mathscr{C})^{op} $\to \mathscr{J}$ -**Tract**(\mathscr{C}) has a left adjoint **Str** that sends each \mathscr{J} -tractable \mathscr{V} -category over \mathscr{C} to its \mathscr{J} -structure. This adjunction is idempotent, and restricts to an adjoint equivalence



The monad-pretheory adjunction

- Given an admissible *J*-pretheory *T*, the *V*-functor
 U^T: *T*-Alg → *C* is strictly monadic, and hence Sem corestricts to the full subcategory Monadic!(*C*) → *J*-Tract(*C*) of strictly monadic *V*-categories over *C*.
- Let \mathscr{J} be amenable. Because $Monadic^!(\mathscr{C}) \simeq Mnd(\mathscr{C})^{op}$, the structure-semantics adjunction yields an idempotent adjunction

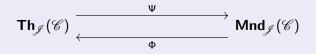
where Φ sends a \mathscr{V} -monad \mathbb{T} to its **Kleisli** \mathscr{J} -**theory**, while Ψ sends an admissible \mathscr{J} -pretheory \mathscr{T} to the free \mathscr{T} -algebra \mathscr{V} -monad on \mathscr{C} .

The monad-theory equivalence

A \mathscr{V} -monad \mathbb{T} on \mathscr{C} is \mathscr{J} -nervous if $\mathbb{T} \cong \Psi \mathscr{T}$ for some admissible \mathscr{J} -pretheory \mathscr{T} (there is also a more technical definition that does not involve pretheories).

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be an amenable subcategory of arities. The idempotent monad-pretheory adjunction $\Psi \dashv \Phi$ restricts to an adjoint equivalence



between \mathscr{J} -theories and \mathscr{J} -nervous \mathscr{V} -monads, which commutes with semantics in an appropriate sense. Also $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Preth}^{a}_{\mathscr{J}}(\mathscr{C})$ is reflective, while $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Mnd}(\mathscr{C})$ is coreflective.

Additional consequences of strong amenability

We now suppose that \mathscr{V} is complete and cocomplete, that \mathscr{C} is cocomplete and cotensored, and that $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is small and strongly amenable.

Proposition

 $\operatorname{Preth}_{\mathscr{J}}(\mathscr{C}), \operatorname{Th}_{\mathscr{J}}(\mathscr{C})$, and $\operatorname{Mnd}_{\mathscr{J}}(\mathscr{C})$ are all cocomplete, and small colimits therein are sent to limits in \mathscr{V} -CAT/ \mathscr{C} by the respective semantics functors.

Monadicity over signatures

A \mathscr{J} -signature is a \mathscr{V} -functor Σ : **ob** $\mathscr{J} \to \mathscr{C}$, i.e. an **ob** \mathscr{J} -indexed family of objects of \mathscr{C} . We have a category $\mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ of \mathscr{J} -signatures, and a forgetful functor \mathcal{U} : $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ defined by

$$\mathcal{U}\mathbb{T}=(\mathcal{T}J)_{J\in\mathscr{J}}$$
 .

Theorem

The functor $\mathcal{U} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ is monadic, and hence every \mathscr{J} -nervous \mathscr{V} -monad has a \mathscr{J} -presentation. Moreover, every \mathscr{J} -presentation P presents a \mathscr{J} -nervous \mathscr{V} -monad \mathbb{T}_P with \mathbb{T}_P -Alg \cong P-Alg in \mathscr{V} -CAT/ \mathscr{C} .

Some other nice consequences

We now also suppose that \mathscr{T} -Alg has conical coequalizers of reflexive pairs for each \mathscr{J} -pretheory \mathscr{T} .

Theorem

Let $H : \mathscr{T} \to \mathscr{U}$ be a morphism of \mathscr{J} -pretheories. Then the algebraic \mathscr{V} -functor $H^* =$ **Sem** $H : \mathscr{U}$ -Alg $\to \mathscr{T}$ -Alg is strictly monadic.

Theorem

Let \mathscr{T} be a \mathscr{J} -pretheory. Then the full sub- \mathscr{V} -category \mathscr{T} -Alg $\hookrightarrow [\mathscr{T}, \mathscr{V}]$ is reflective.

Summary so far...

- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is amenable, then we have a structure-semantics adjunction $\mathbf{Str} \dashv \mathbf{Sem} : \mathbf{Preth}^{a}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J} \cdot \mathbf{Tract}(\mathscr{C})$; a monad-theory equivalence $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \simeq \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$; and the reflectivity of $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Preth}^{a}_{\mathscr{J}}(\mathscr{C})$ and coreflectivity of $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Mnd}(\mathscr{C})$.
- If j: J → C is small and strongly amenable and C, V are sufficiently (co)complete, then we also have a monad-pretheory adjunction Ψ ⊢ Φ : Mnd(C) → Preth_J(C); the (algebraic) cocompleteness of Preth_J(C), Th_J(C), Mnd_J(C); a rich theory of presentations for J-nervous V-monads (and hence J-theories); the strict monadicity of algebraic V-functors; and the reflectivity of *T*-algebras in presheaves (and more).

First example: eleutheric subcategories of arities

- A subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **eleutheric** [10, 12] if every \mathscr{V} -functor $H : \mathscr{J} \to \mathscr{C}$ has a left Kan extension along j that is preserved by each $\mathscr{C}(J, -) : \mathscr{C} \to \mathscr{V}$ ($J \in \mathbf{ob} \mathscr{J}$). For example:
 - ► The full sub-𝒴-category of enriched α-presentable objects in a locally α-presentable 𝒴-category 𝔅 over a locally α-presentable 𝒴.
 - ► The "strongly finitary" subcategory of arities j : SF(𝒴) → 𝒴 consisting of the finite copowers of the terminal object in a complete and cocomplete cartesian closed 𝒴.
 - Just the unit object $\{I\} \hookrightarrow \mathscr{V}$ in any closed category \mathscr{V} .
 - ▶ The "unrestricted" subcategory of arities $1_{\mathscr{C}} : \mathscr{C} \hookrightarrow \mathscr{C}$ in any \mathscr{V} -category \mathscr{C} .
 - ▶ The Yoneda embedding $\mathbf{y} : \mathscr{A}^{\mathbf{op}} \hookrightarrow [\mathscr{A}, \mathscr{V}]$ for any small \mathscr{V} -category \mathscr{A} .
 - ► Any free Ψ -cocompletion $j : \mathscr{J} \hookrightarrow \mathscr{C}$ of a small \mathscr{V} -category \mathscr{J} under a class of small weights Ψ .

First example: eleutheric subcategories of arities

Theorem

Let $j: \mathscr{J} \hookrightarrow \mathscr{C}$ be an eleutheric subcategory of arities. Then \mathscr{J} is amenable.

- We will observe below that most of the above examples satisfy an additional *boundedness* property that also makes them **strongly** amenable.
- If j = 1_𝒴 : 𝒴 → 𝒴, then we recover Dubuc's structure-semantics adjunction [4] between 𝒴-theories and tractable 𝒴-valued 𝒴-functors, and his equivalence between 𝒴-theories and arbitrary 𝒴-monads on 𝒴.
- If C = V and j : J → V is an eleutheric system of arities (i.e. contains I and is closed under ⊗), then we recover Lucyshyn-Wright's equivalence [10] between J -theories and J -ary V -monads on V.

Second example: bounded subcategories of arities

For this example, we make the following background assumptions:

- 𝒱 is complete and cocomplete and has an enriched factorization system (𝔅, ℋ) [9].
- C is cocomplete and cotensored and has a *compatible* enriched factorization system (E_C, M_C) [12], and C has arbitrary E_C-cointersections; moreover, (E_C, M_C) is proper or C is E_C-cowellpowered.

A small subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **bounded** if each $J \in \mathbf{ob} \mathscr{J}$ is bounded (in the sense of [12]). If \mathscr{C} is a locally bounded \mathscr{V} -category [11] over a locally bounded closed category \mathscr{V} , then any small $\mathscr{J} \hookrightarrow \mathscr{C}$ is automatically bounded.

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be a (small) subcategory of arities that is contained in some bounded and eleutheric subcategory of arities. Then \mathscr{J} is strongly amenable, and \mathscr{T} -Alg has small conical colimits for every \mathscr{J} -pretheory \mathscr{T} .

Second example: bounded subcategories of arities

- For example: most of the above examples of eleutheric subcategories of arities are also bounded, and hence strongly amenable. Also, any small subcategory of arities in a locally presentable *V*-category *C* over a locally presentable *V* is contained in a bounded and eleutheric subcategory of arities, from which we recover the monad-pretheory adjunction and monad-theory equivalence of Bourke and Garner [2].
- By dropping the requirement of eleuthericity and strengthening the notion of boundedness in certain ways, we can also obtain further examples of strongly amenable subcategories of arities.

Locally bounded examples

A \mathscr{V} -category \mathscr{C} is \mathscr{V} -**sketchable** if \mathscr{C} is equivalent to the \mathscr{V} -category Φ -**Cts**(\mathcal{T}, \mathscr{V}) of models of a small Φ -theory \mathcal{T} for a class of small weights Φ .

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be any small subcategory of arities in a \mathscr{V} -sketchable \mathscr{V} -category \mathscr{C} over a locally bounded closed category \mathscr{V} . Then \mathscr{J} is strongly amenable. If \mathscr{V} is \mathscr{E} -cowellpowered, then \mathscr{T} -Alg is locally bounded (and hence cocomplete) for any \mathscr{J} -pretheory \mathscr{T} , and \mathbb{T} -Alg is locally bounded for any \mathscr{J} -nervous \mathscr{V} -monad \mathbb{T} .

This provides a second method for recovering the main results of Bourke-Garner [2], because every locally presentable \mathscr{V} is locally bounded and every locally presentable \mathscr{V} -category \mathscr{C} is \mathscr{V} -sketchable [6].

Locally bounded examples

Since $\mathscr V$ itself is $\mathscr V\text{-sketchable, we may take }\mathscr C=\mathscr V$ and obtain the following:

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{V}$ be any small subcategory of arities in a locally bounded closed category \mathscr{V} . Then \mathscr{J} is strongly amenable, and \mathscr{T} -Alg is cocomplete for each \mathscr{J} -pretheory \mathscr{T} .

As shown in [11], we have the following examples of locally bounded closed categories: any locally presentable closed category; any cocomplete locally cartesian closed category with a small generator (e.g. Dubuc's concrete quasitoposes [5] and the convenient categories of smooth spaces of [1]); any topological category over **Set** with its canonical ("separate continuity") symmetric monoidal closed structure (e.g. **Top** and **Meas**); and many convenient (cartesian closed) categories of topological spaces.

In summary...

• We have developed a general framework for enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities. If \mathscr{J} is amenable (every \mathscr{J} -theory has free algebras), then we have a structure-semantics adjunction

$$\mathsf{Str}\dashv\mathsf{Sem}:\mathsf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathsf{op}}\to\mathscr{J}\text{-}\mathsf{Tract}(\mathscr{C})$$

and a monad-theory equivalence $\mathsf{Th}_{\mathscr{J}}(\mathscr{C}) \simeq \mathsf{Mnd}_{\mathscr{J}}(\mathscr{C})$.

• If \mathscr{C}, \mathscr{V} are sufficiently (co)complete and \mathscr{J} is small and strongly amenable (every \mathscr{J} -pretheory has free algebras), then we also have a monad-pretheory adjunction $\Psi \dashv \Phi : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ and a rich theory of presentations and algebraic colimits for \mathscr{J} -theories and \mathscr{J} -nervous \mathscr{V} -monads.

In summary...

- Many previously studied subcategories of arities are (strongly) amenable, from which we obtain many of the enriched structure-semantics adjunctions and monad-theory equivalences already established in the literature.
- Every small subcategory of arities in a 𝒴-sketchable 𝒴-category 𝒴 over a locally bounded closed category 𝒴 is strongly amenable; in particular, we may take 𝒴 = 𝒴 itself. Examples of such 𝒴 include many convenient categories of spaces.

Thank you!

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