Algebraic geometry and Tangent categories (part two)

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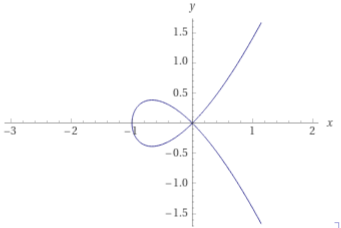


From last time...

Main idea: looking at the tangent category structure of

$$\text{(affine schemes over } R) = cAlg_R^{op}$$

where recall:

- A tangent category is a category $\mathbb{X}$ where each object has an associated "tangent bundle", represented axiomatically by an endofunctor $T : \mathbb{X} \longrightarrow \mathbb{X}$ equipped with certain natural transformations
- The canonical example is $\mathbb{X}$ = smooth manifolds, where TM = tangent bundle of M
- We're focusing on $\mathbb{X} = cAlg_R^{op}$ where

$$TA := \text{Symmetric } A\text{-algebra of (Kahler differentials of } A \text{ over } R)$$

- For example, if $A = R[x, y]/(y^2 - x^3 - x^2)$,

$$TA = A[x, y, dx, dy]/(y^2 - x^3 - x^2, 2ydy - 3x^2xdx - 2xdx)$$

Today

In any tangent category, one can define many analogs of ideas from differential geometry. **What do these look like in this example?**

- Today we'll focus on **differential objects** and **differential bundles** (the analogs of vector spaces and vector bundles).
- We'll also take a quick look at vector fields and de Rham cohomology.
- However, first I need to start with a quick corrigendum and addendum. Last time I stated that in $cAlg_R^{op}$,

$$\text{quotients of algebras} = \text{inclusions of sub varieties/schemes}$$

  which is definitely not true: quotients are only *some* of the inclusions.

Finally, even if you don't particularly care about algebraic geometry or tangent categories, we'll see a result which is of independent interest: a characterization of the *opposite* of the category of R-modules.

Quotients $\subsetneq$ Subobjects in $cAlg_R^{op}$

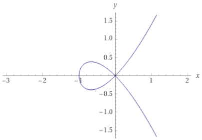
Quotients give important examples of subobjects in $cAlg_R^{op}$, but they are **not** all of them.

- Another *very* important class of subobjects are **localizations**.
- Recall that the localization of a ring/algebra A at a multiplicative set S, $A[S^{-1}]$, adds inverses for every element of S.
- For example, if S is the multiplicative set generated by x in $R[x]$, $R[x][S^{-1}] = R[x, x^{-1}]$ is the ring of Laurent polynomials (in which all powers of x are invertible).
- For any S, the canonical algebra map $A \longrightarrow A[S^{-1}]$ is an epimorphism (!), hence a monomorphism $A[S^{-1}] \longrightarrow A$ in $cAlg_R^{op}$.

What do these subobjects look like?

Localization examples

We saw that the (categorical) points of taking $\mathbb{R}[x, y]$ and quotienting by the ideal generated by $y^2 - x^3 - x^2$ are the solutions to $y^2 - x^3 - x^2 = 0$:



What happens if instead we *localize* $\mathbb{R}[x, y]$ at the multiplicative set S generated by $y^2 - x^3 - x^2$?

- Now its categorical points in $cAlg_{\mathbb{R}}^{op}$ are $cAlg_{\mathbb{R}}$ maps

$$\mathbb{R}[x, y][S^{-1}] \longrightarrow \mathbb{R}$$

- Again, such a map is determined by where it sends x and y, with now the only restriction being that wherever x and y get sent, $y^2 - x^3 - x^2$ must be *invertible*.
- So since our base is a field, this set is $\{(x, y) : y^2 - x^3 - x^2 \neq 0\}$, ie., everything *not* on the above variety!

So we can think of localizations as giving *open* subobjects in $cAlg_{\mathbb{R}}^{op}$.

Localizations and tangent bundles

And in fact thinkig of localizations as open subsets works exactly as one would hope with the tangent bundle:

- Recall that for U an open subset of $\mathbb{R}^n$, $TU = U \times \mathbb{R}^n$; that is,

$$\begin{array}{ccc} TU & \hookrightarrow & T\mathbb{R}^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ U & \hookrightarrow & \mathbb{R}^n \end{array}$$

  is a pullback (in smooth manifolds).

- The same is true for localizations: if S is a multiplicative subset of A, then

$$\begin{array}{ccc} T(A[S^{-1}]) & \hookrightarrow & TA \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ A[S^{-1}] & \hookrightarrow & A \end{array}$$

  is a pullback (in affine schemes $= cAlg_R^{op}$).

That is, **tangent vectors at any point of an open subset are exactly the tangent vectors from the space in which its contained.**

Vector spaces/bundles in a tangent category?

How can we talk about vector spaces and more generally vector bundles in a tangent category?

- The axioms do not assume any "object of scalars" (seemingly necessary for vector spaces).
- The axioms do not assume any "open subsets" (seemingly necessary for the local triviality of vector bundles).

However, we shall see that we can characterize both vector spaces and vector bundles by canonical "lift" maps to their tangent bundles.

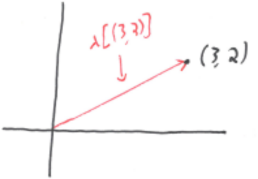
Vector spaces I: lift map λ

What special properties does a vector space V have in the tangent category of smooth manifolds?

- It has a zero and an addition; that is, there are maps

$$\zeta : 1 \longrightarrow V, \sigma : V \times V \longrightarrow V$$

  so that (V, ζ, σ) is a commutative monoid.
- More importantly, however, every $v \in V$ gives a unique tangent vector $\lambda(v)$ at 0, and all tangent vectors at 0 are of this form.

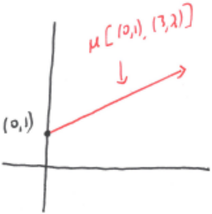


- This gives a map $\lambda : E \longrightarrow TE$.

Vector spaces II: lift map μ

More generally, for any $v \in V$, every tangent vector at v is uniquely represented by some $v' \in V$; this gives an isomorphism

$$\mu : E \times E \longrightarrow TE$$



- λ is a special case of μ: $\lambda(v) = \mu(0, v)$.
- But in fact μ can also be constructed from λ via

$$\mu = \langle \pi_0 0, \pi_1 \lambda \rangle T(\sigma)^1$$

- Our definition of "vector space" will assume a λ and construct μ from it as above (axioms are easier to state using λ instead of μ).

[1] Writing composition in diagrammatic order.

Differential objects

Assume $(\mathbb{X}, T)$ is a tangent category with finite products which are preserved by T.

Definition

A **differential object** consists of a commutative monoid (A, ζ, σ) together with a "lift" map $\lambda : E \longrightarrow TE$ (satisfying various coherences with the tangent structure) and such that

$$E \times E \xrightarrow{\mu := \langle \pi_0 0, \pi_1 \lambda \rangle T(\sigma)} TE$$

is an isomorphism.

Note: This is a alternative axiomatization to the one presented in (Cockett and Cruttwell, 2014).

Example

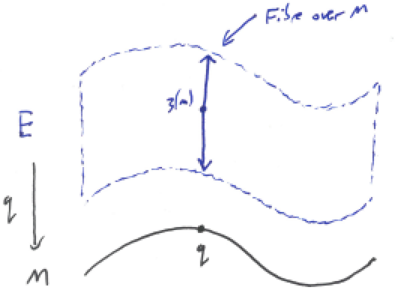
In $\mathbb{X}$ = smooth manifolds, differential objects = vector spaces.

Vector bundles I

A vector bundle is a map $q : E \longrightarrow M$ such that each fibre

$$q^{-1}(m) = \{e \in E : q(e) = m\}$$

is a vector space (plus a local triviality condition).



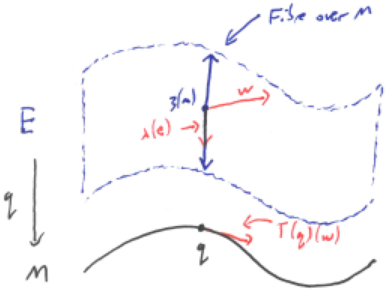
- The addition and zero in each fibre can be represented as asking for a commutative monoid in $\mathbb{X}/M$.
- In particular, this gives a section $\zeta : M \longrightarrow E$ of q, and a "fibred addition" $\sigma : E \times_M E \longrightarrow E$.

Vector bundles II: Lifts

A vector bundle will again have lift maps

$$\lambda : E \longrightarrow TE \text{ and } \mu : E \times_m E \longrightarrow TE.$$

The only difference is that **not all** TE **will be in the image of** μ: eg., w is not in the image of μ:



But we *can* recognize which ones should be in the image of μ: they are exactly the **vertical** tangent vectors: those w for which $T(q)(w) = 0$ (they "cast no shadow").

Differential bundles

Definition (Cockett/Cruttwell 2017)

A **differential bundle over** M is a commutative monoid in $\mathbb{X}/M$; that is,

$$(q : E \longrightarrow M, \zeta : M \longrightarrow E, \sigma : E \times_M E \longrightarrow E)$$

together with a “lift” map $\lambda : E \longrightarrow TE$ (satisfying various coherences with the tangent structure) such that if μ is defined as

$$E \times_M E \xrightarrow{\mu := \langle \pi_0 0, \pi_1 \lambda \rangle T(\sigma)} TE$$

then the following diagram is a pullback:

$$\begin{array}{ccc} E \times_M E & \xrightarrow{\mu} & TE \\ {\scriptstyle \pi_0 q}\downarrow & & \downarrow{\scriptstyle T(q)} \\ M & \xrightarrow[0_M]{} & TM \end{array}$$

(Ie., every *vertical* tangent vector is represented by a $(e, e') \in E \times_M E$.)

Differential bundle properties

Differential bundles in a tangent category $(\mathbb{X}, T)$ enjoy many useful properties:

- For every $M \in \mathbb{X}$, the tangent bundle $p_M : TM \longrightarrow M$ is a differential bundle (essentially axiomatically - the axioms ask for a lift map $\ell : TM \longrightarrow T^2M$...)
- If $q : E \longrightarrow M$ is a differential bundle and

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ M' & \longrightarrow & M \end{array}$$

  is a pullback preserved by T, then $q' : E' \longrightarrow M'$ is a differential bundle.
- If $q : E \longrightarrow M$ is a differential bundle, then $Tq : TE \longrightarrow TM$ is a differential bundle.
- Differential bundles over a terminal object 1 are exactly differential objects.

There is a category of differential bundles over M, where the maps are bundle maps which preserve the lift.

Differential bundles in smooth manifolds?

Vector bundles $q : E \longrightarrow M$ are differential bundles, but they also satisfy a **local triviality condition**: there is a covering of M by open sets $\{U_i : i \in I\}$ such that for each i, there is some n so that

$$q^{-1}(U_i) \cong U_i \times \mathbb{R}^n$$

In 2021, Ben MacAdam showed that the implication goes the other way as well - **every differential bundle in the tangent category of smooth manifolds is a (smooth) vector bundle!**

- This is *remarkable* as it tells us that the existence of the "lift" forces not only a scalar action in each fibre (not assumed) but also the local triviality condition above.
- The maps work out as well: differential bundles maps (those preserving the lift) = vector bundle maps.
- This gives a completely new way to think about smooth vector bundles and their morphisms.

Summing up

So, in the tangent category of smooth manifolds:

- Differential objects = Vector spaces
- Differential bundles = (Smooth) Vector bundles

Differential objects/bundles are generally very important in any tangent category.

So what are they in affine schemes??

Tangent bundles as differential bundles

As a starting point, every tangent bundle is a differential bundle, so for any $A \in cAlg_R^{op}$

$$TA := \text{Symmetric } A\text{-algebra of (Kahler differentials of } A \text{ over } R)$$

is a differential bundle over A, with lift structure $\lambda : TA \longrightarrow T^2A$ given by the algebra map $\lambda' : T^2A \longrightarrow TA$ defined by

$$a \mapsto a$$

$$da \mapsto 0$$

$$d'a \mapsto 0$$

$$d'da \mapsto da$$

But actually the above works for the symmetric algebra of *any* module over A, not just the Kahler differentials over A!

Modules as differential bundles

Definition

If M is an A-module, the **symmetric algebra** of M, $\mathbf{Sym}(M)$, is the free A-algebra generated by M (so freely generated by $a \in A, m \in M$).

Theorem

For $A \in cAlg_R^{op}$, for any A-module M, $\mathbf{Sym}(M)$ is a differential bundle over A, with lift map λ given by the algebra map

$$\lambda' : T(\mathbf{Sym}(M)) \to \mathbf{Sym}(M)$$

defined by

$$a \mapsto a$$

$$m \mapsto 0$$

$$da \mapsto 0$$

$$dm \mapsto m$$

Differential bundles as modules

But we can go the other way as well! Given a diffential bundle $q : E \longrightarrow A$ over A, its lift $\lambda : E \longrightarrow TE$ corresponds to an algebra map

$$\lambda' : TE \longrightarrow E$$

where TE is generated by symbols e, de, and we define M as the image of the de's by λ':

$$M := \mathrm{Im}_d(\lambda) := \{\lambda'(de) : e \in E\}$$

then one can show that M is indeed an A-module.

- It is immediate that if you start with a module M, build its differential bundle via **Sym** then go back as above, you recover M.
- Conversely, if you start with a differential bundle E, build its module as above, then go back via **Sym**, you recover E: but this uses in a crucial way the pullback property of the lift!

Differential bundles and modules

So we have a bijection on objects

$$(\text{Differential bundles over } A) \Leftrightarrow (\text{Modules over } A)$$

given by

$$\textbf{Sym}(M) \Leftarrow M$$

$$(E, \lambda) \Rightarrow \text{Im}_d(\lambda)$$

Interesting side note: Grothendieck called the construction **Sym**(M) "the vector bundle associated to M"[2]! This fell out of favour since the result is not something locally trivial...perhaps what he really had in mind was something like differential bundles (?).

[2] EGA II, section 1.7

Differential bundles $\cong$ modules^{op}

But what about the maps? Something interesting happens:

- An A-module morphism $f : M \longrightarrow M'$ gives a $cAlg_R$ morphism

$$\mathbf{Sym}(f) : \mathbf{Sym}(M) \longrightarrow \mathbf{Sym}(M')$$

  ie., a map in $cAlg_R^{op}$

$$\mathbf{Sym}(M') \longrightarrow \mathbf{Sym}(M)$$

- So we are actually getting a contravariant functor!

Theorem

For $A \in cAlg_R^{op}$, there is an equivalence of categories

$$(\textit{Diff. bundles over } A) \cong (A\textit{-modules})^{op}$$

I don't know of another characterization of A-modules^{op}...

Aside: Slice tangent categories

In might seem a bit strange that we get the same differential bundles if we view A as an R-algebra or as some other R'-algebra. But in fact all these tangent structures are just slices of a single tangent structure:

- Recall that $cAlg_R = R/cRing$
- So $cAlg_R^{op} = (cRing^{op})/R$
- In general, if $(\mathbb{X}, T)$ is a tangent category, for any M, $\mathbb{X}/M$ is again a tangent category
- And the tangent structure of $cAlg_R^{op} = (cRing^{op})/R$ is indeed the slice of the tangent structure on $(cRing)^{op} = cAlg_{\mathbb{Z}}^{op}$

So since differential bundles are a structure "over" an A, it's not surprising their structure is the same in different slices, and really all comes from a single tangent structure on

$$\text{Affine schemes} = cRing^{op} = cAlg_{\mathbb{Z}}^{op}$$

Summing up differential bundles

To sum up:

- **In smooth manifolds, differential bundles = vector bundles**
- **In affine schemes, differential bundles = modules^{op}**

Does this make sense?

- Algebraic geometers often speak of modules (or, in the more general case of schemes, quasi-coherent sheaves of modules) as an important supporting structure that plays a role like that of vector bundles for smooth manifolds.
- More generally, the opposite of "algebraic" categories are "geometric" categories, so it makes sense that the opposite of R-modules is "geometric".

Vector fields in a tangent category

A vector field is a (smooth) choice of tangent vector at each point. More precisely:

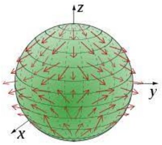
Definition (Rosický 1984)

If $(\mathbb{X}, T)$ is a tangent category, a **vector field** on an object M in $\mathbb{X}$ is a map $\chi : M \longrightarrow TM$ which is a section of p_M, ie.,

$$M \xrightarrow{\chi} TM \xrightarrow{p_M} M$$

is the identity on M.

A non-trivial vector field on the sphere:



Vector fields in $cAlg_R^{op}$

What is a vector field on an affine scheme $A \in cAlg_R^{op}$?

- It is an algebra map $\chi : TA \longrightarrow A$, so is determined by where it sends the a and da elements.
- But since it is a section of the projection, it sends a to a. So it is entirely determined by where it sends da; let $D(a) := \chi(da)$.
- Then

$$D(ab) = \chi(d(ab)) = \chi(adb + bda) =$$

$$\chi(a)\chi(db) + \chi(b)\chi(da) = aD(b) + bD(a)$$

  and D is R-linear.
- This is precisely the requirements of a **derivation**: a R-linear map $D : A \longrightarrow A$ which satisfies the Leibniz rule.

Therefore,

Vector fields on A = Derivations on A

Another nice connection...

Tangent category de Rham cohomology

In a tangent category $(\mathbb{X}, T)$, for every differential object E, there are (two) associated cohomologies:

Definition (Cruttwell/Lucyshyn-Wright 2018)

For an object $M \in \mathbb{X}$ and a differential object E, a **singular form on M (with values in E)** is a map $\omega : T^n M \longrightarrow E$ which is suitably multilinear and alternating. Call the set of such maps $\Omega_n(M; E)$.

Theorem (Cruttwell/Lucyshyn-Wright 2018)

There is a cochain complex

$$\Omega_0(M; E) \xrightarrow{d} \Omega_1(M; E) \xrightarrow{d} \Omega_2(M; E) \dots$$

which we will call here the **tangent de Rham complex**.

Example

In the tangent category of smooth manifolds, with $E = \mathbb{R}$, the tangent de Rham complex is the (ordinary) de Rham complex.

Tangent de Rham cohomology and algebraic geometry

Conjecture: In affine schemes over R,

$$(\text{Tangent de Rham with values in } R[x]) = (\textit{algebraic} \text{ de Rham})$$

(we checked the lower terms...it seems to be just a matter of finding a clean way to write everything down...)

What about other differential objects?

- If R is a field, the differential objects are all R-vector spaces; wouldn't give much new information.
- But if R is not a field, there are many other possible differential objects (eg., non-free modules)...could potentially give other interesting "differential" cohomology theories for (affine) schemes.

There is also another cohomology theory in any tangent category (the cohomology of "sector" forms)...still trying to understand it even in smooth manifolds.

Conclusions

Tons of things still to do!

- Connections in a tangent category: are they the same as existing notions? Not done much in algebraic geometry...
- Curve objects? (Solutions to differential equations)
- Can we define what it means for an object in tangent category to be *smooth*?
- Verify algebraic de Rham conjecture
- Look at tangent de Rham over other differential objects: is this related to crystalline cohomology?

And again further generalizations of this example to non-commutative geometry by Marcello...

References

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