1. Locate each of the isolated singularities of the given function, and tell whether it is a removable singularity, a pole, or an essential singularity. If the singularity is removable, find the analytic extension of the function (the function $g$ in the definition of a removable singularity), and if it is a pole, give the order of the pole.

1. $f(z) = \frac{z^2 - 1}{z^3 - 1}$.
2. $f(z) = \cotan(z)$.
3. $f(z) = \frac{\exp(\frac{1}{z})}{z - i}$.

Solution of Part 1: We have

$$f(z) = \frac{z^2 - 1}{z^3 - 1} = \frac{(z - 1)(z + 1)}{(z - 1)(z - e^{\frac{2\pi i}{3}})(z - e^{\frac{4\pi i}{3}})}.$$ 

Therefore the isolated singularities of $f$ are $z_0 = 1$, $z_1 = e^{\frac{2\pi i}{3}}$, and $z_2 = e^{\frac{4\pi i}{3}}$.

Case $z_1$: Observe that $f$ is analytic on the punctured disc $D_1 = \{ z \in \mathbb{C} : 0 < |z - e^{\frac{2\pi i}{3}}| < 1 \}$. Moreover,

$$f(z) = \frac{g_1(z)}{z - e^{\frac{2\pi i}{3}}} \quad \text{with} \quad g_1(z) = \frac{z^2 - 1}{(z - 1)(z - e^{\frac{4\pi i}{3}})},$$

where $g_1$ is analytic on the ball $b_1(e^{\frac{2\pi i}{3}})$, and $g_1(e^{\frac{2\pi i}{3}}) \neq 0$. Hence $z_1 = e^{\frac{2\pi i}{3}}$ is a pole of order 1 for $f$.

Case $z_2$: Similarly we can show that $z_2 = e^{\frac{4\pi i}{3}}$ is a pole of order 1 for $f$.

Case $z_0$: To determine whether $z_0 = 1$ is a pole or a removable singularity of $f$, we use the theorems in the notes.

$$\lim_{z \to 1} f(z) = \lim_{z \to 1} \frac{(z - 1)(z + 1)}{(z - 1)(z - e^{\frac{2\pi i}{3}})(z - e^{\frac{4\pi i}{3}})} = \lim_{z \to 1} \frac{z + 1}{(z - e^{\frac{2\pi i}{3}})(z - e^{\frac{4\pi i}{3}})} = \frac{2}{(1 - e^{\frac{2\pi i}{3}})(1 - e^{\frac{4\pi i}{3}})}.$$ 

Since the above limit exists and is less than infinity, the point $z_0 = 1$ is a removable singularity of $f$. Moreover, define the function $g(z) = \frac{z + 1}{(z - e^{\frac{2\pi i}{3}})(z - e^{\frac{4\pi i}{3}})}$. Clearly, $g$ is analytic on the disc $b_1(1)$, and $g(z) = f(z)$ for every $z \in b_1(1) \setminus \{1\}$. Hence $g$ is an analytic extension of $f$. 


Solution of Part 2: The isolated singularities of \( f(z) = \cotan(z) \) are the solutions of the equation \( \sin(z) = 0 \), i.e. the set of points \( \{ z_k = k\pi : k \in \mathbb{Z} \} \). Moreover, each \( z_k \) is a pole of \( f \), because

\[
\lim_{z \to k\pi} |f(z)| = \lim_{z \to k\pi} \frac{\cos(z)}{|\sin(z)|} = \lim_{z \to k\pi} \frac{1}{|\sin(z)|} = \infty.
\]

Fix a positive integer \( k \in \mathbb{Z} \). By Taylor’s Theorem, we can write the power-series expansion of \( \sin(z) \) about \( k\pi \). Note that

\[
a_0 = \sin(k\pi) = 0,
\]

\[
a_1 = \cos(k\pi) = (-1)^k,
\]

\[
a_2 = -\frac{\sin(k\pi)}{2} = 0,
\]

\[
a_3 = -\frac{\cos(k\pi)}{3!} = -\frac{(-1)^k}{6},
\]

\[\vdots\]

Therefore for every \( z \in \mathbb{C} \),

\[
\sin(z) = (-1)^k(z-k\pi) - (-1)^k(z-k\pi)^3 + \ldots = (z-k\pi) \left( (-1)^k - (-1)^k(z-k\pi)^2 + \ldots \right).
\]

Now define the function \( g(z) = (-1)^k - (-1)^k(z-k\pi)^2 + \ldots \). The function \( g \) is analytic on \( \mathbb{C} \), \( g(k\pi) = (-1)^k \neq 0 \), and \( \sin(z) = (z-k\pi)g(z) \) for every \( z \in \mathbb{C} \). Now note that

\[
f(z) = \cotan(z) = \frac{\cos(z)}{\sin(z)} = \frac{\cos(z)}{(z-k\pi)g(z)} = \frac{\cos(z)}{g(k\pi)} \frac{1}{z-k\pi},
\]

where \( \frac{\cos(z)}{g(k\pi)} \) is analytic on \( b_1(k\pi) \), and \( \frac{\cos(k\pi)}{g(k\pi)} \neq 0 \). Therefore, \( z_k = k\pi \) is a pole of order 1 for \( f \).

Solution of Part 3: The function \( f(z) = \frac{\exp(z/2)}{z-i} \) is analytic on \( \mathbb{C} \setminus \{0, 1\} \). Note that \( f \) is analytic on the punctured disc \( D_1 = \{ z \in \mathbb{C} : 0 < |z - i| < \frac{1}{2} \} \), and the function \( g(z) = \exp(z/2) \) is analytic on the disc \( b_1(i) \). Moreover, \( g(i) \neq 0 \). Thus \( z_0 = i \) is a pole of order 1 for \( f \). To determine the type of singularity at \( z_1 = 0 \), observe that the limit \( \lim_{z \to 0^+} |f(z)| \) does not exist, because taking the limit on different paths does not result in the same value. For example,

\[
\lim_{x \to 0^+} \frac{\exp(x/2)}{|x - i|} = \infty,
\]

and

\[
\lim_{y \to 0, |y| < 0.5} \frac{\exp(y/2)}{|y - i|} \leq 4.
\]
Therefore \( z_1 = 0 \) is neither a pole nor a removable singularity, i.e. it is an essential singularity.

2. Locate each of the poles of the following functions. For each pole, find the corresponding order and the residue. Justify your answer.

1. \( f(z) = \frac{e^z}{z^3 + 4z} \).
2. \( f(z) = \frac{1}{z} - \frac{e^z}{z(z^2 + 1)} \).
3. \( f(z) = \frac{\pi}{\sin(z)} \).

**Solution of Part 1:** The function \( f(z) = \frac{e^z}{z^3 + 4z} = \frac{e^z}{z(z+2i)(z-2i)} \) has three isolated singularities \( z_0 = 0 \), \( z_1 = 2i \), and \( z_2 = -2i \).

**Case** \( z_0 = 0 \): Observe that \( f \) is analytic on the punctured disc \( D_1 = \{ z \in \mathbb{C} : 0 < |z - 0| < 1 \} \). Moreover,
\[
f(z) = \frac{H_1(z)}{z} \quad \text{with} \quad H_1(z) = \frac{e^z}{(z+2i)(z-2i)},
\]
where \( H_1 \) is analytic on the ball \( b_1(0) \), and \( H_1(0) \neq 0 \). Hence \( z_0 = 0 \) is a pole of order 1 for \( f \), and we have
\[
\text{Res}(f; 0) = H_1(0) = \frac{1}{4}.
\]

**Case** \( z_1 = 2i \): Observe that \( f \) is analytic on the punctured disc \( D_2 = \{ z \in \mathbb{C} : 0 < |z - 2i| < 1 \} \). Moreover,
\[
f(z) = \frac{H_2(z)}{z-2i} \quad \text{with} \quad H_2(z) = \frac{e^z}{z(z+2i)},
\]
where \( H_2 \) is analytic on the ball \( b_1(2i) \), and \( H_2(2i) \neq 0 \). Hence \( z_1 = 2i \) is a pole of order 1 for \( f \), and we have
\[
\text{Res}(f; 2i) = H_2(2i) = \frac{-e^{2i}}{8}.
\]

**Case** \( z_1 = -2i \): Observe that \( f \) is analytic on the punctured disc \( D_3 = \{ z \in \mathbb{C} : 0 < |z + 2i| < 1 \} \). Moreover,
\[
f(z) = \frac{H_3(z)}{z+2i} \quad \text{with} \quad H_3(z) = \frac{e^z}{z(z-2i)},
\]
where \( H_3 \) is analytic on the ball \( b_1(-2i) \), and \( H_3(-2i) \neq 0 \). Hence \( z_2 = -2i \) is a pole of order 1 for \( f \), and we have
\[
\text{Res}(f; -2i) = H_3(-2i) = \frac{-e^{-2i}}{8}.
\]
Solution of Part 2: The isolated singularities of the function $f(z) = \frac{1}{z} - \frac{e^z}{z(z^2+1)}$ are $z_0 = 0$, $z_1 = i$, and $z_2 = -i$.

Case $z_0 = 0$: Observe that $f$ is analytic on the punctured disc $D_1 = \{z \in \mathbb{C} : 0 < |z-0| < 1\}$. Moreover,

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{1}{z} - \frac{e^z}{z(z^2+1)} = \lim_{z \to 0} \frac{z^2 + 1 - e^z}{z(z^2+1)} = \lim_{z \to 0} \frac{2z - e^z}{3z^2 + 1} = -1.$$  

Hence $z_0 = 0$ is a removable singularity, and it is not a pole.

Case $z_1 = i$: Observe that $f$ is analytic on the punctured disc $D_2 = \{z \in \mathbb{C} : 0 < |z-i| < 1\}$. Moreover,

$$f(z) = \frac{H_2(z)}{z-i} \quad \text{with} \quad H_2(z) = \frac{z^2 + 1 - e^z}{z(z+i)},$$

where $H_2$ is analytic on the ball $b_1(i)$, and $H_2(i) \neq 0$. Hence $z_1 = i$ is a pole of order 1 for $f$, and we have

$$\text{Res}(f; i) = H_2(i) = \frac{e^i}{2}.$$  

Case $z_1 = -i$: Observe that $f$ is analytic on the punctured disc $D_3 = \{z \in \mathbb{C} : 0 < |z+i| < 1\}$. Moreover,

$$f(z) = \frac{H_3(z)}{z+i} \quad \text{with} \quad H_3(z) = \frac{z^2 + 1 - e^z}{z(z-i)},$$

where $H_3$ is analytic on the ball $b_1(-i)$, and $H_3(-i) \neq 0$. Hence $z_2 = -i$ is a pole of order 1 for $f$, and we have

$$\text{Res}(f; -i) = H_3(-i) = \frac{e^{-i}}{2}.$$  

Remark: The residue of a function at its removable singularities is always zero.

Solution of Part 3: The function $f(z) = \frac{z}{\sin(z)}$ is analytic on $\mathbb{C} \setminus \{k\pi : k \in \mathbb{Z}\}$. Fix $k \in \mathbb{Z}$, and let $g$ be the function defined in Question 1 Part 2, i.e. $\sin(z) = (z-k\pi)g(z)$ and $g(k\pi) \neq 0$. Therefore

$$f(z) = \frac{H(z)}{z-k\pi}, \quad \text{where} \quad H(z) = \frac{z}{g(z)}.$$
Case 1: If $k = 0$, then the point $z_0 = 0$ is a removable singularity of $f$, because,

$$
\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{\sin(z)} = 1.
$$

Case 2: If $k \neq 0$, then the above function $H(z)$ is analytic on $b_1(k\pi)$ and $H(k\pi) \neq 0$. Hence $k\pi$ is a pole of order one, and

$$
\text{Res}(f; k\pi) = H(k\pi) = (-1)^k k\pi.
$$

3. Use the residue theorem to compute each of the following integrals. All the curves are oriented counter-clockwise.

1. \( \int_{\gamma} \frac{z}{\sin(z)} \, dz \), where $\gamma$ is the circle of radius 5 centered at the origin.

2. \( \int_{\gamma} \frac{e^z}{(z-2)^2(z+3)(z-i)} \, dz \), where $\gamma$ is the circle of radius $\pi$ centered at the origin.

Solution of Part 1: The singularities of the function $f(z) = \frac{z}{\sin(z)}$ are \( \{k\pi : k \in \mathbb{Z}\} \). Note that only $\pi$, $-\pi$ and 0 lie inside the curve $\gamma$. Using the results of Question 2 Part 3, we have

$$
\int_{\gamma} \frac{z}{\sin(z)} \, dz = 2\pi i [\text{Res}(f; 0) + \text{Res}(f; \pi) + \text{Res}(f; -\pi)] = 2\pi i [0 + (-1)^1 \pi + (-1)^{-1} (-\pi)] = 0.
$$

Solution of Part 2: The singularities of the function $f(z) = \frac{e^z}{(z-2)^2(z+3)(z-i)}$ are $z_0 = 2$, $z_1 = -3$, and $z_2 = i$.

Case $z_0 = 2$: Observe that $f$ is analytic on the punctured disc $D_1 = \{z \in \mathbb{C} : 0 < |z-2| < 1\}$. Moreover,

$$
f(z) = \frac{H_1(z)}{(z-2)^2} \quad \text{with} \quad H_1(z) = \frac{e^z}{(z+3)(z-i)},
$$

where $H_1$ is analytic on the ball $b_1(2)$, and $H_1(2) \neq 0$. Hence $z_0 = 2$ is a pole of order 2 for $f$, and we have

$$
\text{Res}(f; 2) = \frac{H_1'(2)}{1!} = \frac{e^2 (3-4i)}{25(3-4i)} = \frac{e^2}{25}.
$$

Case $z_1 = -3$: Observe that $f$ is analytic on the punctured disc $D_2 = \{z \in \mathbb{C} : 0 < |z+3| < 1\}$. Moreover,

$$
f(z) = \frac{H_2(z)}{z+3} \quad \text{with} \quad H_2(z) = \frac{e^z}{(z-2)^2(z-i)},
$$

$$
\text{Res}(f; -3) = \frac{H_2'(-3)}{(-3)!} = \frac{e^{-3} (-2)(3-4i)}{24(3-4i)} = \frac{e^{-3}}{24}.
$$
where $H_2$ is analytic on the ball $b_1(-3)$, and $H_2(-3) \neq 0$. Hence $z_1 = -3$ is a pole of order 1 for $f$, and we have

$$\text{Res}(f; -3) = \frac{H_2(-3)}{0!} = \frac{e^{-3}}{25(-3 - i)} = \frac{e^{-3}}{250}(3 - i).$$

**Case** $z_2 = i$: Observe that $f$ is analytic on the punctured disc $D_3 = \{z \in \mathbb{C} : 0 < |z - i| < 1\}$. Moreover,

$$f(z) = \frac{H_3(z)}{z - i} \text{ with } H_3(z) = \frac{e^z}{(z - 2)^2(z + 3)},$$

where $H_3$ is analytic on the ball $b_1(i)$, and $H_3(i) \neq 0$. Hence $z_2 = i$ is a pole of order 1 for $f$, and we have

$$\text{Res}(f; i) = \frac{H_3(i)}{0!} = \frac{e^i}{(3 - 4i)(i + 3)} = \frac{e^i}{13 - 9i}.$$

All of the singularities of $f$ lie inside $\gamma$. Hence

$$\int_\gamma \frac{e^z}{(z - 2)^2(z + 3)(z - i)}\,dz = 2\pi i [\frac{e^2}{25} - \frac{e^{-3}}{250}(3 - i) + \frac{e^i}{13 - 9i}].$$

4. Let $f$ be a function analytic on a punctured disc $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Prove that if $\lim_{z \to z_0} f(z)$ exists and is less than infinity then $z_0$ is a removable singularity of $f$. (Hint: Use the $\varepsilon - \delta$ definition of limits to show that if $\lim_{z \to z_0} f(z) < \infty$ exists, then $|f(z)|$ is bounded by a constant $M$.)

**Solution:** Suppose that $\lim_{z \to z_0} f(z) = L$. Let $\epsilon = 1$. Then there exists $\delta > 0$ such that $|f(z) - L| < 1$ for every $z \in b_\delta(z_0)$. Hence by triangle inequality $|f(z)| < L + 1$ for every $z \in b_\delta(z_0)$. Therefore, by the theorem proved in the class, $z_0$ is a removable singularity.